Proceedings of the
Eighth European Summer University on
History and Epistemology in Mathematics Education

ESU 8

**Edited by Evelyne Barbin, Uf e Thomas Jankvist, Tinne Hof Kjeldsen,
Bjørn Smestad, Constantinos Tzanakis**

Skriftserie 2019 nr 11

OSLO METROPOLITAN UNIVERSITY
STORBYUNIVERSITETET

OsloMet – Oslo Metropolitan University

Proceedings

of the

Eighth European Summer University

on

History and Epistemology in Mathematics Education

ESU 8

Edited by:
Evelyne Barbin, University of Nantes, France
Uffe Thomas Jankvist, Aarhus University, Denmark
Tinne Hoff Kjeldsen, University of Copenhagen, Denmark
Bjørn Smestad, Oslo Metropolitan University, Norway
Constantinos Tzanakis, University of Crete, Greece

Oslo, Norway
20 - 24 July 2018

OsloMet Skriftserie 2019 nr 11

ISSN 2535-6984 (trykt)
ISSN 2535-6992 (online)

ISBN 978-82-8364-211-7 (trykt)
ISBN 978-82-8364-212-4 (online)

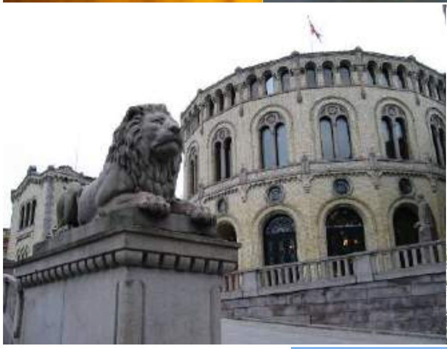
OsloMet – storbyuniversitetet
Læringssenter og bibliotek,
Skriftserien
St. Olavs plass 4,
0130 Oslo,
Telefon (47) 64 84 90 00

Postadresse:
Postboks 4, St. Olavs plass
0130 Oslo

Trykket hos Byråservice

Trykket på Scandia 2000 white, 80 gram på materiesider/200 gram på coveret








ESU8
European Summer University on history and epistemology in mathematics education
At OsloMet
July 20th-24th


Table of Contents

		page
		i
Proceedings presentation		vii

1. Theoretical and/or conceptual frameworks for integrating history and epistemology of mathematics in mathematics education

Plenary Lecture

1.1 H. N. Jahnke	*Hermeneutics, and the question of "How is science possible?"*	3

Workshops

1.2 A. Demattè & D. Guillemette	*Thinking with Levinas about history of mathematics and mathematics education*	23
1.3 A. G. Hitchcock	*Niels Abel: 'So many ideas …'- A workshop on using theatre to bring episodes in the history of mathematics to life in the classroom*	43
1.4 S. Lawrence	*What can art teach us about mathematics?* (abstract)	65
1.5 X. Wang, J. Zou, Z. Yue & Z. Shen	*HPM and in-service mathematics teachers' professional development in China* (abstract)	67

Oral Presentations

1.6 É. Barbin	*Using ancient instruments in the teaching of geometry with Bachelard's phenomeno-technology*	69
1.7 R. Capone, M. R. Del Sorbo & V. Ninni	*Using artifacts and dynamic geometry software in primary school inspired by Montessori method*	81
1.8 D. Guillemette	*Being in research and doing research on history and mathematics education in a dialogical perspective*	91
1.9 U. T. Jankvist & E. Geraniou	*Digital technologies as a way of making original sources accessible to students*	107
1.10 A. Mutanen	*On mathematical reasoning*	131
1.11 P. Bonissoni, M. Cazzola, P. Longoni, E. Rottoli, G. Riva & S. Sorgato	*Philosophical and didactic practice in the universe of fractions: Trace and icon*	143
1.12 S. Schorcht & N. Buchholtz	*Different facets of pre-service teachers' beliefs on the history of mathematics*	155
1.13 Z. Shen	*HPM lesson study in the context of an HPM learning community: A case study in Chinese senior high school*	177
1.14 D. Sun	*A categorization model of the "Hows" of using history in mathematics education: An empirical study* (abstract)	191

2. History and epistemology in students and teachers mathematics education: Curricula, courses, textbooks, and didactical material of all kinds - their design, implementation and evaluation

Plenary Lecture

2.1 I. Witzke	*Epistemological beliefs about mathematics – Challenges and chances for mathematical learning: Back to the future*	195

Plenary Panel Discussion

2.2 C. Vicentini (coordinator), N. Chevalarias, K. M. Clark, M. Roelens	*History, epistemology and teaching mathematics: A challenging partnership?*	205

Workshops

2.3 C. Can, M. E. Aktas, J. H. Barnett & K. M. Clark	*Leonhard Euler's differentials: An attempt to restructure teaching of the derivative concept* (abstract)	221
2.4 M. K. Clark, G. Stoffels, I. Witzke & H. Struve	*Capturing student beliefs at, during, and because of the transition from school to university mathematics: Evidence of influence of the historical development of geometry* (abstract)	223
2.5 C. Guillet, M-L. Moureau & I. Voillequin	*Mathematics and experiment: How to calculate areas without formulas?*	227
2.6 H. Languereau, Hombeline & A. Michel-Pajus	*Using French websites to find useful online material to integrate the history and epistemology of mathematics into our teaching*	245
2.7 A. Popotis, & K. Nikolantonakis	*The contribution of the Chinese abacus to the development of the number sense*	251
2.8 M. Roelens	*The bicylinder or birdcage or móuhéfāng gài: Combining a cultural approach with many other goals of mathematics education*	273
2.9 G. Stoffels	*(R)evolutions in probability theory: Students reflecting their own beliefs about mathematics by dealing with original sources from 20th century development of probability theory* (abstract)	287
2.10 D. Tournès, N. Daval & M. Mouyssinat	*Learning arithmetic with counting boards and jetons* (abstract)	289

Oral Presentations

2.11 N. Chevalarias	*Some elements on the training in history of mathematics for teachers in France*	291
2.12 T. Deligiannidis & K. Nikolantonakis	*Developing geometric proportional thinking to 6th grade students with the use of a historical instrument of Errard de Bar le Duc*	303
2.13 B. Durmaz	*Mathematics and science history contexts of mathematics textbooks in Turkey* (abstract)	315
2.14 H. Gu B. Hou	*Using history to teach complex number* (abstract)	317
2.15 U. T. Jankvist, M. Sánchez Aguilar & M. Misfeldt	*Tschirnhaus' transformation: mathematical proof, history and CAS*	319
2.16 E. Lappa & K. Nikolantonakis	*The teaching of logarithms in upper secondary school from a historical perspective*	331
2.17 X. Li	*A practical study of using the history of mathematics in a flipped classroom* (abstract)	343
2.18 P-H. Liu	*An international comparative study on how mathematical culture is implemented in the textbooks*	345
2.19 Y-K. Man	*Using a historical problem in a mathematics problem*	355

	solving class (abstract)	
2.20 X. Wang	*The use of historical materials in mathematics teaching: the case of logarithms* (abstract)	357
2.21 Q. C. Yan	*What specialized content knowledge do senior high teachers have about trigonometry from the perspective of HPM? An exploration and case study* (abstract)	361
2.22 Z. Yue, Z. Shen, X. Wang & J. Zou	*Research on factors affecting mathematics teachers' HPM lesson study* (abstract)	363

Short Oral Communications

2.23 T-S. Chen	*Researching high school students' strategies for solving the Chinese rings puzzle* (abstract)	367
2.24 A. J. Lemes	*Potentialités de l'histoire des mathématiques dans la formation des enseignants de mathématiques* (abstract)	371

3. Original historical sources in teaching and learning of and about mathematics

Plenary Lecture

3.1 F. Métin	*Between words and artefacts: Implementing history in the math class from kindergarten to teacher training*	375

Workshops

3.2 S. Bella & M. Blanco	*Quelle rigueur pour enseigner l'analyse? Ce que nous apprend le calcul des différences (1696-1768)* (abstract)	397
3.3 P. Blaszczyk	*On Euler's formula - between standard and non-standard analysis: An interpretation of Euler's "Introductio in analysin Infinitorum"* (abstract)	399
3.4 A. Boyé & X. Lefort	*Un éclairage historique pour l'enseignement des nombres négatifs* (abstract)	401
3.5 R. Chorlay	*Why bother with original sources?*	403
3.6 M. Moyon	*Teaching mathematics and algorithmics with recreational problems: the Liber Abaci of Fibonacci*	417

Oral Presentations

3.7 M. Chiorescu	*Engaging with Primary Sources in a Mathematics for the Liberal Arts Course*	437
3.8 , I. Guevara Casanova & C. Puig-Pla	*Reversed procedure and Kuttaka method: The calculation of Indian Mathematics (ganita) in Aryabhatiya and Brahma-sphuta-siddhanta*	449
3.9 M. Mauntel	*A case study of the implementation of primary sources in undergraduate mathematics*	463
3.10 M. O'Reilly	*"What is maths without a challenge!": Reporting on how undergraduate mathematics students in an Irish university worked with original sources in a novel context* (abstract)	481
3.11 H. Pinto & T. C. Clain	*Histórias com Ciência na Biblioteca Escolar [Histories with Science in the School Library]: A project to bring topics of History of Science to secondary schools in*	483

	Aveiro (Portugal)	
3.12 E. Zubillaga Guerrero, M. T. González Astudillo & F. M. Rodríguez Vásquez	*Jordan's isomorphism concept in the work "Traité des substitutions et des équations algébriques"*	499

4. Mathematics and its relation to science, technology, and the arts: Historical issues and socio-cultural aspects in relation to interdisciplinary teaching and learning

Plenary Lecture

4.1 S. Lawrence	*The art and architecture of mathematics education: A study in metaphors*	515

Workshops

4.2 F. Métin	*17th century fortification and geometry: A military and mathematical revolution*	531
4.3 P. Ransom	*The geometry of the Dambusters: A cross-curricular approach using history in the mathematics classroom with students and teachers*	545
4.4 J. M. Rodin	*How to use culturally relevant trans-disciplinary activities to improve student attitudes and learning in school mathematics* (abstract)	557

Oral Presentations

4.5 M. G. Adesso, R. Capone, O. Fiore & F. S. Tortoriello	*Discovering neglected synthetic geometry on social networks: Learning maths as in the historical Italian academies*	561
4.6 A. Affan & M. Fried	*Potential for collaboration between history and mathematics teachers: An investigation and framework based on a text by Abū'l-Wafā᾿ Buzj'ani*	575
4.7 A. Bernard	*Borel's approach to mathematics, probability and citizenship*	585
4.8 D. Calandrino, M. Cecchi, A. Ferrini, L. Isolani, V. Natali & C. Tognaccini	*The magic of the East – from the Alhambra to Sammezzano Castle: Symmetries in mathematics, nature and art*	601
4.9 A. G. Hitchcock	*Nothing left to be desired: The naming of complex numbers*	609
4.10 L. Kvasz	*The concept of space in the history of mathematics and in the history of painting* (abstract)	619
4.11 C-C. Liao	*How the counting rod configuration affects the presentation of the method of "fangcheng" in Qin Jiushao's "Shushu jiuzhang"* (abstract)	621
4.12 L. Rogers	*Technology, radical education, and applications of mathematics in the pre-industrial period in Baconian England (1580-1750)* (abstract)	623
4.13 F. Romero Vallhonesta & M. R. Massa-Esteve	*Sources from the 16th century for the teaching and learning of mathematics*	627
4.14 C. Tzanakis	*Time Measurement as an interdisciplinary subject in Mathematics Education*	641

5. Topics in the history of mathematics education

Plenary Lecture

5.1 M. Menghini	*The fusion of plane and solid geometry in the teaching of geometry: Textbooks, aims, discussions*	679

Oral Presentations

5.2 M. C. Almeida	*The 'New math' in mathematics teachers training in Portugal (1957-1969)* (abstract)	695
5.3 D. Basyal	*Description of old Nepali mathematics books and their potential in improving current day teaching and learning*	697
5.4 K. Bråting	*Development of school algebra – a comparison between the 1980 and 2011 Swedish mathematics curricula*	711
5.5 K. Karpińska	*Mathematics teaching in gymnasia and real schools in Poland in the years 1795-1918: Schools with Polish and German as the language of instruction - comparison*	727
5.6 E. Lakoma	*On the main milestones in developing mathematics in Poland prior to the XIX century through the lens of mathematics education*	759
5.7 D. M. Narváez	*The didactical contract, its effects and clauses: A historical Study* (abstract)	773
5.8 L. Puig	*Joseph Zaragoza's "Arithmetica Universalis" and the teaching of algebra in Spain in the second half of the 17th century* (abstract)	775
5.9 M. K. Siu	*Equations in China: Two millennia of innovation, transmission and re-transmission*	777
5.10 J. J. Tattersall	*A Cambridge Correspondence Class in arithmetic for women*	793
5.11 G. Vanpaemel & D. De Bock	*New Math, an international movement?*	801

6. History of mathematics in the Nordic countries

Plenary Lecture

6.1 A. Christiansen	*The first Norwegian textbooks in mathematics: A story of independence and controversy*	815

Oral Presentations

6.2 K. Bjarnadóttir & B. V. Halldórsson	*The Norse Treatise Algorismus preserved in Manuscript GKS 1812 4to*	841
6.3 R. Guitart	*Problems and methods in elementary geometry, according to Julius Petersen* (abstract)	859
6.4 J. Pejlare	*Infinite sums and the calculation of π, as presented by the Swedish mathematician Anders Gabriel Duhre in the early 18th century*	861

The International Scientific Program Committee	875
The Local Organizing Committee	877
Author Index	879
Participants/Countries	883

PROCEEDINGS PRESENTATION

The 8th *European Summer University on the History and Epistemology in Mathematics Education* (ESU-8) took place at Oslo Metropolitan University, Pilestredet Campus in Oslo, Norway from 20 to 24 July 2018. This was the eighth meeting of this kind since July 1993, when, on the initiative of the French IREMs[1], the first *European Summer University on the History and Epistemology in Mathematics Education* took place in Montpellier, France. The next ESUs took place in Braga (Portugal, 1996; conjointly with the *HPM*[2] Satellite Meeting of ICME 8), Louvain-la-Neuve and Leuven (Belgium, 1999), Uppsala (Sweden, 2004; conjointly with the *HPM* Satellite meeting of ICME 10), Prague (Czech Republic, 2007), Vienna (Austria, 2010) and Copenhagen (Denmark, 2014).

Since its original conception and realization, ESU has been developed and established into one of the major activities of the HPM Group. From 2010 onwards, it is organized every four years, so that every two years at least one major international meeting of the HPM Group takes place; namely, ESU and the HPM Satellite Meeting of ICME.

The purpose of ESU is not only to stress the multifarious role that history and epistemology can play in the teaching and learning of mathematics, in the sense of a technical tool for instruction, but also to reveal that mathematics should be conceived as a living science, with a long history and a vivid present. And in this way to become more deeply aware of what mathematics as a discipline is and how it grows; more specifically, that mathematics:

- has undergone changes over time, underscored by shifting views of what mathematics is and how it should be taught and learnt;
- has been in constant dialogue with other scientific disciplines, technology, philosophy and the arts;
- has constituted a constant force for stimulating and supporting scientific, technical, artistic and social developments;
- is the result of contributions from many different cultures.

In this connection, ESU has a threefold aim as an international meeting: (i) to provide a school for working on a historical, epistemological and cultural approach to mathematics and its teaching, with emphasis on actual implementation; (ii) to give the opportunity to mathematics teachers, educators and researchers to share their teaching ideas and classroom experiences related to a historical perspective in teaching; and (iii) to motivate further collaboration along these lines, among teachers of mathematics and researchers on history, epistemology and education of mathematics in Europe and beyond.

In this sense, ESU is more a collection of intensive courses than a conference for researchers. It is a place where teachers and researchers meet and work together and where beginners, more experienced researchers and teachers present their teaching experiences to the benefit of the participants from whom they get constructive feedback. ESU refers to all levels of education – from primary school, to tertiary education and including in-service teachers' training – with focus preferably on work and conclusions based on actual classroom experiments and/or produced teaching and learning materials.

[1] *Institut de Recherche sur l'Enseignement des Mathématiques*.
[2] An abbreviation for the *International Study Group on the Relations between the History and Pedagogy of Mathematics*, affiliated to the *International Commission on Mathematical Instruction* (ICMI).

Publishing the Proceedings of the ESUs has always been a major task, since in all cases they have become standard references in this domain[3]. The present volume collects papers or abstracts stemming from all types of activities that were accepted and included in the scientific programme of ESU 8, and is divided into six sections corresponding to each of the six main themes of ESU 8; namely:

Theme 1: Theoretical and/or conceptual frameworks for integrating history and epistemology of mathematics in mathematics education;

Theme 2: History and epistemology in students and teachers mathematics education: Curricula, courses, textbooks, and didactical material of all kinds - their design, implementation and evaluation;

Theme 3: Original historical sources in teaching and learning of and about mathematics;

Theme 4: Mathematics and its relation to science, technology, and the arts: Historical issues and socio-cultural aspects in relation to interdisciplinary teaching and learning;

Theme 5: Topics in the history of mathematics education;

Theme 6: History of mathematics in the Nordic countries.

For each main theme, one plenary lecture was delivered and its text appears in the corresponding section. The same holds for the panel discussion, which was also delivered in a plenary session. There are also papers coming from workshops, which are a type of activity of special interest, making focus on studying a specific subject and having a follow-up discussion. The role of the workshop organizer is to prepare, present and distribute the historical/epistemological (2-hour workshops) or pedagogical/didactical material that integrates historical elements (1.5-hour workshops), which motivates and orients the exchange of ideas and the discussion among the participants, after they have read and worked on the basis of this material. Finally, there are texts and abstracts based on 30-minute oral presentations and abstracts of 10-minute short oral communications. Each paper or abstract appears under the main theme to which the corresponding activity during ESU-8 was mostly (though not exclusively) related.

Each paper for a workshop or an oral presentation has been reviewed by one or two members of the Scientific Program Committee at the usual international standards. In almost all cases authors were asked to amend their papers. Papers that have been finally accepted are included in the present volume. In all other cases in which either the text was not accepted, or no full text has been submitted, only an abstract of the corresponding contribution appears. In total there are 51 papers and 28 abstracts corresponding to the 81[4] activities accepted and included in the scientific program of ESU-8[5], authored by 112 contributors coming from 25 countries worldwide. Moreover, ESU-8 was attended by 113 registered participants from 27 countries, including secondary school teachers, university teachers and graduate students, historians of mathematics, and mathematicians, all interested in the relations between mathematics, its history and epistemology, its teaching, and its role today and in the past. It is important to note that many of them participated for

---

[3] For detailed reference see the HPM website http://www.clab.edc.uoc.gr/hpm/about HPM

[4] There were two cases in which the authors contributed to the program with two activities, but merged their reports into one final paper in each case.

[5] For various reasons another 9 activities that had been accepted were withdrawn by their authors before ESU-8 took place.

the first time in a meeting organized in the context of the HPM Group[6]. We thank all of them. Special thanks go to those 63 members of the International Scientific Program Committee, (see pp. 875-877), who willingly reviewed the submitted papers, thus contributing essentially to the scientific quality of this volume. We are particularly grateful to all members of the Local Organizing Committee (see p. 877), who succeeded to make ESU-8 an insightful and interesting scientific event that took place in a warm and friendly atmosphere. We warmly thank the Oslo Metropolitan University for hosting and financially supporting ESU-8, and its personnel for their help and kindness.

Evelyne Barbin, Université de Nantes, France
Uffe Thomas Jankvist, Aarhus University, Denmark
Tinne Hoff Kjeldsen, University of Copenhagen, Denmark
Bjørn Smestad, Oslo Metropolitan University, Norway
Constantinos Tzanakis, University of Crete, Greece

[6] A list of the participants, as well as an author index is found at the end of the Proceedings (pp. 879-885).

Theoretical and/or conceptual frameworks for integrating history and epistemology of mathematics in mathematics education

HERMENEUTICS, AND THE QUESTION OF “HOW IS SCIENCE POSSIBLE?”

Hans Niels JAHNKE

Faculty of Mathematics, University of Duisburg-Essen, Essen, Germany
njahnke@uni-due.de

ABSTRACT

The present paper discusses the educational potential of reading historical sources in the mathematics classroom under the viewpoint of hermeneutics. In a first step, key concepts of Gadamerian hermeneutics will be discussed, ‘application’, ‘hermeneutic circle’ and ‘prejudice’. Taken in their entirety they characterize what Gadamer calls the ‘historicity of understanding’ which reflects the intellectual situatedness of any reader and provides a foundation to the idea that any text allows a multitude of possible interpretations. In a second step, three readers of ancient Greek texts on astronomy are studied in order to exemplify the functioning of Gadamer’s hermeneutic conception. The readers are students (grade 9), the pedagogue Martin Wagenschein (1896 – 1988) and the philosopher Karl Raimund Popper (1902 - 1994). We show their different intellectual backgrounds, different questions and problems they were interested in and the quite different ways of reading the sources. Nevertheless, there exists an important point of convergence. They all understood the epistemological relevance of their ancient sources culminating in the question of “How is science possible?”. Indeed, this question is and should be a central theme of education.

1 Introduction

The present paper discusses the educational potential of reading historical sources in the mathematics classroom under the viewpoint of hermeneutics. In recent times, reading historical sources has become a fruitful avenue of theoretical and practical educational activity on all levels of the teaching of mathematics. Some references might highlight the scope and variety of work (Arcavi & Isoda 2007, Kjeldsen & Blomhøj 2012, Barnett et al 2013, Jahnke 2014, Chorlay 2016). A nearly overview on these activities is given by Jahnke et al (1990), a more recent survey can be found in Clark et al(2016)

Beyond that, there is in many countries a trend towards a stronger emphasis on activities of modelling. Frequently, this requires tasks with extensive texts for detailing circumstances and conditions of the modelling problem in question. Therefore, interpreting and writing texts enters the mathematics classroom from this side, too. In sum, language, reading and writing will take up a more extended space in today’s teaching of mathematics.

We approach our problem by looking in more detail on the process of reading and interpreting a text and ask what hermeneutics, that is the theory and methodology of interpreting texts, has to contribute to this problem. We shall see that hermeneutics will allow a twofold perspective. There is a *dynamical* perspective insofar reading and interpreting changes the text and the reader. After the very act of reading, the text is not the same as before, and also the reader has changed. Thus, hermeneutics makes understandable the necessary *variety of possible interpretations* which is not an outcome of vague thinking but of a process involving rigour and imagination.

The argument of this paper will be unfolded in four steps. First, some key concepts of Gadamerian hermeneutics will be discussed. Afterwards, we study three (types of) readers illuminating Gadamer’s concept of the ‘historicity of understanding’ and leading to the theme of the variety of possible interpretations. We study students (grade 9) reading an ancient source on astronomy, a pedagogue and a prominent philosopher. This part of the paper has some relations to Fried (2018) though its conceptual frame is different. In the end we shall return to the educational potential of reading sources in the mathematics classroom.

2 Gadamer's hermeneutics

Modern hermeneutics originated from methods used for interpreting theological and legal texts. It was only in the 17th and 18th centuries that some authors formed the idea of a general theory and method of interpretation independent of the subject to which it was applied. In Germany, probably for the first time, the Latin term 'hermeneutica' in this general sense can be found with Johann Conrad Dannhauer (1603–66) whereas Johann Martin Chladenius (1710–59) und Georg Friedrich Meier (1718–77) wrote influential books on this topic. (Joisten 2010, ch. 6). It was Chladenius who introduced the concept of 'Sehe-Punkt' as translation of the Latin 'scopus' (engl. 'point of view') which aimed at the idea of different interpretations depending on the 'Sehe-Punkt' of the reader. (Joisten 2010, 88ff, Gadamer 1990, 186). Modern hermeneutics then as an established field of study emerged in the beginning of the 19th century with the romantic movement and historicism in the work of theologian F. Schleiermacher (1768-1834) (cf. Schleiermacher 1977) and others.

Gadamer's (1900 – 2002) 'philosophical hermeneutics' can be considered as an attempt to mediate between the 'hermeneutical philosophy' of young Heidegger (1889 – 1976) and traditional hermeneutics of the 19th century. The relation between these two points of reference still seems to be a philosophically unsolved problem (Scholz 2011, 444). Thus, it is not without difficulties to refer to Gadamer's *Truth and Method* (Gadamer 1990, originally published in 1960, English translation Gadamer 2004)[1] as a theoretical frame for analysing processes of understanding texts. The present paper tries to circumvent systematic philosophical problems by pragmatically limiting and concentrating the discussion on three key concepts of Gadamer's hermeneutics, namely 'application', 'hermeneutic circle', and 'prejudice'. These concepts are introduced and elaborated in the middle part of his book in chapter II 'Elements of a Theory of Hermeneutic Experience' (TaM, 203ff) ('Grundzüge einer Theorie der hermeneutischen Erfahrung' (WuM, 270ff)).

Application. With religious and legal texts the problem of arriving at an unambiguous and 'correct' interpretation receives a special urgency. Laws should be binding rules of behaviour within a community or a state which are valid for everybody. The same is true for religious texts. Obviously, a community can only function when laws are applied to any concrete case in the same way. But what could 'the same way' mean? Already in the 14th century, Italian jurist Bartolus distinguished between three possible types of interpretation, a 'declarative' one explaining only the meaning of the words, a 'restrictive' one limiting the set of cases to which the law was to be applied, and an 'extensive' interpretation which widens the number of cases. (Schröder 2011, 206). In principle, when the meaning of the words did not suffice to apply a law jurists had to take into account the 'sense' of the law. In such a situation, jurists would appeal to 'reason' or to general

[1]We refer to Gadamer (1990) as: WuM, to Gadamer (2004) as: TaM. We shall quote only the English translation TaM to which we add the page numbers of the German original WuM.

principles of jurisdiction and frequently arrived at rather extensive interpretations. But from the times of renaissance up to the 19th century there happened a shift of argument and jurists would point more and more to the will of the legislator and the special circumstances under which the law was issued. This shift contributed considerably to a growing awareness that understanding a text must take into account its author and his times and, thus, has a necessary historical dimension. This development substantially furthered the emergence of general hermeneutics (cf. Schröder 2011).

Gadamer is well aware of this development, but in his considerations on 'application' he stresses another point of view. He considers 'application' "just as integral a part of the hermeneutical process as are understanding and interpretation" (TaM, 318; WuM, 313). As he explains, the decisive point with 'application' is

> "that understanding always involves something like applying the text to be understood to the interpreter's present situation. Thus we are forced to go one step beyond romantic hermeneutics, as it were, by regarding not only understanding and interpretation, but also application as comprising one unified process" (loc.cit.).

The insight that understanding a text necessarily involves considering the historical situation and the author as a historical person was already reached in 19th century romantic hermeneutics and historicism. To Gadamer the new element was the application of a text "to the interpreter's present situation". This point of view he exemplifies with several cases. An interpreter in a negotiation should not simply repeat what one of the partners says in the discussion he is translating, but he should say it in a way that seems most appropriate to him, since he alone knows both languages (loc.cit.).

In a similar vein a reproduction of a historical piece of music is more than a mere reproduction. Rather, the player has to take into account "that the stylistic values of one's own day… set limits to the demand for a stylistically correct reproduction" (TaM, 320; WuM, 315).

Returning to the relation of theological and legal hermeneutics to general hermeneutics he says:

> "The fact that philological, legal, and theological hermeneutics originally belonged closely together depended on recognizing application as an integral element of all understanding. In both legal and theological hermeneutics there is an essential tension between the fixed text—the law or the gospel—on the one hand and, on the other, the sense arrived at by applying it at the concrete moment of interpretation, either in judgment or in preaching. A law does not exist in order to be understood historically, but to be concretized in its legal validity by being interpreted. Similarly, the gospel does not exist in order to be understood as a merely historical document, but to be taken in such a way that it exercises its saving effect. This implies that the text, whether law or gospel, if it is to be understood properly—i.e., according to the claim it makes—must be understood at every moment, in every concrete situation, in a new and different way. Understanding here is always application" (loc.cit., 318/9; WuM, 314)

Thus, the core of application is the fact that a text must be understood "at every moment, in every concrete situation, in a new and different way." Whether we read a love poem or a historical document on an administrative act we inevitably relate it to our situation, our ideas, concepts, emotions, phantasies, former experiences, former studies etc. That is, we apply the text. And by applying the text we add to it connotations and dimensions of meaning the author necessarily could not have thought of.

Hermeneutic Circle. In Jahnke (2014, 84pp) and Fried et al (2016, 216ff) the reader will find a short account of the hermeneutic circle applied to reading sources in the mathematics classroom. Hermeneutics distinguishes systematically between the author and the reader of a text and their different perspectives. This causes an experience of dépaysement (Barbin 1994). Gadamer speaks of the 'temporal distance' ('Zeitenabstand', TaM, 303; WuM, 296) and of different 'horizons'. Understanding amounts in his terminology to a 'fusion of horizons' (TaM, 310ff) ('Horizontverschmelzung', WuM, 305ff). In hermeneutics the process by which the fusion of horizons occurs is described by a spiral, the so-called 'hermeneutic circle' which points to the necessity of already possessing an interpretation of a text in order to gain a new interpretation. A reader starts with a certain expectation what the text might be about. Then s*he reads the text and realizes that some aspects of the expectation do not agree with what is said in the source. Thus, s*he has to modify the expectation, read again, modify and so on until s*heis satisfied with the result. Here is one of Gadamer's descriptions of the spiral:

> "We know this from learning ancient languages. We learn that we must 'construe' a sentence before we attempt to understand the linguistic meaning of the individual parts of the sentence. But the process of construal is itself already governed by an expectation of meaning that follows from the context of what has gone before. It is of course necessary for this expectation to be adjusted if the text calls for it. This means, then, that the expectation changes and that the text unifies its meaning around another expectation. Thus the movement of understanding is constantly from the whole to the part and back to the whole. Our task is to expand the unity of the understood meaning centrifugally. The harmony of all the details with the whole is the criterion of correct understanding. The failure to achieve this harmony means that understanding has failed" (TaM, 303/4; WuM, 296)

We single out from this quotation two aspects.

(1) The hermeneutic circle is a process of adaption. Successful interpretation means that the harmony between the expectations of the reader and the text is step by step enhanced. This might be visualized by a diagram.

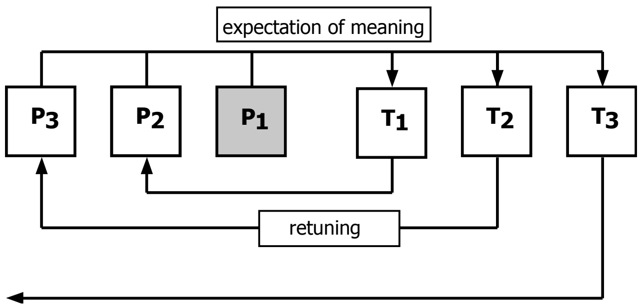


Figure 2.1: Hermeneutic Circle (cf. Glaubitz 2011, 61)

(2) Gadamer describes the process of adaptation as a dialectical oscillation between whole and part. This might refer to the interplay between the meaning of a single word and the meaning of a phrase in which a word occurs. In further steps the reader has to take

into account the meaning of, say, a paragraph in its interplay with the whole text in front of her/him. In this way ever larger units of text have to be taken into account, say, a collection of texts up to reconstructing, for example, the philosophical thinking of Greek antiquity. The dialectic of part and whole is a principle problem of understanding, experienced when reading a piece of literature as well as of mathematics.

Seen from the dialectic of part and whole it is teachable to consider a subtle difficulty in the English translation above. The English speaks of a ‘change’ of expectation whereas the German original uses the word ‘umstimmen’ instead of ‘change’. The literal translation of ‘umstimmen’ is ‘retune’, a musical metaphor which expresses much better the intended holistic meaning.

In regard to the hermeneutic circle I would like to point at the fact that it is quite analogous to the spiral of modelling and can be considered as a process in which a hypothesis is put up, tested against the (empirical) data, modified, tested again and so on until the reader arrives at a satisfactory result. With modelling, too, it is an important point of view that it aims not only at a better and better representation of the problem in question, but that it is also dependent of the situation and the needs of the creator of the model.

Considered in this perspective hermeneutics can be related to the “hypothetico-deductive method”. This is basically the argument by Foellesdal (1979) who elaborates it by interpreting a piece of literature (Ibsen’s Peer Gynt). Gadamer, too, in his late years agreed that there is a certain reconciliation of his philosophical hermeneutics and certain directions of thinking in modern analytic philosophy (Gadamer 1976, 1070).

The conception of the hermeneutic circle was already well known in the 19th. But people at that time thought that in the hermeneutic spiral the reader approaches better and better the ‘real meaning’ of a text by putting oneself better and better into the historical situation of its author. Though ‘real meaning’ might only be a regulative ideal which can never be reached, it was the aim of interpretation in the eyes of Schleiermacher and Dilthey. Gadamer’s conception is radically different. He thinks that interpretation and understanding necessarily involve also an adaptation of a text to the thinking of its reader such that a new meaning emerges which the original author never would have thought of. This is a motive we have already seen in the concept of ‘application’. It will become fully clear with the concept of ‘prejudice’.

Prejudice. With the concept of prejudice Gadamer takes up an essential element of the philosophy of his teacher Heidegger. There is a whole word field of interrelated concepts stemming from Heidegger and being used by Gadamer. They are: Vor-Urteil = pre-judgement, Vor-Meinung = fore-meaning, Vor-Entwurf = fore-projection, Vor-Sicht = fore-sight, Vor-Griff = fore-conception. The basic message enshrined in these concepts is “to elevate the historicity of understanding” to the status of a hermeneutic principle (TaM, 284; WuM, 270). Historicity here does not mean to consider a text as an historical document, but the fact that the very act of understanding itself happens at a certain moment in time and is an act in history. Any act of understanding starts with a certain “fore-conception” or “pre-judgement” (the P1 in fig. 1).

> “A person who is trying to understand a text is always projecting. He projects a meaning for the text as a whole as soon as some initial meaning emerges in the text.

> Again, the initial meaning emerges only because he is reading the text with particular expectations in regard to a certain meaning. Working out this fore-projection, which is constantly revised in terms of what emerges as he penetrates into the meaning, is understanding what is there." (TaM, 285/6; WuM, 271)

Thus, only "the recognition that all understanding inevitably involves some prejudice gives the hermeneutical problem its real thrust" (TaM, 286; WuM, 274). Any reader of a text has his personal intellectual history which itself is embedded in the culture of his time. This determines the perspective under which s*he approaches a text. Prejudices are not an obstacle, but a condition of understanding. However, prejudices become a problem when we remain unconscious of them. "It is the tyranny of hidden prejudices that makes us deaf to what speaks to us in tradition" (TaM, 286; WuM, 274).

In order to become conscious of our prejudices we have to uncover them as much as possible.

> "Indeed, what characterizes the arbitrariness of inappropriate fore-meanings if not that they come to nothing in being worked out? But understanding realizes its full potential only when the fore-meanings that it begins with are not arbitrary. Thus it is quite right for the interpreter not to approach the text directly, relying solely on the fore-meaning already available to him, but rather explicitly to examine the legitimacy—i.e., the origin and validity—of the fore-meanings dwelling within him" (TaM, 284; WuM, 272).

Hermeneutic understanding requires from the interpreter "to be open to the meaning of the other person or text" (TaM, 285; WuM, 273) and to be permanently aware of the alterity and temporal distance of the text (or the other person).

In a separate chapter Gadamer discusses "The discrediting of prejudice by the Enlightenment" (TaM, 288ff; WuM, 276ff) and by historicism and maintains that being conscious of our prejudices neither "involves … 'neutrality' … nor the extinction of one's self, but the foregrounding and appropriation of one's own fore-meanings and prejudices" (TaM, 286; WuM, 274). The term "self-extinction" had been used by historian Ranke to designate the neutrality of a researcher in doing history (TaM, 229; WuM, 215). Gadamer uses a whole chapter to uncover the hidden prejudices in the views of historians Ranke and Droysen in order to show the illusionary character of the idea of a 'neutral' observer (TaM, 215-233; WuM, 177-221).

On the basis of the crucial role of fore-meaning and prejudice Gadamer proceeds to a rehabilitation of authority and tradition (TaM, 292ff; WuM, 305ff) and the principle of 'history of effect' ('Wirkungsgeschichte') (TaM, 292ff; WuM, 305ff). In this way, he thought to have established a new foundation for the human sciences different from Dilthey's. We are embedded in tradition, and tradition suggests concepts and questions we pose in regard to the texts we study.

To sum up, according to Gadamer 'application' and 'prejudice' are inherent components of every act of interpretation and responsible for the 'historicity of understanding' and explain the necessity and legitimacy of different understandings of a text. Interpreting a text is not adequately described as a reconstruction of the original ‘true’ meaning, but a construction of a new meaning the original author of the text would not have thought of.

Gadamer is vague in delimiting ‘application’ from ‘prejudice’. When a reader thinks of an example for a statement in a text, does s*he apply the text or embed the text in her/his intellectual background and, thus, refer to her/his prejudices? A first approximation might be provided by looking at the human sciences as a whole. On the one hand there is the internal functioning of science, and a scientist studying a text would first ask for the state of research and embed her/his research problem into this context. On the other hand, there is the ‘application’ of human sciences which consists above all in contributing to societal debates on, say, ethics, aesthetics, politics and culture in general.

Résumé. (a) In a principle way Gadamer’s conception of hermeneutics leads to an unlimited *variety of interpretations* of a text. This is a necessary consequence of his concepts of **application** and **prejudice** and the resulting **historicity of understanding** which reflect the intellectual situatedness of the reader. Since in many cases it is difficult to distinguish between application and prejudice we shall henceforth only talk of the historicity of understanding.

(b) The variety of interpretations does not mean that interpretations are arbitrary. On the contrary, implicit in Gadamer’s approach is a high demand on the **internal quality of argument**. We mention three points. (1) The spiral process of the ‘hermeneutic circle’ is in principle infinite and requires ever subtler and precise arguments for reaching harmony between part and whole, and may require the consideration of ever larger collections of texts. Gadamer does not give “rules” of interpretation, and it is a matter of judgement whether harmony is reached and an interpretation is successful. (2) Any interpreter is subject to the requirement that he has to get conscious of his prejudices and to make them explicit. (3) Any interpretation has to respect the temporal distance and the ‘alterity’ of the text. In a principle way, the distance between the text and an interpretation cannot be reconciled. Out of these insights we conclude that in a very high amount interpretation requires what a mathematician should have learnt, namely rigour.

In the following we study three different types of readers in order to explore the relevance of Gadamer’s concept of ‘historicity of understanding’, and begin with schoolchildren.

3 Students (grade 9) as Readers

The following analysis refers to a teaching experiment described in Glaubitz & Jahnke (2003a). The experiment was conducted in a classroom of 26 students aged 15 to 16 years and comprised 6 lessons. The students were to read two fragments from an ancient Greek booklet on astronomy, ‘The Heavens’ by Cleomedes (English transl.: Bowen & Todd 2004; German transl.: Kleomedes 1927). Since nothing is known about the author the dates of his life are quite controversial, estimates running from 100 BC to 400 AD (the latter guess by O. Neugebauer). Today, there exists consensus that he should have lived in the second century AD. Textual analysis shows him as an adherent of stoic philosophy (Bowan & Todd 2004, 5ff). Cleomedes’ booklet gives a survey on astronomy for the educated public of his time, themes ranging from

Figure 3.1: The source

shape, position and size of the earth over movements of moon, sun, planets, fixed stars to special questions like f.e. the difference between solar and sidereal day. The students read fragments about (1) the shape of the earth and (2) the method of Posidonius to determine the size of the earth. On concrete experiences with reading sources on ancient astronomy and about the importance of this topic cf. Hosson 2015 and Tzanakis 2016.

(1) In regard to the shape of the earth Cleomedes discusses the alternatives of being a plane, a bowl, a cube, a pyramid, or a sphere (Kleomedes. 1927, 26ff; Bowen & Todd, 2004, 65ff). We used an abbreviated version of the respective passages in Cleomedes' book. On account of observations on the visibility of sun, moon and stars the first four alternatives are excluded, thus the earth is a sphere. If, for example, the earth were a plane the sun and the stars would rise and set at all places at the same time. Since, however, it is well known that the sun rises in Persia four hours earlier than in Spain this cannot be the case.

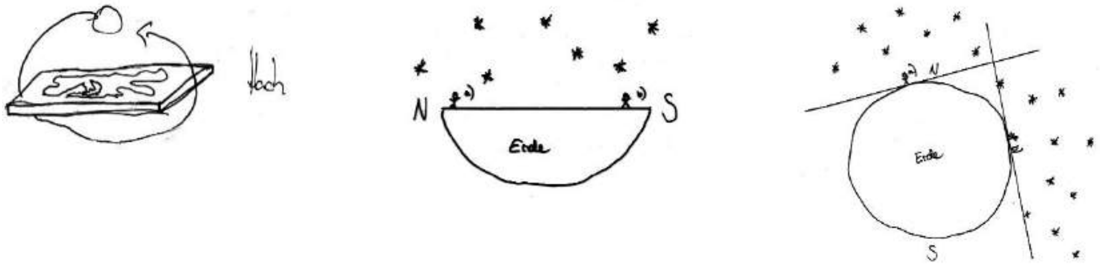


Figure 3.2: Student drawings. The earth as plane, bowl, sphere

(2) Posidonius' (135 - 51 BC (?)) method for determining the size of the earth uses the bright fixed star Canopus (Kleomedes, 1927, 33ff; Bowen & Todd, 2004, 78ff). At the time of its declination it is elevated 7.5° above horizon in Alexandria, whereas in Rhodos it is just visible at horizon. Since Alexandria and Rhodos have approximately the same longitude it follows that these cities have a difference of latitude of 7.5°. According to Posidonius/Cleomedes the distance between Alexandria and Rhodos amounts to 5 000 stades which implies a circumference of the earth of 340 000 stades. Figure 3.3 shows the situation with the simplification of parallel rays from Canopus to Alexandria and to Rhodos, whereas figure 3.4 is a student drawing with non-parallel rays from a point-shaped Canopus.

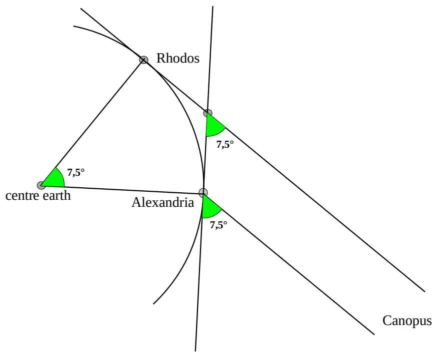


Figure 3.1: Parallel rays

The students had substantial difficulties to deal with the simplification that the rays from Canopus to Alexandria and to Rhodos can be considered as 'practically parallel' (figure 3.3) because of the enormous distance of the fixed stars from the earth (cf. similar experiences in Hosson 2015). They were not ready to accept it as legitimate since it seemed to contradict the absolute precision characteristic of mathematics. On the other hand, it seemed much more acceptable to them to consider Canopus as a mathematical point (figure 3.4) in spite of its enormous size. Of

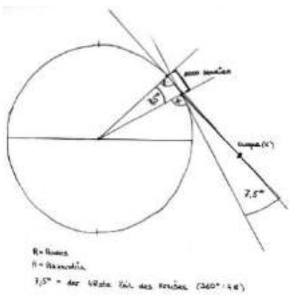


Figure 3.2: Non-parallel rays, drawing by a student

course, this is understandable because fixed stars appear to us visually like points at the sky (Glaubitz & Jahnke 2003a, 83pp).

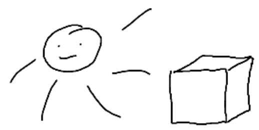


Figure 3.3: sun and cube-shaped earth

Even more difficult was the discussion on Cleomedes' argument against a cube-shaped earth. In this case, according to Cleomedes, the sun would be above horizon at every place only for six hours a day (figure 3.5). But this is only true when the sun at every moment illuminates only one face of the cubic earth, and this again is valid only when the sun moves on a special circle which touches the edge of the cube at the moments of rising or setting. But in general the sun would illuminate three faces at the same time. Thus, Cleomedes' argument is highly problematic. Even more problematic are his reasons against a bowl-shaped earth (Glaubitz & Jahnke 2003a, 81). Thus, the students found themselves in a situation which required from them a critical evaluation of a text which at first they must have considered as a scientific authority simply because their teachers had given it to them.

The source required from the students in a considerable amount to revise their expectations or in Gadamerian terms their prejudices. The students expected that the ancient Greeks considered the earth as a plane. The source taught them that they considered it as a sphere. The students expected that there was only the alternative plane vs. sphere. The source discussed five alternatives, plane, bowl, cube, pyramid, sphere. Against their expectations of an earth travelling around the sun, according to the source the earth doesn't move. Angles are not measured in degree, but in ratios of the zodiac. The students thought that Cleomedes was an authority whose arguments are always correct. Instead of this, they had to accept that he used superficial, even incorrect arguments. Students were convinced that mathematics is always exact. They learnt that (applied) mathematics makes use of simplifications.

Résumé. The students had to cover a considerable distance from their original expectations to an understanding of the source. There was a permanent tension between their modern astronomical knowledge, their prejudices, and the statements and arguments of the source. In a certain amount they were aware of the historicity of the source. Their guiding question however seemed more that they wanted to learn about the astronomical problems discussed than to learn about history. Obviously, they applied the text. Nevertheless, the historical distance was present to them. This became clear from a small questionnaire they were asked to fill in. There one can find several statements of the type: "I found fascinating how somebody could arrive at the idea to prove that the earth is not flat by such simple things." (Glaubitz & Jahnke 2003a, 88pp) This shows, some students realized that reading such texts is also teachable under an epistemological point of view.

4 A Pedagogue as Reader

The pedagogue considered is Martin Wagenschein (1896 – 1988). He took a PhD in Experimental Physics at the university of Gießen in 1921. From 1924 to 1933 he worked at the 'Odenwaldschule', a boarding school and prestigious project of 'Reformpädagogik'. In the 1920s the school had a high reputation with many visitors from abroad. Children of quite a number of prominent people attended the school. The time there became the pedagogically formative period of Wagenschein's life. In his autobiography he spoke of the "magic of this school" (Wagenschein1989,34). "At the Odenwaldschule 'exchange', not ‘instruction’ became for me the unshakable base of teaching.” (loc.cit., 38) In 1927, also mathematician Otto Toeplitz visited the school for two days, and attended among others a course by Wagenschein (loc.cit., 34).

In 1933, the founder and director of the Odenwaldschule, Paul Geheeb, emigrated from Germany, together with some teachers and students, in order to found a new school in Switzerland, and Wagenschein changed to a position at a state driven school. In his autobiography he described the Nazi-time in a chapter headed “Wartezeit” (‘waiting period’), and the time immediately after the war by the verb “aufatmen” (‘drawing a deep breath’). He continued teaching at school until 1957, and gave courses for teachers of physics and mathematics at the Technical University of Darmstadt, the University of Tübingen and other institutions of teacher training quite until his old age. It was only after 1950 that he became a visible and prolific writer of articles and books on didactics of physics and mathematics.[2]

Obviously, Wagenschein had a strong and coherent vision of good teaching, and he described it in a language full of metaphors. He taught at a regular school, but criticised the established school system in a rather fundamentalist manner speaking f.e. of the “tragedy of the teaching of mathematics” (UVeD I [1961], 417-428;) which consisted in his eyes in an one-sided emphasis on ‘passive knowledge’ (he used the German word ‘Stoff’). According to him, students frequently cannot connect their knowledge to real phenomena, and he complained about too early a formalization.

Figure 3.4: Martin Wagenschein in 1983.
Source: Wikipedia

It is not easy to describe Wagenschein’s approach to teaching for the simple reason that he never developed a ‘system of pedagogy’. This might have been a consequence of the fact that he never held a professorship at a university, but it is more probable that creating a ‘system’ did not fit to his way of thinking and his personality. Thus, it seems appropriate to begin with some observations in order to get an image of his visions.

Figure 3.6 shows Wagenschein in a teaching situation at his age of 87. One sees children dealing with some artefacts of a physical experiment (the theme was Pascal’s barometer) and, presumably, describing what they observe. Wagenschein himself is attentively listening, and this is the message of the picture. A teacher should, first of all, be able to remain silent and to listen to his students. This is so because for Wagenschein the main problem of teaching was to connect concepts and theoretical insights with the phenomena they are to explain. In order to become really aware of the phenomena there should be broad opportunities for students to describe them in their own language, and independent of the ‘official’ language of science.

Out of this motive Wagenschein experimented with language. A wonderful example worth reading is a small piece of two pages with the title “Das große Spüreisen” (“The big

[2] Many of Wagenschein’s papers are reprinted in Wagenschein 1970a (=UVeD I) and Wagenschein 1970b (=UVeD II). For reasons of space we do not mention the title of every paper, but always add in square brackets the year of its first publication. All translations into English by the author.

feeling iron”) (UVeD I [1951], 175pp) which at the time of its publication caused a real scandal among physicists and teachers of physics (Wagenschein 1989,79). The German word ‘Spüreisen’ is a creation by Wagenschein whose English translation is only a first approximation. The word is intended to arouse a connotation of magic like f.e. the English word ‘divining rod’.

The ‘feeling iron’ is an oversized magnetic needle of 1 meter length whose slow oscillation around the north-south direction Wagenschein described in a completely animistic language attributing something like a free will to it. The description ends with the sentence “It lasted nearly a quarter of an hour until our feeling iron came to rest. It had to work hard to find its peace.” (UVeD I [1951], 176).

Wagenschein has written many of his texts in a pronounced emotional and existentialist language, take as an example the title of another paper from 1951 “Mind and heart in the acquisition of exact scientific knowledge” (UVeD I [1951], 181ff). Therefore, it is no wonder that he frequently made reference to Simone Weil’s book (1949) on “Enracinement”, since the ‘enracinement’ (‘taking roots’; ‘Einwurzelung’) of phenomena in the minds of students was his great theme. (f.e. UVeD II [1966], 60-62)

In the course of time Wagenschein standardized his approach by using the triad of concepts “genetic – Socratic – exemplary” (see Wagenschein 1968). The three concepts have a long tradition in pedagogy, as a triad they might be adequate to characterize his thinking. Wagenschein explains: “Pedagogy has to deal with genesis: the growing human being and … the genesis of knowledge inside him. The Socratic method is involved since genesis, the awakening of his intellectual forces, happens most effectively in a conversation or a dialogue. The exemplary principle is involved because a genetic-Socratic procedure must and also can confine itself to limited themes (‘Themenkreise’)” (UVeD II [1966], 68). In regard to a Danish reception of Wagenschein’s ideas on exemplary teaching cf. Blomhøj & Kjeldsen (2009).

The language motive for reading sources. In the course of his life Wagenschein seems to have read quite a number of original texts by famous scientists, f. e. Aristarch, Foucault, Galilei, Leonardo da Vinci, Kepler, Lichtenberg, Newton. The list is by far not complete. In his autobiography, in the chapter on his university studies he deals extensivelywith reading such sources under the revealing heading ‘Lichtenberg and other masters of language’ (Wagenschein 1989, 25-30). Language and the distance to textbook knowledge seemed to interest him most. As a consequence, historical sources became part of his intellectual life, and reading them was subordinated to his search for authentic, peculiar, even idiosyncratic linguistic representations of observations and phenomena far off the standardized language of modern science.

Wagenschein stated at several places “Genesis is not history” quoting explicitly Otto Toeplitz (f.e. UVeD II [1965], 78). “The history of his science … is for the teacher not a mere subject but helps him to take the questions of his students as serious as they are meant.” (loc.cit.) At another place one finds a sentence like this one. „We recognize this less from textbooks than out of the history. … The ‘old’ researchers are in reality the young ones, the early ones.” (UVeD II [1967], 26;).

There is a piece of eight pages presumably written in 1962 and published as appendix C to Wagenschein (1995) under the title “Genetic teaching (history of human ideas). The didactical significance of studying sources, shown by the example of the law of inertia”. To my knowledge this is the only paper where he explicitly discussed a possible significance of sources for teaching. The general message of this paper is twofold. (1) He addresses the teacher, not the student as a possible reader of a source. (2) For the teacher sources should be a key to the questions and the thinking of his students. As far as I can see, he never used sources in teaching at school, but there is at least one reference to a seminar he gave to future teachers on “Didactical suggestions from the writings of the scientific pioneers of the 17th century” (Wagenschein 1995, 288).

The epistemological motive for reading sources. Beyond his interest in the not standardized language of historical sources there is also another fundamental reason why he was interested in the history of physics and mathematics. This is an epistemological one and closely connected to the great importance he attributed to ancient astronomy as a subject of teaching mathematics and physics. Time and again he returned to Aristarchos’ method of determining the relative distances of sun and moon from the earth (cf. Jahnke 1998). As is well known, this method consists in observing sun and moon at the moment of half-moon in a case when moon and sun are both visible. From the angle moon (M) – earth (E) – sun (S), one can determine the ratio of the distances of moon and sun from the earth (Figure 3.7).

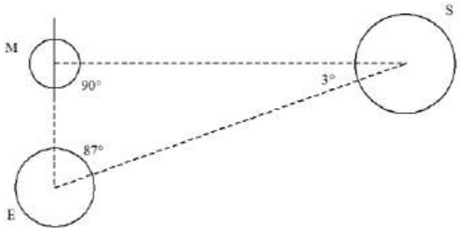


Figure 3.5: Aristarchos‘ method

Wagenschein is not so much interested in the numerical side of Aristarchos’ method, but in the underlying geometric and qualitative understanding of the phases of the moon. Many students and many adults after a long time of attending school have a lot of astronomical textbook knowledge. They know about the sun as the centre of the solar system, the earth travelling around the sun, the moon travelling around the earth, the size of the sun being many times larger than the size of the moon and the earth, etc. However, when asked to explain the phases of the moon a majority of students and adults would refer to the shadow of the earth as a cause for the crescent figure. Since on the other hand it is very easy by observing sun and moon when they are simultaneously visible at the sky to realize that there is no shadow of the earth involved and to arrive at the ‘right’ idea this is paradigmatic for the basic failure in the teaching of mathematics and physics. In a sense textbook knowledge prevents students from observing and thinking (UVeD II [1966], 59). At some places Wagenschein uses the concise wording “verdunkelndes Wissen” (‘obscuring knowledge’) to designate this phenomenon (UVeD II [1966], 58). Thus, we have to regain “the primary phenomenological reality” (Wagenschein 1995, 291) and Wagenschein calls this process the “genetic metamorphosis of science” (UVeD II [1965], 87).

To regain the primary phenomenological reality meant to „let students understand how human beings can come to know such things“or to answer the question **“How is science possible?”** (Wagenschein 1995, 292). This Kantian-type question was a running theme in his pedagogical thinking, and he explained it in a pathetic, but illuminating way in an early publication:

> “The relation of human beings to nature is mysterious since it touches deeply the enigma of their own existence: they belong to it [to nature] and, yet, are able to pose themselves opposite to it. Therefore, the question of how we gain and pass on scientific knowledge is a humanist question; it concerns the whole human being … science is the trace of a one-sided … expression of human nature… It concerns all

> the more the whole human being since s*he must know this one-sidedness when s*he wants to remain whole…” (UVeD I [1951], 182)

Thus, answering the question of how human beings can come to know is a requirement of humanism, and this in turn implies the fundamental importance of ancient astronomy for teaching science and mathematics. In this area one can observe with the naked eye or with very simple instruments and derive from these observations by means of simple geometry far-reaching consequences about the structure of our universe.

This means in regard to the example of the phases of the moon that the teacher should not start with the above figure 7, or alternatively, with a lamp (representing the sun) being moved around a tennis ball (the moon) and try to let students see the crescent figure. According to his experience, this will not work. Instead: when we suppose that moon and sun are really physical bodies (which was not obvious to the Greeks) and that the sun is a shining body whereas the moon receives light from the sun, then we can conclude by pure imagination a fundamental fact: **always** is one hemisphere of the moon illuminated by the sun, and one hemisphere is dark. Only on the rare occasions of an eclipse of the moon this is not the case.

With this fundamental idea in mind students are asked and guided to observe the moon every day during a certain period, say 2 weeks. In fact, they are not asked to observe only the moon, but to observe the pair moon – sun. When both are visible simultaneously, that is during day-time, this is no problem. But what about when it is night? In this case we have to add the sun in our mind’s eyes to the visible configuration by looking in which direction the illuminated hemisphere of the moon is pointing. When we prolong this direction beyond horizon we will intuitively get an estimate of the position of the sun and after a time intuitively realize that the sun must be distant from the earth many times farther than the moon, and, since its apparent size is equal to that of the moon, must be many times larger than the moon. Thus, in a combination of imagination, thinking and observation anybody can get, after some training of his eyes and his imagination, a correct qualitative intuition of the configuration earth – moon – sun without any measurement, any technical instrument and without any numerical calculation. Obviously, measuring and calculating are necessary further steps leading to new insights and to new surprises (UVeD I [1951], 184ff).

The phases of the moon are a fundamental paradigm for Wagenschein’s views on history of science and of ‘regaining the primary phenomenological reality’. No wonder then, that he was interested in authentic descriptions of this phenomenon he found in historical sources. In an Italian – German edition of Leonardo da Vinci’s “philosophical diaries” he hit upon a short remark of da Vinci’s (Ms. Arundel 94r) on the phases of the moon he liked so much that when quoting it he rearranged it in a way that it looked like a poem (UVeD II [1966], 67). Here it is.

„Der Mond hat kein Licht von sich aus, und soviel die Sonne von ihm sieht, soviel beleuchtet sie; und von dieser Beleuchtung sehen wir soviel, wieviel davon uns sieht.“	“La luna non ha lume da sè, se non quanto ne vede il sole, tanto l’allumina; della qua lluminosità, tanto ne vediamo quanto è quella cheve de noi.”	“The moon has no light out of herself, and as much as the sun sees of her, as much he illuminates; and of this illumination we see as much as much of it sees us.”

We come back to the motive of ‘poem’ in the next section of this paper.

Résumé. Wagenschein had a strong vision of good teaching formed by his personal experiences at the ‘Odenwaldschule’ and he was immersed in the pedagogical tradition of ‘Reformpädagogik’. He quoted Toeplitz’ statements implying that genesis is different from history. He saw historical sources as a valuable component of the intellectual life of a teacher of mathematics and physics, but, presumably, did not think about reading sources with children and students at school. His central problem was the regaining of the primary phenomenological reality and in this context attributed a high relevance to the history of science and, especially, to ancient astronomy. He considered answering the question of “how is science possible” as a requirement of humanism. He was also very much interested in the linguistic dimension of historical sources, and, by this, also the literary and artistic quality of sources became important.

5 A Philosopher as Reader

The philosopher considered is Karl Raimund Popper (1902 - 1994). Since 1918 he took courses on mathematics, physics, psychology, philosophy at the University of Vienna, and in the early 1920s he engaged in a social-democratic student organisation, worked in street construction and made an apprenticeship as cabinet maker. In 1924 he passed an examination as a teacher at elementary schools, andin 1928 he earned a PhD in psychology under Karl Bühler. From 1929 to 1937 he worked as a teacher of mathematics and physics at a secondary school. In 1937 he obtained a position as lecturer of philosophy at the university of Christchurch/New Zealand. In 1945 he changed to the London School of Economics, and in 1949 was appointed professor of logic and scientific method at the University of London (cf. Popper 1976b and the article on Popper in the English Wikipedia).

Popper was one of the most influential philosophers of the 20th century who is well-known to a broader public far beyond specialized philosophy. His book *The Open Society and Its Enemies’*, published for the first time in 1945, in which he criticized the in his view totalitarian components of the philosophies of Plato, Hegel and Marx, fitted to the situation between west and east in the 1950s and, surely, contributed a lot to this public visibility.

In 1934 appeared his major work *Logik der Forschun*g (‘The logic of scientific discovery’) which established his philosophy of ‘critical rationalism’ and, biographically, helped him to get the position in New Zealand and to emigrate shortly before the Nazis entered Austria.

Base and starting point of Popper’s ‘critical rationalism’is his fierce rejection of traditional inductivism and sensualism. Instead, he held the view that theories can never be (finally) confirmed by empirical evidence – a philosophical position which came to be called ‘fallibilism’. Popper considered scientific progress as guided by a permanent interplay between the creation of theories as bold conjectures and the search of scientists torefute them, be it by critical argument or by falsifying evidence (cf. the title of Popper1976a).

Relation to ancient astronomy. According to his own testimony Popper was interested in and deeply impressed by the Presocratic philosophers since the time when he was 16 years old, and he cultivated this interest all of his life (Popper 2006. 88). But it was only in 1956 that he published a paper entitled 'Back to the Presocratics' in his *Conjectures and Refutations* (1976a, 1st edition in 1956). Since the 1970s,on advise of the later editor A. Petersen, Popper began to think seriously about writing a book on Parmenides and the Presocratics (Popper 2006. 9ff). He wrote and rewrote a number of papers including a preface, but the book itself appeared only after his death (Popper 1998).

Popper read fluently ancient Greek, all translations of Presocratic Sources into English used in the book are his own. He knew much of the philological and philosophical literature on Presocratic philosophy and did not hesitate to enter even philological arguments where he found this necessary.

Parmenides (520? – 450? BC) was a student of Xenophanes, the teacher of Zeno and lived in the newly founded Greek colony of Elea in Southern Italy (Popper 1998, 139).We know of Parmenides' philosophical thinking by a poem in hexameters in the style of Homer and Hesiod of which only 180 lines out of estimated 800 lines have been passed on to us. According to Popper, Parmenides' work "is beset with problems that perhaps will never be solved" (loc.cit.). Nevertheless, Popper tried to develop an interpretation whose tentative character becomes clear from the fact that his book on Parmenides contains several different attempts.

An overarching motive in Poppers interpretation was the thesis that Parmenides was "essentially a cosmologist" (Popper 1998, 143), in contrast to many modern philosophers who consider him above all as an 'ontologist'.

Parmenides' poem contained a revelation from the goddess Dike in two distinct parts. In the first part, the goddess reveals the truth about what really exists. This is called the **way of truth**. In the second part, the goddess speaks about the world of appearances, the illusory world of movement, change, development (loc.cit., 138). This way is called the **way of opinion**. The following is a short account of Popper's interpretation.

The way of truth proceeds **purely rational** and by **logical proof**. It is the way of the Gods. The Way of Opinion is the way of the mortals who believe in **sensory perceptions**. Popper interprets Parmenides as a rationalist who rejects explicitly sensualism (loc.cit., 103).

The **real world** which the way of truth uncovers by rational deduction is very simple. It is a universe without change or movement. "This universe consists of one well-rounded spherical block that is completely homogeneous and structureless. It has no parts: it is one. It has no origin and thus no cosmogony, and it always was and is and always will be at rest, changeless and colourless." (loc.cit., 140)

By contrast, the **appearing world** as it is seen by the mortals is a universe in which there is change, movement and development. The Goddess describes also this world in detail, and this part of the poem contains important and original ideas, such as the doctrine of the spherical shape of the earth and a theory of the moon. (loc.cit., 139). Thus, any interpreter of Parmenides' poem is confronted with the paradoxical situation that the best achievements of Greek astronomy of the time which in part might have been reached by Parmenides himself are attributed to the illusory world of appearances. To say it in Popper's words: "Why does the goddess expound part 2 at all, stressing that it is mistaken?" (loc.cit., 124)

According to Popper “Parmenides was the first who consciously placed reality and appearance in opposition and consciously postulated one true unchanging reality behind the changing appearance” (loc.cit., 140).

How did Parmenides arrive at this distinction and to understand its importance? Popper’s attempt to answer this question points to the *phases of the moon*. This is an essential issue in his book. There are three chapters around this idea with slightly modified titles and representing different attempts at describing the idea and its context. The titles are “*How the Moon might shed some of her light upon the Two Ways of Parmenides*” (1992), “*How the Moon might throw some of her light upon the Two Ways of Parmenides*” (1989) and “*Can the Moon throw light on Parmenides' Ways?*” (1988).

Popper’s explanation is rather straightforward. “Parmenides discovered that the observation … that the Moon — Selene — waxes and wanes during the course of time is false. … She does not change in any way. Her apparent changes are an illusion.” (loc.cit., 108). “The moon does not change. It is a material sphere of which one half is always illuminated, the other half is always dark.” (loc.cit.) Therefore, in eternity the moon does not change. This eternal moon is ‘being’. On the other hand, the changing shape of the moon from new moon to crescent to half-moon to full moon is mere appearance. It does not really exist; it is ‘not being’.

Of course, to a modern reader and, may be, also to an ancient reader, this case might at best be an example making understandable the distinction between reality and appearance. It is a stupendous step from there to Parmenides’ radical conception of a rigid universe without movement and change. Taking into account the fragmentary state of Parmenides’ poem it is clear that much subtler arguments as are given here are required to bridge this gap, andany attempt at interpreting Parmenides is necessarily highly hypothetical as Popper himself stressed.

There is another important point. To realize that a hemisphere of the moon is always illuminated seems to be only a small step. But we have seen in the last section how excited Wagenschein was about it, so much so, that he rearranged a short remark of da Vinci expressing this in the form a poem. The same excitement we find with Popper/Parmenides. When reporting how,as a boy of 16 years, he hit upon the Presocratics Popper said: “The verses that I liked best were Parmenides' story of Selene's love for radiant Helios (DK 28 Bl4—15). … before reading Parmenides' story it had not occurred to me to watch how Selene always looks at Helios' rays …

> Bright in the night
> with the gift of his light,
> Round the Earth she is erring,
> Evermore letting her gaze
> Turn towards Helios' ray”

(loc.cit., 88/89, Popper’s translation of Parmenides). And he added: “Since the day when I first read these lines (in Nestle's translation), 74 or 75 years ago, I have never looked at Selene without working out how her gaze does indeed turn towards Helios' rays (though he is often below the horizon).” (loc.cit., 89). At another place he added: “„I personally am indebted to him [Parmenides] for the infinite pleasure of knowing of Selene‘s longing for Helios…“ (loc.cit., 130)

To consider Parmenides’ phrases as a love poem about Selene (the moon) and Helios (the sun) is only weakly suggested by the Greek wording. Only the half sentence that

Selene is always looking for Helios' rays might be in favour of this interpretation. Therefore, it is a (de)construction by Popper, similar to Wagenschein's poetic deconstruction of da Vinci's short remark. Both, Wagenschein and Popper, show a sensitivity to the artistic quality of their sources and, thus, create a personal and individual relation to them.

Résumé. Popper's book on Parmenides and the Presocratics is a contribution to the history of human ideas. The author sees himself as part of the historiographic and philosophical tradition and meets philological standards. His guiding problem is the search for ideas which are essential in the context of his own philosophical outlook, critical rationalism. A key element of his argument is the thesis that the Presocratics, and, especially, Parmenides can best be understood from the point of view of ancient astronomy. Beyond the main line of argument, as a certain surprise, the artistic quality of Parmenides' poem plays an important part.

6 Conclusions

Gadamer's concept of the 'historicity of understanding' which he concretized in the two concepts of 'prejudice' and 'application' implies that there is necessarily an unlimited variety of legitimate interpretations of a text. Any interpretation depends on the intellectual background and the situatedness of the reader. Any new interpretation by a new reader deepens our understanding of a text, understanding is the 'sum' of all existing interpretations and, thus, open to future. At the same time, reading requires rigour and judgement.

Our three readers: the students, the pedagogue and the philosopher have, of course, different intellectual backgrounds, different questions and problems they were interested in and quite different ways of reading their sources. Nevertheless, there was an important point of convergence. They all understood the epistemological relevance of their sources. Indeed, to answer the question of "How is science possible?" is and should be a central objective of education. Ancient astronomy (not necessarily Greek as Hosson 2015 shows) is particularly suitable for this aim.

Wagenschein has made us sensitive to the special role of language. This is a particular quality of reading historical sources which cannot be replaced by any other educational activity. The language of a historical source provides opportunities of reflection which cannot be arrived at by the standardized language of textbooks. Additionally, language, particular words or phrases, might cause the reader to feel touched and addressed in a special way. Then s*he will build up a relation to a text which may last over a long time as the 'poems' by the pedagogue and the philosopher show.

REFERENCES

Arcavi, A., & Isoda, M. (2007). Learning to listen: From historical sources to classroom practice. *Educational Studies in Mathematics, 66*, 111-129.

Barbin, E. (1994). Préface. In Commission Inter-Irem Épistémologie et Histoire des Mathématiques (Ed.), *Quatrième université d'été d'histoire des mathématiques* (pp. ii–iii). Lille: IREM.

Barnett, J., Pengelley, D. a., & Lodder, J. (2013). The Pedagogy of Primary Historical Sources in Mathematics: Classroom Practice Meets Theoretical Frameworks. *Science and Education, 23*(1), 7-27.

Blomhøj, M., & Kjeldsen, T. H. (2009). Project organised science studies at university level: exemplarity and interdisciplinarity. *Zentralblatt für Didaktik der Mathematik, 41*(1), 183-198. doi:10.1007/s11858-008-0102-3

Bowen, A. C., & Todd, R. B. (2004). *Cleomedes' Lectures on Astronomy. A Translation of 'The Heavens' With an Introduction and Commentary*. Berkeley et al.: University of California Press.

Chorlay, R. (2016). Historical sources in the classroom and their educational effects. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *History and Pedagogy of Mathematics. Proceedings of the 2016 ICME Satellite Meeting- HPM 2016* (pp. 5-23). Montpellier: IREM de Montpellier.

Clark, K., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang, X. (2016). History of Mathematics in Mathematics Education. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *History and Pedagogy of Mathematics. Proceedings of the 2016 ICME Satellite Meeting - HPM 2016* (pp. 135 - 179). Montpellier: IREM de Montpellier.

Foellesdal, D. (1979). Hermeneutics and the Hypothetico-Deductive Method, *Dialectica 33,* 319 - 336

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education, 10*, 391-408.

Fried, M. N., Guillermette, D., & Jahnke, H. N. (2016). Panel 1: Theoretical and/or conceptual frameworks for integrating history in mathematics education. In: L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *History and Pedagogy of Mathematics. Proceedings of the 2016 ICME Satellite Meeting - HPM 2016* (pp. 211 - 230). Montpellier: IREM de Montpellier.

Fried, M. N. (2018). Ways of Relating to the Mathematics of the Past. *Journal of HumanisticMathematics, 8*(1), 3-23.

Gadamer, H. G. (1976). Artikel 'Hermeneutik'. In: J. Ritter & K. Gründer (Eds.), *Historisches Wörterbuch der Philosophie* (Vol. 4, pp. 1061-1073). Darmstadt: Wissenschaftliche Buchgesellschaft.

Gadamer, H.-G. (1990). *Wahrheit und Methode. Grundzüge einer philosophischen Hermeneutik* (6. ed.). Tübingen: Mohr. [quoted as WuM]

Gadamer, H.-G. (2004). *Truth and Method* (J. Weinsheimer & D. G. Marshall, Transl. 2 ed.). London/New York: continuum. [Quoted as TaM]

Gallin, P., & Ruf, U. (1990). *Sprache und Mathematik in der Schule. Auf eigenen Wegen zur Fachkompetenz. Illustriert mit sechzehn Szenen aus der Biographie von Lernenden*. Zürich: Verlag Lehrerinnen und Lehrer Schweiz.

Glaubitz, M. R. (2011). *Mathematikgeschichte lesen und verstehen. Eine theoretische und empirische Vergleichsstudie. Dissertation*. Universität Duisburg-Essen. http://duepublico.uni-duisburg-essen.de/servlets/DocumentServlet?id=25416. Accessed 23 July 2019.

Glaubitz, M., & Jahnke, H. N. (2003a) Die Bestimmung des Umfangs der Erde als Thema einer mathematikhistorischen Unterrichtsreihe. *Journal für Mathematikdidaktik, 24*(2), 71-95.

Glaubitz, M., & Jahnke, H. N. (2003b). Texte lesen und verstehen. Eine Erwiderung. *Journal für Mathematikdidaktik, 24*(3/4), 252-260.

Hosson, C. d. (2015). Promoting an interdisciplinary teaching through the use of elements of Greek and Chinese early cosmologies. In E. Barbin, U. T. Jankvist, & T. Hoff Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the Seventh European Summer University ESU 7* (pp. 571-583). Copenhagen: Danish School of Education.

Jaeger, H.-E. H. (1974). Studien zur Frühgeschichte der Hermeneutik *Archiv für Begriffsgeschichte, 18* 35-84

Jahnke, H. N. (1998). Sonne, Mond und Erde oder: wie Aristarch von Samos mit Hilfe der Geometrie hinter die Erscheinungen sah. *mathematik lehren* 91, 20 - 22, 47 - 48.

Jahnke, H. N. (2014). History in Mathematics Education. A Hermeneutic Approach. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics & Mathematics Education: Searching for Common Ground* (pp. 75 – 88).

Dordrecht: Springer
Joisten, K. (2010). *Philosophische Hermeneutik*. Berlin: Akademie Verlag.
Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning meta-discursive rules. *Educational Studies in Mathematics, 80*, 327-349.
Kleomedes. (1927). *Die Kreisbewegung der Gestirne. Übersetzt und erläutert von A. Czwalina* (Vol. 220). Leipzig: Akademische Verlagsgesellschaft.
Popper, K. R. (1968). *The logic of scientific discovery* (Rev. ed.). London [u.a.]: Hutchinson.
Popper, K. R. (1976a). *Conjectures and refutations: the growth of scientific knowledge* (4. ed., rev., reprint. ed.). London [u.a.]: Routledge and Kegan Paul.
Popper, K. R. (1976b). *Unended quest: an intellectual autobiography*. La Salle, Ill.: Open Court.
Popper, K. R. (1998). *The World of Parmenides: Essays on the Presocratic Enlightenment. Edited by A. F. Petersen with the assistance of Jörgen Mejer*. London & New York: Routlegde.
Popper, K. R. (2009). *Vermutungen und Widerlegungen : das Wachstum der wissenschaftlichen Erkenntnis* (2. ed.). Tübingen: Mohr Siebeck.
Popper, K. R. (2006). *Die Welt des Parmenides : der Ursprung des europäischen Denkens* (Ungekürzte Taschenbuchausg., 2. ed.). München [u.a.]: Piper.
Schleiermacher, F. D. (1977). *Hermeneutik und Kritik. Hrsg. u. eingel. v. M. Frank* Frankfurt
Schönert, J.& Vollhardt, F. (2011). *Geschichte der Hermeneutik und die Methodik der textinterpretierenden Disziplinen*. Berlin: De Gruyter.
Scholz, O. R. (2011). Die Vorstruktur des Verstehens. Ein Beitrag zur Klärung des Verhältnisses zwischen traditioneller Hermeneutik und philosophischer Hermeneutik. In J. Schönert & F. Vollhardt (Eds.), *Geschichte der Hermeneutik und die Methodik der textinterpretierenden Disziplinen* (pp. 443-461). Berlin: De Gruyter.
Schröder, J. (2011). Entwicklungstendenzen der juristischen Interpretationstheorie von 1500 bis 1850. In J. Schönert & F. Vollhardt (Eds.), *Geschichte der Hermeneutik und die Methodik der textinterpretierenden Disziplinen* (pp. 203-219). Berlin: De Gruyter.
Tzanakis, C. (2016). Mathematics and physics: an innermost relationship. Didactical implications for their teaching and learning. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *History and Pedagogy of Mathematics. Proceedings of the 2016 ICME Satellite Meeting – HPM 2016* (pp. 79 - 104). Montpellier: IREM de Montpellier.
Wagenschein, M. (1968). *Verstehen lehren: genetisch, sokratisch, exemplarisch* Weinheim [u.a.]: Beltz.
Wagenschein, M. (1970a). *Ursprüngliches Verstehen und exaktes Denken I* (2. Auflage ed.). Stuttgart: Klett. (= UVeD I)
Wagenschein, M. (1970b). *Ursprüngliches Verstehen und exaktes Denken II*. Stuttgart: Klett. (= UVeD II)
Wagenschein, M. (1989). *Erinnerungen für morgen : eine pädagogische Autobiographie* (2., erg. u. durchges. Aufl. ed.). Weinheim [u.a.]: Beltz.
Wagenschein, M. (1995). *Die pädagogische Dimension der Physik* (1. new ed.). Aachen-Hahn: Hahner Verl.-Ges.
Wagenschein, M. (1995). *Naturphänomene sehen und verstehen: genetische Lehrgänge* (3rd. ed.). Stuttgart: Klett.
Weil, S. (1949). *L'Enracinement: Prélude à une déclaration des devoirs envers l'être humain.* Paris: Gallimard.

THINKING WITH LEVINAS ABOUT HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION

Adriano DEMATTÈ
University of Genoa & Liceo Rosmini, via Malfatti 2, 38122Trento, Italy
adriano.dematte@scuole.provincia.tn.it

David GUILLEMETTE
Université du Québec à Montréal, Département de Mathématiques, C. P 8888, succursale Centre-Ville, Montréal (Québec), H3C 3P8, Canada
guillemette.david@uqam.ca

ABSTRACT

For the French philosopher Emmanuel Levinas, human beings reveal themselves in relation to the Other (ethical perspective). In mathematics education, Levinas' philosophy helps to constitute theoretical frameworks in which one could avoid drawing on a private, self-regulated and autonomous (in a rationalist and dualist perspective) subject. The phenomenological reflections of Levinas on Alterity support our reflection on history and mathematics education, particularly with respect to the history of mathematics for a nonviolent mathematics education. Data from empirical studies and from our own research with secondary school students (15-18 years old) and prospective teachers have been provided in order to stimulate in vivo reflection and feedback of the group. A dialog took place during the Workshop. Excerpts from the written, short essays produced by participants and their oral remarks are presented and commented in the present article.

1 Introduction

The importance of Levinas' thought in the field of education is widely known and confirmed by several publications (*see* Atweh & Brady, 2009; Ernest, 2012; Radford, 2008; 2012). Indeed, Levinas has helped us, as teachers (and researchers, as we will see), to develop a useful rethinking of our relation with our students and mathematics. We became aware of the presence of the "third party" (Levinas, 2010, p.213) - persons outside the I - Other relations who require us to perform a list of duties. That fact tends to overcome our being available to accept the infinity in the student. For instance, the students have to pass national exams; therefore, my professional duty is to make them store quite a lot of notions. Moreover, the acceptance of that infinity, we believe, can for example imply that I, as a teacher, do not fall either into the temptation of a definitive judgment about the students' performance or into not admitting that they could get another chance in case of failure. In connection with that, Levinas helped us to develop the concept of violence as the simplistic solution to reduce the student to our criteria of judgment.

On a larger perspective, Levinassian ethics suggested us a new line of reflections about some aspects of education that are inherent both to the problem of motivation, engagement, willingness of deepening the subjects, etc. and to the question, 'How should a student get involved in the study of mathematics?'

To be honest, each of these points recalls some points of Levinassian philosophy but not all, indeed. That philosophy refers to our subjective experience, and suggests us how to use part of it in the attempt of contextualizing it in our teaching.

Moreover, Levinassian ethics helped to realize that students live in the “postmodern ethics”, after the title of (Bauman, 1993), which is radically different from the traditional ethics dealing with virtues, duties and principles by means of which to take decisions in actual situations.Levinassian ethics avoids “the temptation and the illusion that would consist of finding again by philosophy the empirical data of positive religions” (*Trace of the Other*). On the contrary, the presence of the Other is “meeting and friendship” (*Alterity and Transcendence*, p. 56).

We had the idea to discuss the work of Emmanuel Levinas within a workshop. The point was that this workshop might lead us to share with the participants how Levinas' phenomenological reflection on otherness supports our respective research on history and teaching-learning of mathematics.

We thought that this workshop could engage the participants to think with us about the learner’s subjectivity, our relation to history, our experience of history and the role of history in mathematics education having in mind this Levinassian ethics. Even if there was certainly a didactical and educational dimension in this workshop, our approach and the discussions that took place were more theoretical and philosophical.

2 A workshop: setting the scene

This paper is a resume of the presentations, activities and discussion that took place during the two-hour workshop entitled *Thinking with Levinas about history of mathematics in mathematics education* that took place during ESU-8.

The workshop involved about 30 participants. The present report aims to describe the activities and considers the outcomes as contributions of/for the participants. We are aware that the limited number of hours did not permit to take into account the possibility to improve the inquiry by qualitative research, for instance by interviewing the participants who wrote their opinions following the question form we provided them with. In any case, various techniques (dialogue, short oral interviews, written answers, informal exchange of opinions) were used.

Therefore, final conclusions can consider the specific outcomes about Levinas’ philosophy with respect to educational use of the history of mathematics compared with the limited time at disposal. Considering the involvement of the teachers and researchers taking part in ESU, we think that it is of great importance to have their opinions taken into account. Therefore, we believe that a useful discussion should be started on how to stimulate and collect the participants’ contributions either in the one and a half or in the two hours ESU workshops.

Our idea of a workshop led us to exclude a long introductory talk and to introduce the activities right away. The proposal to the participants was to discuss educational problems, not only to examine excerpts from Levinas’ works, and that seemed to complicate the task by extending the range of questions. Anyway, one must also consider that an almost mandatory choice, in view of both ESU8 topics and the need to stick the aspects of which the participants are specialized in.

We chose to propose the workshop as an opportunity for a dialogue on Levinas’ philosophy. The importance of this author stands on the idea of alterity in mathematics education. The seminal character coexisted with the need to get reflections and contributions from the participants, establishing a dialogue in which the participants who already knew Levinas could get involved successfully with those who approached the French philosopher for the first time. Mathematical content in historical documents was not the main focus but it was a resource to introduce some important educational problems inherent to its use. Levinas's philosophy, in fact, could help us to explain the importance of history in mathematics education, with some questions in the background such as, 'Why history?', 'For what purposes', 'What can history add to mathematics teaching?'

We considered the actual behavior of the participants as one of the aims of the workshop and in particular we observed both the discussion of some of the topics included in the trace form we gave them and the participants' request of explanations about materials and excerpts from Levinas' works and their suggestions for ways to compare Levinas' philosophy with school education and classroom activities. The aim of the dialogue, however, made the outcomes of the workshop hardly predictable.

Our aim was not to get a critical point of view of Levinas' works, but on the contrary, it was that of finding out the main ideas that his thought can offer. Even if his philosophy is today encountering a wider and wider acceptance, we believe that it contains a lot of aspects that could be perceived as paradoxical and should be clarified: the idea of face, for example, considered not (only) as a part of the body but as the way to transcendence. These aspects, we believe, can be the subject of discussions. The workshop activities were planned looking to establishing a dialogue between conductors and participants as well as among the participants themselves.

Before the workshop started, it was indeed impossible to know who would be the participants, and also which should be their expectations about the proposed activities. We did not know if they had previously studied Levinas' works or made any reference to him in their works, etc. A priori, we could expect that, for some of the participants, the workshop could be an introduction to Levinas' philosophy, while for others the deepening of aspects of his thought, and for others a useful exchange of ideas with participants who already knew Levinas, and finally for others just a rhetorical exercise. All considered, the outcome was not predictable in view of the participants' prior ideas and knowledge of the problems. It would have been impossible for us to write the present paper before the workshop had been held!

The workshop, which focused on theoretical concerns, proposed an introduction to Emmanuel Levinas' philosophy (1989, 2010, 2011a, 2011b) by highlighting how the phenomenological reflections of Levinas on Otherness support our (the authors') own respective research activities on history and mathematics education. Several elements were expected to be discussed with regard to Levinas' perspective, more specifically:

- History of mathematics for a nonviolent mathematics education
- History of mathematics as an experience of radical otherness
- Reflections on humanism and "antihumanism" with regard to the history of mathematics
- Reconsiderations of our ground pedagogical objectives related to the introduction of history of mathematics in the classroom
- Reflections on how to operationalize these elements in research.

As we will explain in more detail below, data from empirical studies and from our own research with secondary school students and prospective teachers were expected to be analysed and discussed by taking the Levinassian perspective. More concretely, we planned:

- An introduction at a theoretical level to Levinas's philosophy
- Readings of carefully chosen short excerpts from Levinas' texts
- Discussions in dialogical form aimed to formulate questions and stimulate the reflections of the participants
- Data analysis of empirical studies with the aim to reflect on Levinas' perspective in a more "applied" or "practical" way
- Reflections upon a list of questions aiming at collecting the participants' ideas that could lead to short writing activities performed in small groups.

The following sections explain how those activities were carried out and a number of objectives were reached.

3 The Levinassian perspective

Our first objective was to introduce the group to Levinas's philosophy by defining both the main philosophical projects of Levinas and our own understanding of the main concepts that was introduced by Levinas in his philosophical investigation.

A short PowerPoint presentation was made in order to introduce Levinas and some major concepts in his work. Then, the readings of some excerpts of Levinas's main texts were organised. Finally, a group discussion of those readings took place.

3.1 The presentation of Levinas' philosophy

The presentation of elements of Levinas' philosophy took about 25 minutes. It was pointed at some key elements such as: 1) the phenomenological positioning of the work of Levinas; 2) his project of a phenomenological investigation of otherness; 3) major concepts arisen from this project;4) and some reflections on the meaning of the encounter with the past.

During the presentation, the philosophy of Levinas was situated in the continental tradition and more precisely in phenomenology. Phenomenology was succinctly presented as a way to conduct philosophical investigation that is characterized by a certain rehabilitation of the mind sensitivity through the work of Edmund Husserl (1859-1938). In a bold move against metaphysics, phenomena are here considered as objects of consciousness, things toward which we orient ourselves, observe and reflect upon. This capacity of the mind to be "affected" (in the deep sense of a capacity to be influenced bodily, emotionally, cognitively, etc) leads to a philosophy that finds its way through idealism and empiricism. In a way, phenomenology was presented as a method for those who want to orient themselves in the field of thinking or a way to develop a philosophical thinking aimed at the description of how things give themselves (im)mediately. Phenomenology, at the time of Husserl, was taking a strong stance against positivism and psychologism.

The work of Levinas (who was a student of Husserl and Heidegger, himself a student of Husserl) was presented as situated in that phenomenological tradition. That has permitted scholars to view the philosophical project of Levinas as the phenomenological investigation of otherness in its fundamentals.

Otherness is taken by Levinas not exactly as a simple object of phenomenology in the continuity of tradition, but as a particular thematic enabling him both to enter into discussions with his masters Husserl and Heidegger and to criticize some of their results and to renew the phenomenological tradition.

As Levinas put it in his several phenomenological essays, the notion of Otherness is taken as the central part and the core of the human being. In his work, Levinas overturned the traditional (from Plato to Heidegger) ontological way of thinking of the human being. For him, the philosophical inquiry on the human being does not begin from his nature (the ontological perspective), but from his relation to the Other (the ethical perspective). In other words, ethics here is not understood as a "satellite" element of human existing, it is rather the central and the determinant field of reflections and research. Levinas opened a very new perspective on the human being which is no longer perceived as an isolated subject beset by phenomena, an *ipseity* thrown into reality, as Heidegger would say, but an ethical subject revealing himself in relation with the Other.

In this perspective, Levinas introduces the concept of Face (in French *visage*) which could be considered the authority of the Other, the injunction of the Other who imperatively asks for a careful relation. Levinas would say that Kant's categorical imperatives are in the Face of the Other. That leads Levinas to see the other human being as something infinitely-other, something that cannot be totalised, that cannot be filled with intuition.

To introduce a reflection on historical images and documents, some quotations from (Levinas, 1989) were given. Levinas introduces the concept of Resemblance, referring to the fact that we have the opportunity to get to know the thing through its image. The image is not to be considered a mere simulacrum and resemblance, not even a vague link to reality: "We will say the thing is itself and is its image. And that this relationship between the thing and its image is resemblance […] there is the simultaneity of a being and its reflection. […] Resemblance is […] the very structure of the sensible as such […]".

Fig. 3.1 shows a person who is doing an action or, better, who is going to do an action. We say that the person shows an intention, like in every figurative work of art: "[E]ternally Laocoon will be caught up in the grip of serpents; the Mona Lisa will smile eternally. Eternally the future announced in the strained muscles of Laocoon will be unable to become present. Eternally, the smile of the Mona Lisa about to broaden will not broaden".

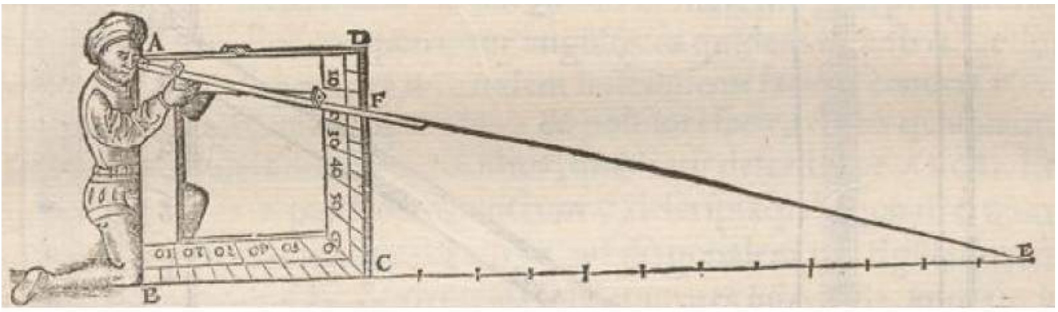


Figure 3.1: Oronce Finé, 1532, *Protomathesis*, Lib. II, fo. 66.

But "the immobile statue has to be put in movement and made to speak…". That image requires the student, who analyses it, to carry out the action: mentally or practically. The students of the second class of the upper secondary school Liceo Rosmini of Trent – Italy used that image to be able to understand how to measure a distance of a point far away by means of the above instrument, the quadrant. It was used in the Middle Ages and in the Renaissance for topographic and astronomical measurements. It is made of a wooden square having sides of about one meter. The index can turn around the point A until reaching the direction AE. E is the point we want to know the distance of, which is calculated by considering that the triangles AFD and ABE are similar.

A few other Levinassian concepts were introduced in order to give further insights into his philosophy, above all in order to help the participants in the reading of the excerpts. These concepts will be introduced in the next section, where we present the excerpts that have been chosen for the reading activities.

3.2 Readings of some excerpts from Levinas' main texts

After this short exploration of Levinas' philosophy, we proposed a reading activity to the group. The objectives were not only to deepen the exploration of the Levinassian perspective, but also to give a hint of the style of analysis that is deployed in that work. It was also a way to "enter" Levinas' work and to start a deeper discussion about it. We will try to highlights here how, more specifically, the selected excerpts refer to the following notions:

- Alterity and Violence(in a way, understanding the Other is to commit violence against him)
- Ethics and Hermeneutics (the Other and interpretation)
- Face of the Other (to be nude in front of the Other, not in an oblique standing)
- Metaphysical Desire (the total towardness to the Other, the dis(inter)ested relation with the Other)

Concerning the notion of Alterity and Violence, the introducing "In what sense […] does the *absolutely other* concern me?" (Levinas, 1986) can be borrowed as the fundamental question with regard to the reading of a historical document considered as an experience of alterity. Reflections aimed at looking for some answers to that question can be found in the following reports of educational activities with students of different levels (secondary school students and prospective teachers, see section 4). Necessarily, each of those answers will be partial, even if we refuse to consider the psychological specificity of the students. In some lines ahead in the same work, Levinas insists on the relation with the other and puts forward the danger of "transmutation of the other into the same". Levinas deepens the meaning of the previous sentences speaking of the reduction of "an alien world to a world whose alterity is converted into my idea". On the contrary, he opposes the "thought which is […] thought of itself […] *a movement of the same unto the other which never returns to the same*", that is to "the myth of Ulysses returning to Ithaca we wish to oppose the story of Abraham who leaves his fatherland forever for a yet unknown land". That reduction of the other to a mine is what Levinas calls *violence* (Guillemette, 2017). In that we can also see the "ambiguous nature of knowledge" which is “representation and movement […], the dynamism of the infinite and the fullness of actuality” (Levinas, 1999, p. 58).

Concerning Ethics and Hermeneutics, “The manifestation of the other” (Levinas, 1986) is, in a sense, not different from every “signification”. The cultural context illuminates the other. Here Levinas makes a comparison with a text and its context. In philosophical hermeneutics, the context has a central role in understanding a text, specifically both in the fusion of the horizon with the author and, through the reference to specificity and whole, in the hermeneutic circle. Levinas points out that “the comprehension of the other is thus a hermeneutic and an exegesis”. The other appears in his corporeity, linguistic acts, artistic gestures. “The other is given in the concept of the totality to which he is immanent”. The use of the term “totality” produces the expectation that Levinas would introduce another level into his discourse; in fact, reminding the title of his masterpiece *Totality and Infinity,* we can expect that he goes beyond that term derived from ontology in order to introduce infinity. That is made with reference to the *face*.

Concerning the concept of the Face of the Other, Levinas emphasizes that“[T]he phenomenon which is the apparition of the other is also a *face* […] the other does not only come to us out of a context, but comes without mediation” (Levinas, 1986). The face opens to infinity through his independent signification. But how can the infinity of the face appear in a phenomenon? The face, like “every entity, when it enters into immanence, that is, when it exposes itself as a theme, is already dissimulated” (Levinas, 1986). The Other (“the absolutely other”) can be friendship and can be the way through which the *I* can open to infinity. “The face-to-face is a relation in which the *I* frees itself from being limited to itself” (Levinas, 1999, p.56). In that lies the pedagogical value of that relation. The ethical relation depends on the fact that the approach to the other establishes “an experience different from that in which the other is transmuted into the same” (Levinas, 1986). The temptation to make the other a mine - to become his owner, to commit violence against him - is not possibly related to that situation.

Concerning the concept of Metaphysical Desire, Levinas emphasizes that “The metaphysical desire tends toward *something else entirely*, toward the *absolutely other*” (Levinas, 2010, p.33). It passes through the acceptance of the alterity of the other and also through the openness to walk on a path without end, without aim, through the impossibility to anticipate what is being desired. “It is a desire that cannot be satisfied”. When a student abandons any reference to the teacher’s requests and follows his personal interest for a topic, he desires to enter alterity without wondering where to go; it is the situation of somebody who does something without thinking of constraints or rewards; it is, in any case, what happens - also as the consequence of a request by the teacher - when a student has to understand a mathematical concept. Necessarily, he/she abandons, for a moment, any reference to the teacher’s requests, and begins an unpredictable process in order to get the desired acquisition. So, even if the student can outline the desired acquisition as a performance - for example as the solution of exercises – he/she must follow, at least for a moment, a blind desire. If the student completely lacks that desire, he renounces to enter the process of understanding.

Having a complementary role, the excerpts from page 275 to page 282 of (Levinas, 1997) aimed at discussing with the participants in what sense the abandoning of reference to the other in science can get start to negative consequences. In fact, Levinas problematizes the role of science (Levinas, 1999, p.71) by speaking of "its incompleteness" as regards the idea of infinity. The statement of the central place of "Man" and the "respect for the person, both in itself and in the Other" (Levinas, 1997, p.275) contrasts with the "anti-humanism" of the 20th century. We wanted to submit to the participants those reflections in order to recall the context in which Levinas' thought originated (a Hebrew, a prisoner during World War II, his father and brothers killed by the SS…) and to focus on the importance of looking for the other in scientific documents as a way to meet "man" while speaking of mathematics. In a wider sense, we would underline that science is meant for mankind.

4 Levinas and the convocation of historical elements in the mathematics classroom

After the exploration of Levinas' philosophy with the group, our objective was to highlight how the phenomenological reflections of Levinas on Otherness support our (of the authors) own respective research activities on history and mathematics education as well as to discuss it with the group. Two activities based on the data analysis from empirical studies were simultaneously organised. Participants had the possibility to work on the situations of their choice and were encouraged to reflect upon them. A few questions were given to the group in order to help the participants to start their reflection.

4.1 Data analysis concerning Luca Pacioli's work

A data analysis activity with the group was organized on Luca Pacioli's work.
Some excerpts from Pacioli's *Summa* (Fig. 4.1 and Fig. 4.2) were interpreted by students (17-18 years old).

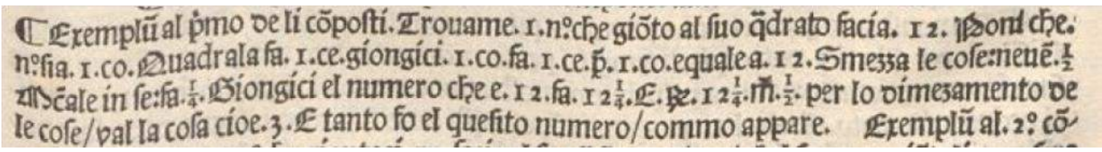


Figure 4.1: *Pacioli* 1.

[For the following English translations: "co" literally means "thing"; "ce" means "census".]

"[…] Find me a number that, if joined to its square, makes 12. Imagine that the number be a thing. Square it. It makes 1 census. Join 1 thing. It makes 1 census plus 1 thing equals 12. Halve the things. It becomes ½. Multiply by themselves. It makes ¼. Joint the number which is 12. It makes 12¼ . And the square root of 12¼ minus ½, because of the halving of the things, equals the thing that is 3. And the required number makes this amount, as it appears. […]".

Luca Pacioli, 1494, Summa, folio 145.

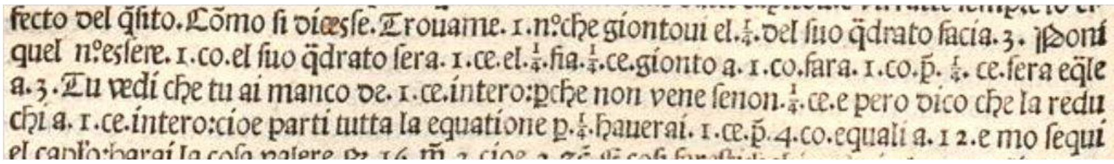

fecto del q̃sito.Como si dicesse.Trouame.1.n°che giontoui el.¼.del suo q̃drato facia.3.Poni
quel n°essere.1.co.el suo q̃drato sera.1.ce.el.¼.fia.¼.ce.gionto a.1.co.fara.1.co.p̃.¼.ce.sera eq̃le
a.3.Tu vedi che tu ai manco de.1.ce.intero:pche non vene senon.¼.ce.e pero dico che la redu
chi a.1.ce.intero:cioe parti tutta la equatione p.¼.hauerai.1.ce.p̃.4.co.equali a.12.e mo sequi

Figure 4.2: *Pacioli* 2

"[…] Find me a number such that, if ¼ of its square is added, makes 3. Let that number be 1 co; its square will be 1 ce. Its ¼ be ¼ ce which added to 1 co will make 1 co p[lus] ¼ ce, it will be equal to 3. You see that you have less than 1 whole ce because it results ¼ ce, but I say that you [can] reduce it to 1 whole ce, that is divide all equation by ¼; you will have 1 ce p 4 co equals 12 […]"

Luca Pacioli, 1494, Summa, folio 146.

Students synthesized their difficulties through questions or utterances, such as: "Where is the question? Where does the solution begin? No modern symbols!" This suggests their discouragement. The experiment is described in (Demattè, 2015). The same documents were presented to the group of participants.

In this study, we wanted to highlight just the students' behavior in front of their difficulties. They renounced in planning actions in order to reconstruct the meaning of specific parts with reference to each other and to the whole text, neither to take into account the global meaning as a resource to understand single parts. In the Levinassian perspective, that suggests students' incapability to see the other in the text and to consider that the comprehension of the other is hermeneutics. In the meanwhile, that suggests the importance to help students in creating the "cultural whole" (Levinas, 1986) in which the other is present. In classroom activities, the original text and a brief oral presentation by the teacher have not been enough to put the students in the trace of the other. In that presentation, the reference to mathematical aspects (quadratic equations) was prevalent. This, directly or indirectly, led – teacher and students - to violent behaviors, namely to neglect the richness the documents offered and, instead, to use it only for the specific purpose to train students with problems and equations. In this way, alterity has been converted into a narrow idea that came from the teacher. That situation could have hindered the students' approach to the other in the document, considering also the question mark we can put on the possibility they share the usefulness of that purpose.

4.2 Data examination concerning Fermat's work

A second data analysis activity was proposed to the group. It was around some of the data from a study (Guillemette, 2017) concerning the actual experience of prospective teachers engaged in the reading of historical texts.

Here is the material that was given to the participants:

In the next two excerpts from video analysis, we can see how future teachers' mathematical activity interacts with Fermat's minima and maxima method, and how it is interpreted. The activity concern Fermat's general description of his method and the first example given. He finds the maximum or minimum of a given term f(x) by "adequating" the two expressions f(x) and f(x + e), reducing and clearing remaining "e-terms". The example analysed here (divide a line AC at a point E such that rectangle ACE area is maximized) involve a term in the form of $f(x) = bx - x^2$.

First excerpt (Katia and Mitia):

Mitia says he doesn't understand why his [Fermat's] approach works. Katia says the same.

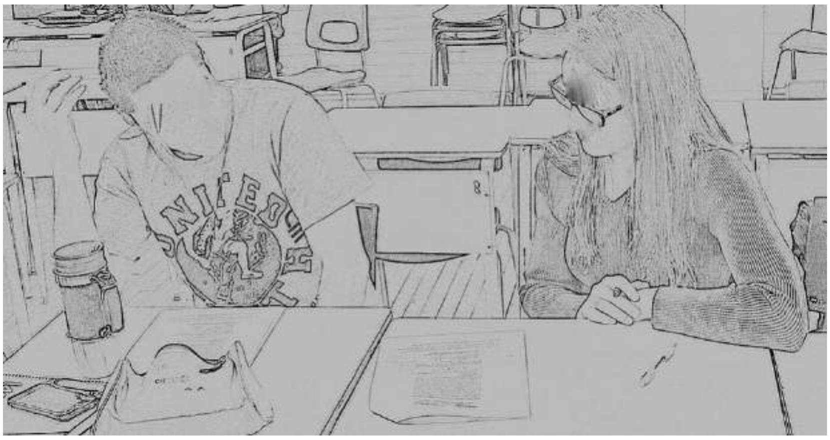


Figure 4.3: Katia and Mitia reading Fermat

Mitia then read the first paragraph aloud and tries to know the meaning of the unknown e.

Katia gives the hypothesis that e is a variable and that one must find for which value of e the area is maximum. Mitia doesn't see what it comes about. Katia then represents the function to be maximized and a second one whose side is increased by e. She emphasizes that it is difficult to represent the division by e with this geometrical representation.

Mitia notices that e is 0. Martha, from the other side of the classroom, indicates that Fermat previously divides by e. Mitia wonders, "How could he divide by 0?". He concludes that "e is not worth 0, but not far". Katia continually tries to illustrate the procedure geometrically, while Mitia invalidates her reasoning, claiming that the value of e is zero. Katia disagrees.

Second excerpt (Martha, Aliocha and Ninotchka):

Martha is saying that e is a very small value.

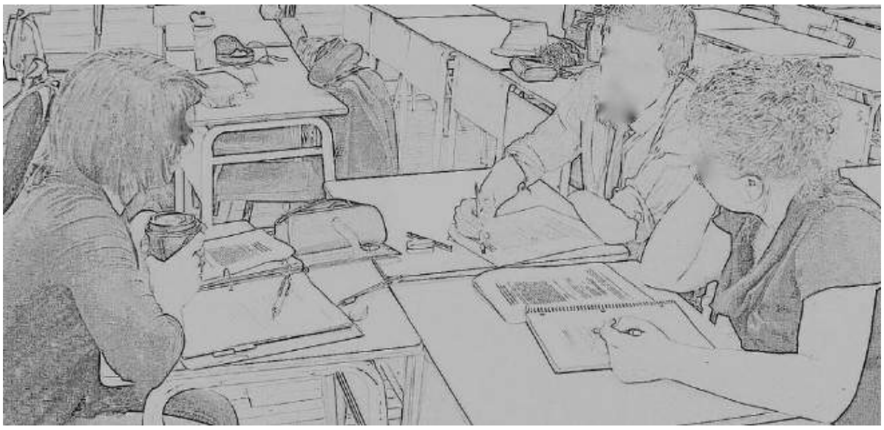


Figure 4.4: Martha, Aliocha and Ninotchka reading Fermat

Aliocha is trying to reconcile Fermat's method with the basic elements of modern calculus. He asks: "if adequating means to subtract the terms". Ninotchka answers that "adequating means simply to equalise".

Then Aliocha asks how she relates Fermat to modern calculus. Ninotchka shows his calculations and Aliocha concludes that their reasoning is equivalent.

After a few moments, Martha points out that Fermat removes e. Aliocha indicates that "e is almost 0, so the multiplication by e also gives almost 0". Martha asks herself whether the reader should "decide on the value of e". Aliocha replies, "Yes". Martha emphasizes that there is something missing in the reasoning. Aliocha asks why Fermat is using symbol of inequality, and concludes that adequating means to reduce to the minimum.

In this study, the investigation points out two interrelated experiences lived by the prospective teachers: *otherness* and *empathy*. Participants have shown serious efforts to understand the historical texts without uprooting them from the context in which they were produced. Indeed, the associated experience of otherness in mathematics seems rough from a cognitive and affective viewpoint, and it may lead to violent responses. For Levinas, violence is a "thematization of the Other", a reification of the Other, a way to make the Other a "Mine". Empathy and violence have been observed. Indeed, the subjectivity of the authors is sometimes arduously preserved. The students not always maintain an empathic relation with the authors. This violence can result in the disappearance of the empathic relationship. The authors are then dispossessed of their peculiarities; they are translated, summarized and reified. For us, in a Levinassian perspective, there is a violence of synchronization (Levinas, 1987).

Looking to engage the participants of the workshop in a reflection around these elements, the group was asked the following questions aimed at suggesting a trace of reflection in order to prepare the written work (see section 5):

- Would you agree that their way to engage in the reading of Fermat's text is related to ethics? In what sense?
- In what sense did they (or did not) actually show an ethical relationship regarding Fermat in the excerpts?
- How Levinas can help us to understand the encounter with Fermat (from the researcher's or the educator's viewpoint)?

Globally, the ethics to which we refer in these excerpts are that of the interaction I-Other. Indeed, some ethics that subvert the traditional principles! In a sense, the Levinassian approach goes back to the origins of the traditional ethics. We do not made reference to the ethical responsibility of the teacher as in Boylan (2016).The students showed their ethical involvement in reading the historical documents deepening their interpretation, not limiting to the comprehension of the procedural aspects. In that way, they started an infinite process, as Dilthey pointed out when speaking of the possibility to know the writer and considering "the limits of all interpretation, which is able to fullfil its task only up to a certain point. For all understanding always remains partial and can never be completed. Individuum estineffabile" (1996, p.249).

5 Final writing activity

For the final writing activities, participants were invited to work together in small groups. Most of them accepted that proposal but some preferred to work individually. The suggested task was to answer, in a written form, the questions below. During the work, we talked to the participants. Some participants preferred to only orally discuss their opinions with others and not to write down their answers. After the workshop, we (the authors) met and shared some oral remarks of the participants.

5.1 List of questions

The questions to the group were:

1. From the educational point of view, what are the differences between Pacioli's and Finé's documents? Does the visualization of people help in the Levinassian perspective?
2. How can a student find the "presence" of the Other in a historical document?
3. Describe your agreement about the following statement: 'For students, analysing historical documents is an experience of extreme alterity, positive *per se*. Students, though, are sometimes attracted by that experience and sometimes discouraged by it".
4. Suppose that instead of debating about Fermat's *Methodus ad disquirendam maximam et minimam* and posing questions about the sense of the procedure, Katia and Mitia might, for instance, have only tried to use the method in another example. Do you agree that their behavior is related to Levinas' ethics?
5. What did you treasure of Levinas' philosophy in this workshop?
6. Which links can you make between Levinas' philosophy and the history of mathematics in mathematics education? What has ethics to do with history of mathematics in mathematics education?
7. Do you agree that not considering mathematical and, generally, scientific knowledge related to the Other's involvement leads to negative consequences? Do they regard mathematics education? (See excerpt from *Difficult Freedom*)

The rationales for each question are the following.

Question 1 aims at discussing the relationship between secondary school students and authentic historical documents. The question requires comparing the document of Pacioli with the one of Finé. After the presentation of Levinas' philosophy, we assumed that the participants could have submitted answers related to the fact that Pacioli's document is somewhat obscure so that students could hardly find *another* in it. On the contrary, Finé's document shows a person acting with an instrument that suggests the presence of mathematics.

Question 2 is connected with question 1 and proposes a more general reflection inherent in the hermeneutical task to comprehend the other. Any reference to Levinas is missed out so that the question does not have a rhetorical role. Instead it is proposed to the participants in order to gather their opinions.

Question 3 completes the previous two and focuses on the concrete problem of the class. On the whole, the three questions propose different levels of reflection: class, historical documents, Levinas' philosophy. Sometimes the focus is apparent and sometimes the problem remains open to different references and elaborations leading to the answer.

Question 4 compares both a real and a hypothetical situation in order to reflect on the meaning of ethical involvement of students who read a historical document. In that case, the final question can be considered rhetorical because it requires to relate the educational situations with what we consider the first point of Levinas's philosophy.

Through question 5 we wanted to inquire to what extent our proposal, in its general aspects, passed on to the participants.

Question 6ideally completes question 4 and question 5. After the previous referred both to the class (with prospective teachers, in that case) and to Levinas, the focus is now on history in mathematics education.

Question 7 has a complementary role with the main points of the Levinassian philosophy we presented. It broadens the reflection to the responsibility of science and to the role of mathematics education: a different way to consider the role of ethics.

During the workshop, we realized that some participants already knew Levinas' philosophy. Therefore, they were able not only to give their opinions about the questions and the situations we submitted to them, but also to establish connections with works of other authors and to come up with suggestions about the workshop activities.

5.2 Resume of the writings

The following points are the transcription of the answers written by the participants during the workshop (the numbers correspond to the previous List of questions; the alphabet letters identify each participant. Please note that some of them did not answer all the questions).

1.

A. I think it is the "appearance" of Infinity and Totality.

B. Arithmetic – Algebra vs. Geometry. Visualization. Has to do with Levinas perspective, is a way of seeing, understanding – cure of other.

C. Yes, absolutely the visualization helps.

D. Pacioli's task is described via words. However, Finé's document offers a picture to help understand. There is no doubt that visualization helps students to read.

E.---

2.

A. He finds a lot of strange and unknown methods, ideas and practices.

B. The other is in expression, symbolism, other ways of written a text which is different from ours.

C. Maybe to put in context, it is the representation in this time, the discussions about the infinity concept in these times…

D. Different persons offer diverse viewpoints.

E.---

3.

A.---

B. Of course, it is a "dépaysement", they are trained on that they prefer little text, clear question.

C. Yes, I agree. It is like try to understand other languages. So the visual representations can be useful in this task. For instance, to know what kind of artefacts have in these times helps to understand the kind of productions awaredby the mathematicians in determined times.

D. Students might be interested in how historical situations came about. Their curiosity can be served as a stimulus to solve the problem themselves. Meanwhile, due to the era differences, students can also be discouraged by the solution mathematicians figured out. They vary from different notation to ideas.

E. (*Translation of the original in French.*) I totally agree with the statement: for a student (a majority of them) analysing historical documents is an experience of extreme alterity (differently from one another).More generally, learning something new is related to an absolute (more or less high) alterity. The student or pupil is tackled within his convictions, ideas which he lived, heard, learned before. Ideas that don't fit with the teaching that has occurred. The evidence acquired is now in question. What to do? Either to say no to the past for an alterity absolutely disturbing or to submit ourselves at a reality that is not ours? Here the work of the teacher begins and goes beyond the simple learning of notions.

To confront with historical text, not because they are ancient, but because they go beyond a conceptual level, because they destroy barricades, because the barricades had to be destroyed, and they access to other barricades that have to be destroyed, and so on, a path that the student will have to take himself.

4.

A.---

B. No, because they are in procedural way and not conceptual.

C. ---

D. Their behaviours are related to the Levinas's ethics as students (also us teachers) learn more from different approaches. In the process of listening and discussing with others, we learn.

E.---

5.

A. I was attracted very much and I have to mention that this was my first contact with his work. I leave from the workshop full of questions, quite interesting.

B. Alterity – otherness, subjectivity, resemblance

C.---

D. Otherness brings us with fresh ideas, approaches to activate our thinking.

E. The quest for the absolute: tiring, useless and alienating. No thanks! Indeed, the statement that teaching is, in a way, violence upon the other seems to me an idea more than interesting, more than judicious.

6.

A. In an aspect, we face several forms of relationships, the notion of power and how we use it.

B. Understanding the others (different period texts, ideals, goals)

C.---

D. Levinas' philosophy puts emphasis on otherness, which promotes students with abundant opportunities to get familiar with problems and inspire them to think more as alternatives are provided with.

E.---

7.

A. In some way I agree.

B. Yes.

C. Absolutely yes! The students tend to think that the mathematics algorithms or procedures are impersonal and disconnected with humanity problems. Other bad beliefs which are consequence of this “not considering” is, for instance, the mathematicians never wrong and all problems have a only one solution way…

D.---

E.---

F. (Not a particular question, just thoughts.)

Students usually want a method they can follow, which can be violence. But from the excerpts they are m??? wondering, trying to understand, finding obstacles while doing it. Then they adjust and renew their own understanding.

They need to respect the others understanding by not just taking it over.

They learn by making their own understanding and keeping the other alive with respect.

When you get to know an other you need to care about the other person and not make the other yours by thinking you understand them.

5.3 Resume of the oral commentaries

Some participants who chose not to write their answers preferred to give oral commentaries. Just after the workshop, we took note in a debriefing.

Participant G. He posed the problem of identifying the Otherand highlighted the difficulties to “choose” proper manifestations of the Other for ourselves during a history-based activity. He pointed out the example of the hat of the person depicted in the image from Finé, underlining that it probably had nothing to do with mathematics. From that, he raised the pedagogical problem of how to lead students to approach a historical document, and to help them to focus on mathematical concepts.

H. She noted that Levinas can help to think about mathematics itself as something that has inherently to be done with that experience of otherness. This helps one thinking of mathematics as a way to be-with-the-others.

I. He insisted on the question “What/Who is the Other?” From a more strictly philosophical point of view, he spoke of the necessity of a “mutual transformation” of identities.

L. He made a remark about the notion of trace. Presence and absence are modalities of the manifestation of the Other. This implies the responsibility for the teacher to make the otherness of the Other to appear.

M. She insisted on a non-violent relation with the other as something necessary for learning. There is the necessity of making room to the Other within the act of learning, by letting the Other show himself up.

N. She referred to Arthur Rimbaud’s “Je est un autre” [I is anOther], as a way to think about human subjectivity as something multiple, permeable,etc.

O. He referred to Jean-Luc Godard “Si vous m’avez bien compris, c’est que je me suis mal exprimé” [If you understood me well, it means that I didn’t express myself well], as a way to think about the violence of the act of understanding.

P. She considered multicultural class issues, where history is a way to think about different possibilities of teaching mathematics and different capabilities of students in learning it.

6 Discussion about the activities

Just after the workshop, we reflected on the fact that participants' answers could surely have been influenced by the reference to the Levinas' thought: we feared that it might even have happened in contrast with their real convictions. To be honest, in some sense the questions have to be considered a sort of exercises with respect to Levinas's works. In such perspective, it would have been better to express question 1, for example, in a more direct form: "[…] What aspect of Levinas' thought do you consider inherent to the visualization of …".

Here, we recall the previous answers focusing on some points, looking for connections and possibly sketching unitary discourses. This has the ideal purpose to broaden the dialogue we had with the participants also to the readers of these Proceedings.

About question 1, participants A and B use terms that are particularly meaningful in the Levinassian perspective, that is: "Infinity" and "other"; the latter sketches a reasoning about images as connections with the Other. Participants C and D prefer to highlight the use of images in mathematics, specifically in geometry, in order to facilitate students' understanding.

Question 2. After B, students could track the other through the unusual symbols and expressions, and also through particular ways to write the text they see in the historical document. From the previous historical document, two examples of particular ways to write a text can be the absence of a typographic distinction between problem and solution in Pacioli's excerpts (Fig. 4.1 and Fig. 4.2) and Fermat's use of equalities in the "*adequating*" method. After A, the student can learn that also in mathematics there are different viewpoints. C highlights the pedagogical opportunity to refer to the context in which mathematical concepts were used, so that they appear as the expressions of a specific historical period.

Question 3.The answers underline the possibility that students be discouraged while working with historical documents, but individual differences can be found. By means of the keywords "little" and "clear", the answer of B gives a hint to discuss the ways by which students are used to approach mathematical tasks. We think that this can suggest that students are not used to tackle complexity. In addition, C (speaking of understanding "languages") introduces the theme of the difficulties that students meet and have to overcome. Both these answers pose an educational problem that is how the teacher can help student facing non-trivial mathematical problems (we consider, among them, also the interpretation of historical documents). Students' curiosity or specific tools such as visualizations can be resources to be exploited. E discusses how students live alterity: his general utterance seems to derive from the specific cases of documents interpretation. Alterity is identified with "learning something new" and the necessity for students to always overcome new "barricades". This suggests reflecting on an attitude for school success, i.e. the willingness to tackle difficulties. After Levinas (2010, p. 213, English edition) "[t]he relation with the Other, discourse, is not only the putting in question of my freedom, the appeal coming from the other to call me to responsibility, is not only the speech by which I divest myself of the possession that encircles me by setting forth an objective and common world, but is also sermon, exhortation, the prophetic word". We would like to read this passage in these terms: "the relation with the Other", present in the historical document, who offers him/herself to me through a discourse inherent in mathematics, "is not only the putting in question of my freedom", i.e. staying in my closure and ignorance, without attention for Other's, conditioning, proposal, "the appeal coming from the other to call me to responsibility", i.e. to being open to the content of the document and being ready to tackle difficulties and surmount obstacles that I can meet in understanding, "is not only the speech by which I divest myself of the possession that encircles me", by which I put in discussion my previous knowledge about historically contextualized mathematics, "by setting forth an objective and common world", by accepting to discuss the problems the document presents to me, "but is also sermon, exhortation, the prophetic word": the Other, I recognise inside the document, is the mirror of the *Third Party* that is the community of mathematicians. The Other offers himself to me without violence, shows me a goal, leads me through his/her proposal and so gives me motivation. So, a problem with the class rises: are we able to help students to see in mathematics (in history and documents) the Other who can "exhort" them in their effort with mathematical problems?

Question 4. D proposes a general reflection on Levinas' ethics referred to the dialogue with the Other in learning. He relates the behaviour of Katia and Mitia with Levinas' ethics. B seems to give the reason for considering their behaviour as ethical: because they are working with reference to conceptual aspects and not (only) to procedures.

Question 5. The term "Otherness" is present in two answers (B and D). The last contribution (E) mainly refers to violence (to deepen this, see Guillemette, 2017).

Question 6. The answers confirm that this question could be considered a synthesis of the previous five. Again, they propose the theme of alterity ("relationship", "other", "otherness") again. We note references to history and mathematics education in the use of the following key words recalling the answers to question 2: problems and alternatives, different period texts.

Question 7. The answers report three levels of agreement. The last one focuses on educational problems and suggests a few more examples presumably regarding mathematics as a socio-cultural process. We expected that the document from Levinas's *Difficult Freedom* could lead participants' reflection toward explicit remarks regarding the social use of science.

Participant F opens and closes his "thoughts" by referring to the concept of violence. The first reference recalls what Levinas says in his Preface to *Totality and Infinity*: violence toward persons is "making them carry out actions that will destroy every possibility of action". Let us consider students who are requested to perform methods or procedures they do not understand: will they be able to act by using them in a new situation, for example in solving a problem? We believe the answer is no. Nevertheless, we, as teachers, too often force our students to follow rules that, in their eyes, have neither justification nor usefulness. Insisting on our reflection on the fact that students "want a method" to follow, we can say that they seem to have internalized the customs of the person (teacher) who commits violence against them. They seem to require behaving like that person even if, in that way, they suffer violence.

We propose here a categorization of written and oral answers, considering the different kinds of elaborations that the participants offered. In brackets, for the written answers, the question numbers are written close to the participants' capital letters; for the oral commentaries, O is followed by the participants' letters. Underlined are those answers that touch one or more aspects - out of a, b, c, d, e, f – in case they do not match the corresponding question. We consider that question 1 is inherent in the aspects a, f; question 2 in a, d, f; question 3 in a, d, f; question 4 in a, d, f; question 5 in a; question 6 in a, f; question 7 in a, e. Please, note repetitions highlighting the fact that some answers touch different aspects; remind that F wrote some thoughts without reference to any specific questions. Some answers that express agreement without any specifications do not appear in the following categorization.

Categorization of written and oral answers:

a. References to the Levinas' thought (1A, 1B, 4D, 5A, 5B, 5D, 5E, 6A, 6B, 6D, OG, OH, OI, OL, OM, ON, OO)

b. Connections with other authors (ON,OO)

c. References to mathematical contents (1B)

d. References to students or class (1B,1C,1D, 3B, 3C, 3D, 3E, 4B,6D,7C, F, OG, OP)

e. References to mathematics education (3C,3E,4D,5E, F, OG, OH, OI, OL, OM, OO)

f. References to the history in mathematics education (2A, 2B, 2C, 2D, 3C, 3E, 6A, OG, OP)

The above categorization suggests some reflections about the outcomes of the dialogue that took place during the workshop. The added categories b and c touch aspects that we recognize absolutely relevant to the topic of the workshop. The small number of participants who referred to them does not diminish their value, being suggestions that only indirectly derive from the questions we posed. *A posteriori*, we recognize that the presence of b in some questions could have led the participants to a critical position with our presentation of Levinas' thought. Concretely, the reference to other authors could have been limited to the quotations of their points of view, excluding a further comparison between their theories. The presence of c in some questions could have given more concreteness to the reflection, but it is not to forget that the aim of the workshop was to focus on methodological aspects.

The fact that some participants did not choose to write their answers, preferring instead oral interventions, suggests the limits of writing as recording tool with respect to others, i.e. audio or videotaping.

It is noteworthy that one of the participants helped us, by email, to 'interpret' his own written answers a few weeks after the end of the workshop.

The answers are often short and seem only to give strength to what the corresponding questions suggested. We can note that, in the cases in which the answers are wider, the participants either chose to answer only some of the questions or left out the reference to specific questions. That fact could suggest the organizers to further diminish the number of the questions and to make them less analytic. For example, it would have been preferable to unify the first two questions rather than refer to the historical documents given during the presentation of the workshop, and also should have been more important to insist on alterity in historical documents. Question 5 should have occupied the first place and, moreover, question 6 should have contained the final line of question 4. A different possibility might have been the one of substituting all the questions with keywords, with an introductory statement requiring the participants to write down some reflections or suggestions inherent in them, or their free observations about the contents and structure of the workshop. Question 7 could have been deleted because the reflection on mathematics education it proposes seems too wide and even too vague.

7 Conclusions

When we got the announcement for ESU8, at the beginning we thought to submit a proposal of an oral presentation, but to be structured in a different way that is in dialogical form. We thought that brevity would help us to reduce complexity in our experiment. Dialogue would give the possibility to introduce ‘Levinas in action’, as interactions between the same and the others. This would subvert the traditional structure of oral presentations in ESU Conferences and would contrast with the participants’ expectation. We concluded that a workshop could be the best location for a dialogue on Levinas and for approaching his thought, in case for the first time. Moreover, organisers suggested a two hours’ workshop, instead of one and a half-hour one. We were aware of our responsibility in organizing more complex activities. We were also aware that participants would tackle a difficult task, even having a longer time at their disposal. We were afraid for the fact that none of them might have ever approached Levinas before: what could be the best way to introduce him? Would the traditional presentation be enough? What kind of activities to involve participants? And in case of a not immediate positive response?

According to the seminal character of the workshop, a more realistic aim was arousing interest in the theme and stimulating involvement in the activities. We consider that this aim was reached. Moreover, in the last part of the workshop we were asked by a participant for some suggestions about how to deepen Levinas’ thought. We concluded that articles and books regarding mathematics teaching or history in mathematics education could help to focus Levinas’ role in those perspectives (*e.g.* Boylan, 2016; Maheux, 2013; Radford, 2012; Roth, 2011; Roth & Radford, 2011). Moreover, other secondary sources about Levinas’s works could help to see him inside the history of philosophy, comparing him, for example, with Descartes, Husserl or Heidegger, in order to focus on the aspects of innovation in his thought and on his prophetic message.

REFERENCES

Atweh, B., & Brady, K. (2009). Socially response-able mathematics education: Implications of an ethical approach. *Eurasia Journal of Mathematics, Science & Technology Education*, *5*(3), 267–276.

Bauman, Z. (1993). *Postmodern Ethics*. Oxford UK and Cambridge USA: Blackwell Publishers.

Boylan, M. (2016). Ethical dimensions of mathematics education. *Educational Studies in Mathematics, 92*(3), 395–409.

Demattè, A., & Furinghetti, F. (2011). History, figures, and narratives in mathematics teaching. In V. Katz, & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education. MAA series* (pp. 103-112). Washington, DC: Mathematical Association of America.

Demattè, A. (2015). History in the classroom: educational opportunities and open questions. In É. Barbin, U. T. Jankvist & T. H. Kjeldsen (Eds.), *Proceedings of the Seventh European Summer University,* first edition (pp. 335–352). Danish School of Education, Aarhus University.

Dilthey, W. (1977). *Descriptive psychology and historical understanding*. The Hague: Martinus Nijhoff. Partly in: http://www.westminster.edu/staff/nak/courses/documents/Dilthey_Other_Persons.pdf.

Ernest, P. (2012). What is our first philosophy in mathematics education? *For the learning of mathematics*, *32*(3), 8–14.

Guillemette, D. (2017). History of mathematics in secondary school teachers' training: towards a nonviolent mathematics education. *Educational Studies in Mathematics*, *96*(3), 349-365.

Levinas, E. (1986). The Trace of the Other. In M. C. Taylor (Ed.) *Deconstruction in Context*, (pp. 345-359). Chicago: University of Chicago Press.

Levinas, E. (1987). *Time and the Other*. Pittsburgh: Duquesne University Press.

Levinas, E. (1989). Reality and Its Shadow. In S. Hand (Ed.) *The Levinas's Reader*, First edition (pp. 129-143). Oxford: Basil Blackwell Inc.

Levinas, E. (1997). *Difficult Freedom: Essay on Judaism*. Baltimore: Johns Hopkins.

Levinas, E. (1999). *Alterity and Transcendence*. London: The Athlone Press.

Levinas, E. (2010). *Totalité et infini: essai sur l'extériorité* [Totality and infinity: an essay on exteriority]. Paris: Librairie Générale Française.

Levinas, E. (2011a). *Autrement qu'être: ou au-delà de l'essence* [Otherwise then being: or beyond essence]. Paris: Librairie Générale Française.

Levinas, E. (2011b). *Le temps et l'autre* [Time and the Other]. Paris: Presses Universitaires de France.

Maheux, J.-F. (2013). Offering and differing. *For the Learning of Mathematics*, *33*(1), 17–19.

Radford, L. (2008). The ethics of being and knowing: towards a cultural theory of learning. In L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics education: epistemology, history, classroom and culture* (pp. 215–234). Rotterdam: Sense Publishers.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics*, *80*(1), 101–118.

Roth, W.-M. (2011). *Passibility: at the limits of constructivist metaphor*. New York: Springer.

Roth, W.-M., & Radford, L. (2011). *A cultural historical perspective on teaching and learning*. Rotterdam: Sense Publishers.

NIELS ABEL: '*SO MANY IDEAS …*'

A workshop on using theatre to bring episodes in the history of mathematics to life in the classroom

Gavin HITCHCOCK
SACEMA, University of Stellenbosch, 19 Jonkershoek, Stellenbosch, South Africa
aghitchcock@gmail.com

ABSTRACT

This workshop explored the creative re-construction of historical events in theatrical form, and practical ways of using such plays in mathematics classroom contexts with minimal rehearsal. Participants were invited to stage cooperatively a 30-minute pre-scripted play featuring Niels Abel and contemporaries, fronted by a historian and an educator, aiming to place Abel's mathematical work in the context of his personal life. The dialogue is designed to display his courage and audacity, and the frustrations and pathos of his brief life. Poignant speeches drawn from his correspondence give insight into personal hopes and fears, publication challenges, and the intellectual excitement of creating new mathematics. Afterwards, participants were invited to critique the play, share ideas or experiences of using such theatrical tools in classroom settings, and then discuss in groups how they might go about weaving highlights from primary sources and biographical materials (distributed to groups) into dialogues for classroom use.

1 Why are mathematical plays important? Why the Abel play?

Niels Henrik Abel (1802-1829) was a Norwegian mathematician educated in Christiania (Oslo). During his short life Abel made ground-breaking contributions in many fields of mathematics. This workshop had a dual aim: to pay tribute to Abel and celebrate his life and work, but also to experience and reflect on ways that the devices of dialogue and theatre can bring mathematical history to life in the classroom. Participants were challenged to construct their own dialogues, drawn from primary sources and biographical materials, and encouraged to use theatre in the classroom to contextualise and enliven the teaching of mathematics, thus engaging learners in the excitement of mathematics-making and celebration of the people and stories behind the symbols, concepts and theorems. The story of Abel's personal life, linked with his extraordinary mathematical achievements, makes for excellent and unforgettable drama, constituting a good example of how such a theatrical experience can be both engaging and mathematically inspiring.

Framing the challenge for this workshop

This subsection contains my introduction to the workshop at ESU8:

My talk on Saturday introduced the idea of using dialogue or theatre as a communication tool. This is not a new idea – it is as old as humanity, and has long been used for communicating mathematical, scientific and philosophical ideas, by Plato, Galileo, and many more. But live dialogue form is sadly neglected in current curriculum-driven, time-constrained educational systems. Films with mathematical, historical & biographical themes are great, but I am convinced, having observed the effect of spontaneous drama and improvisation on young people of various ages, that positively *involving them in the re-enactment* of historical episodes is the key to getting their attention, and engaging their hearts and their minds. This workshop will make the case and (I hope) be inspirational too, by demonstrating theatre in action, involving all of you in the production and enactment. I will then invite you to share ideas and any similar experiences, and also to discuss in smaller groups how to use primary source and biographical material to generate different plays and dialogues for different levels, and to use the power of theatre in your own teaching. In summary, our workshop aim is: *observe, enact, critique, create*!

Now I am not primarily a playwright or a historian, but a mathematician. I gradually stopped going to topology conferences and started going to mathematical history and education conferences. Since then I have had plays enacted in whole or in part at a number of ESUs, HPM meetings, and other conferences, in Toronto, Braga, Stockholm, Oxford, Uppsala, Dublin, Leeds, Washington, Montpellier, Oslo, as well as in some African countries: Zimbabwe, Lesotho and South Africa. Some of these involved teacher trainees, and some of my plays have been used in classrooms or mathematics-camps elsewhere by others. But I am an academic, and though passionate about using history and mathematical competitions in mathematical enrichment in schools, I have had limited opportunity to mount plays under real classroom constraints, and so I am not the expert! I am hoping some of you may be, or may become, experts.

I challenge you to become co-creators of an exciting educational tool and art-form! I want you to run with the ball, catch the vision, and translate your own love of mathematics and its history into magic theatrical moments in your own classrooms. It's not so daunting – to convince you of this is a major goal of this workshop. We'll put this play on now with minimal fuss, minimal props and rehearsal, and (I expect) have lots of fun! Here's the plan for the workshop: [*displayed on a slide*]

Welcome & introduction – 5 minutes
Allocation of parts & distribution of scripts – 15 minutes
Rehearsing in corners – 15 minutes
Performance of play – 30 minutes
Reactions & critique – 15 minutes
Discussion in groups with source material – 20 minutes
General feedback – 15 minutes
Conclusion and thanks – 5 minutes

First we will allocate parts. Please don't be shy to volunteer or to nominate somebody: 2 stage-crew, 1 sound person, 3 directors, 3 major parts, 4 smaller parts, 6 one-speech parts, 6 mime/impro parts; others may help with direction or staging.

2 Casting, rehearsing & stage setting

Colour-coded name cards were hung around the participants' necks as they were cast; actors and directors of each mini-scene were given the same colour, so they could find each other quickly. A few props were handed to appropriate people with their name cards: manuscripts, envelopes, etc. Full scripts were given to participants with major parts, and to directors so they could coach their actors in entries and exits. Partial scripts were given to others, as necessary. Each script had the relevant part highlighted. The parts were listed in order on-screen as below, with colour coding to match the name cards.

Stage crew & Sound person

Directors of mini-scenes

- *Abel and friends in Copenhagen*
- *Abel & Cauchy in Paris*
- *Abel, Crelle & Crelle's Secretary in Berlin*
- *Abel & Christine Kemp & Maid in Frolen*

Major parts:

- MARIA (narrator, historian of mathematics)
- EMMY (co-narrator, mathematics teacher)
- NIELS HENRIK ABEL (young Norwegian mathematician)

Smaller parts:

- AUGUST CRELLE (German mathematician and journal editor)
- AUGUSTIN-LOUIS CAUCHY (renowned French mathematician)
- CHRISTINE KEMP (Abel's young Danish fiancée)
- BERNT HOLMBOE (Norwegian mathematics teacher & textbook writer)

One speech:

- LEONHARD EULER (German-Swiss mathematician)
- JOSEPH-LOUIS LAGRANGE (Italian-French mathematician)
- CARL FRIEDRICH GAUSS (German mathematician)
- GIFFEN WILSON (English mathematician)
- FERDINAND DEGEN (Danish mathematician)
- CHRISTOPHER HANSTEEN (Norwegian mathematician)
- CRELLE'S SECRETARY (German)

Improvisation / mime

- FRU HANSTEEN (Hansteen's wife)
- FOUR FELLOW STUDENTS & FRIENDS OF ABEL
- CHRISTINE'S MAID (Danish/Norwegian)

The stage crew co-operated with directors in placing chairs, tables, desk and couch in place for the mini-scenes. The scripted 'curtain' was simply imagined. The 'sound' was an optional feature and was dispensed with to minimise technical problems.

Casting was done within the allocated 15 minutes, with very little pressure. The use of colour-coded name cards on string necklaces was important in achieving this. Rehearsal and stage-setting were completed in the next 15 minutes. The staging and delivery suggestions in the scripts were kept to a minimum to emphasise the informal, light-hearted and impromptu nature of the production. No suggestion was made (even for short speeches) that people should try to memorise lines. This, together with the minimal guidelines, helped to allay any anxiety and allow freedom for creative improvisation. Participants responded well, and there were sounds of much enjoyment and hilarity in the mini-scene rehearsing; people with short speeches took the time to run through their parts, often reading to each other.

Figure 2.1: A mini-scene in rehearsal. From left to right:
Marcela Chiorescu (USA), Peter Ransom (UK), Fàtima Romero-Vallhonesta (Spain)

3 THE PLAY

[*Maria & Emmy appear front stage, seated to one side*]

MARIA: Hello everyone! Welcome to our celebration of one of Norway's greatest mathematicians, Niels Abel! [*points to slide showing portrait of Niels Abel*]

Figure 3.1: Niels Henrik Abel, *Lithograph after a drawing by Johan Gorbitz, 1826*
From Encyclopaedia Britannica,
Courtesy of the Royal Norwegian Embassy, Washington, DC

My name is Maria, and I am a historian – I am very interested in the story behind mathematics. And this is Emmy, who is a mathematician and teacher.

EMMY: Hello! [*smiles and waves*]

MARIA: Emmy, tell us – what is Niels Abel most famous for?

EMMY: Well, he did *many* amazing things during his short working life in that second decade of the nineteenth century, but perhaps it's his resolution of the problem of the quintic!

MARIA: Ah yes, conquering the quintic equation! Let's set the scene, mathematically. One of the great achievements of the sixteenth century was the cracking by Italian mathematicians of the long-standing problem of finding general algebraic rules for solving cubic equations, and also quartic equations – fourth degree equations. But to find a rule for quintics – fifth degree equations – taxed the ingenuity of the *greatest* mathematicians. Here is Leonhard Euler writing in 1767, expressing his disappointment:

[*cameo appearance of Euler, aged 60, frowning and shaking his head*]

EULER: All the pains that have been taken in order to resolve equations of the fifth degree, and those of higher dimensions, ... or, at least to reduce them to inferior degrees, have been unsuccessful; so we *cannot* give any general rules for finding the roots of equations which exceed the fourth degree.[1]
[*EXIT*]

MARIA: Four years later, Joseph-Louis Lagrange surveyed and analysed the efforts of mathematicians over the three centuries before his time, and reported pessimistically:

[*cameo appearance of Lagrange, aged 35, sighing*]

LAGRANGE: The result of these reflections is that it is *very* doubtful that the methods of which we have just spoken can give the complete solution of equations of the fifth degree. And, for even stronger reasons, those of higher degree. This uncertainty, together with the length of the calculations that the methods display, must put off in advance all those who might be tempted to make use of them to solve one of the most celebrated and important problems in Algebra.[2]
[*EXIT*]

MARIA: The great Carl Gauss himself, as the nineteenth century began, was pretty sure that the thing was impossible, and said so in the final chapter of his *Disquisitiones*. First he gave a masterly analysis of solutions of what he called pure equations – in particular what we now call cyclotomic equations [*slide appears*], and then he continued:

Gauss's cyclotomic equations: $x^n - 1 = 0$
Gauss's 'pure equations': $x^m - A = 0$

[*cameo appearance of Gauss, aged 23*]

[1] Based on (Euler, 1770, p. 286), quoted in (Katz, 2009, p. 665).
[2] Based on (Lagrange, 1771, p. 140); English translation in (Stedall, 2008, p. 348).

GAUSS: Everyone knows that the most eminent geometers have been unsuccessful in the search for a general solution of equations higher than the fourth degree, or (to define the search more accurately) for the reduction of mixed equations to pure equations. And there is little doubt that this problem is not merely beyond the powers of contemporary analysis but proposes the impossible ... [3]
[*shrugs and EXITS*]

MARIA: During the eighteenth century, British mathematicians, too, had been battling with this problem. Listen now to the English mathematician, Giffen Wilson, reading a paper to the Royal Society of London, at the close of the century. You can feel the frustration and puzzlement caused by prolonged failure to resolve the problem of equations of higher degree.

[*cameo appearance of Wilson, showing his frustration; he uses a formal, even pompous style, suitable for the Royal Society; italics are mine*]

WILSON: The difficulties under which the higher branches of algebra still labour are generally known. […] *[pause]* No degree of equations beyond the second is yet perfectly resolved. Cubics present frequently an irreducible case; biquadratics have, by several methods, been reduced to cubics; but no formula exhibiting to the eye the actual resolution of a biquadratic has yet appeared. And, for the fifth degree, and all upwards, not even a *clue* to general resolution has been found, by the combined labour and ingenuity of mathematicians for several centuries. This failure has been *very puzzling* and mortifying to the cultivators of algebra […] *[pause]*

Whether any method will ever be discovered, it is not easy to say. Whoever may be fortunate enough to discern a method for fifth degree and higher, will have the honour of removing an important and inexplicable barrier, which has so long obstructed the further improvement of algebra.[4]
[*EXIT*]

MARIA: That honour would go, twenty-seven years later, to the young Niels Henrik Abel.

EMMY: Except that his insight led, not to the algebraic solution of quintic equations, but to a proof that a general method did not exist! However, his work opened up wonderful new directions of research and discovery.

MARIA: Yes, indeed! Though I should mention that the Italian mathematician, Paolo Ruffini, produced a slightly earlier impossibility proof that was obscure and flawed but probably influenced Abel.
EMMY: We call it the Abel-Ruffini theorem today!

MARIA: And so the long quest to solve the quintic ended, as mathematical quests usually do, in many more questions. But what questions they were!

---

[3] Based on (Gauss, 1801 & 1966, 445), quoted in (Katz, 2009, 722).
[4] Based on (Wilson, 1799, 265).

EMMY: Did the young Niels show his talent early in life?

MARIA: Oh yes, I think so! He was born in Norway in 1802, the year after Gauss published his great *Disquisitiones*. His father, with a degree in theology and philology, taught him until he was thirteen, then he attended the Cathedral School in Christiania – now called Oslo. In his mid-teens, he was recognised as a potentially great mathematician by a new teacher, Bernt Holmboe.

EMMY: [*smiles*] Once again, the teacher is key to the unlocking of gifts in the child! Did nobody recognise his brilliance earlier?

MARIA: Sadly, no. Maybe because the opening of the new University of Christiania a few years earlier had drained the Cathedral School of its best teachers.

EMMY: [*sighs*] If only the best teachers were encouraged, and content, to be where they can be most influential, instead of wasting their gifts becoming deputy heads, heads, and academics!

MARIA: Well, a good head can be very influential! And so can an academic, for students and for creative research –

EMMY: [*emphatically*] Good *heads*, good *academics* and good *teachers* all have distinct combinations of gifts. Helping people find their true vocation must be one of the most under-rated professions in the world.

MARIA: Here is Bernt Holmboe:

[*cameo appearance of Holmboe*]

HOLMBOE: [*beams*] What a privilege it was to have this splendid young genius Niels under my charge! I not only helped him win a scholarship to continue at school after his father died, but encouraged him to read works by the greats: Euler, Newton, Lalande, d'Alembert, Lagrange and Laplace, even before he went to the University of Christiania at the age of 19.[5]
[*EXIT*]

EMMY: So the teenage Abel would have been aware of Lagrange's and even Cauchy's work, and the outstanding problem in the theory of equations – the quintic equation!

MARIA: He even thought he had found a general method of solution of the quintic, and hoped that his paper might be published by the Royal Society of Copenhagen. Fortunately, his paper *was* read carefully by a Danish mathematician, Ferdinand Degen, who demanded numerical examples. In looking for these, Abel realised his proof was faulty.

[5] This speech is my own invention, based on sources such as the biography in the MacTutor archive.

EMMY: Ah – it's good to see an instance of helpful reviewing! Sometimes new and ground-breaking work can be damned out of hand! But a good referee can make all the difference! So, what was Abel's reaction?

MARIA: Undaunted, he put the theory of equations on a back-burner, while he launched into a study of elliptic functions and other things.

EMMY: How did he pick up that elliptic functions and elliptic integrals would be critically important fields for research?

MARIA: It was that reviewer, Ferdinand Degen!

[*cameo appearance of Degen, reading slowly and carefully from a draft of his letter*]

DEGEN: The development of elliptic integrals would have the greatest consequences for analysis and mechanics. ... A serious investigator with suitable qualifications for research of this kind would by no means be restricted to the many beautiful properties of these most remarkable functions, but could discover a Strait of Magellan leading into wide expanses of a tremendous analytic ocean.[6]
[*EXIT*]

EMMY: Abel would certainly have discovered the work of his fellow sailor, the German mathematician Carl Jacobi!

MARIA: Yes, and found that Jacobi's results followed from his own. But he was very curious to know how Jacobi arrived at his.

EMMY: Did Abel publish anything at that stage?

MARIA: A scientific journal had just been launched at the University of Christiania by the professor of astronomy, Christopher Hansteen. From 1823, Abel published in Hansteen's journal papers on functional equations and integral equations.

EMMY: Ah, so he found another mentor at the University!

MARIA: Yes! Here is Hansteen:

[*cameo appearance of Hansteen with his wife Fru Hansteen*]

HANSTEEN: *[looking pleased with himself]* Ah, never was there another student of mine to compare with the young Niels Abel! I encouraged him in his work, and I also worked hard to find him a stipend. And my wife, Fru Hansteen, treated him like her own son! [*he acknowledges her and she smiles, nods, ad libs agreement*][7]
[*EXIT*]

---

[6] From the biography of Abel in the online *Dictionary of Scientific Biography*.
[7] This speech is my own invention, based on standard biographical sources.

EMMY: Meanwhile, I expect, Niels would have kept struggling with the elusive problem of quintic equations.

MARIA: And longing to visit the great European centres of mathematics, like Paris and Berlin! Here he is, a year later:

[*cameo appearance of Abel, holding manuscript*]

ABEL: I am convinced I have proved the impossibility of general solution of the quintic! I have written it out, but the printing costs are so expensive. I have had to leave out most of the details. Now I am told nobody will understand it or believe it. [*pause*] I am going to have to write a better, longer version in German … [*surveys his manuscript, sadly*]

But how can I get the right people to look at it?
[*EXIT*]

MARIA: This paper would finally be published in a new Berlin journal, edited by a man who gave Abel much-needed support, August Crelle. The journal became known simply as *Crelle's Journal*. It played a major role in the development of effective mathematical communication in the nineteenth century.

EMMY: Isolation can kill genius – so can poverty. It's wonderful that Abel had perceptive and supportive teachers and mentors at school and University, and then found a man of such influence in the mathematical establishment!

MARIA: Yes indeed – many poor, young geniuses are never recognised and encouraged!

EMMY: Did Abel ever meet August Crelle?

MARIA: It happened like this. In 1826, having worked at learning French and German, and raised some funds, he went off, aged 24, on what we might call a mathematical pilgrimage, with a group of friends...

[*Abel and four friends cross the stage, talking (in mime) and laughing*]

First, they visited mathematicians in Norway and in Denmark. Once in Copenhagen, they had to decide whether to go to Paris – Abel's heart's desire – or to Berlin, where his friends wanted to go ...

[*Abel and four friends return to mid-stage and mime an argument, pointing in different directions*]

Professor Hansteen had urged Abel to make for Paris, but he stayed with his friends and went to Berlin...

[*they go off-stage together, but Abel stops briefly and confides to audience, shrugging*]

ABEL: ... I am so constituted that I cannot endure solitude. Alone, I am depressed, I get cantankerous, and I have little inclination to work.[8] I will go with my friends to Berlin! Then to Paris, I hope!
[*EXIT*]

MARIA: Before leaving Copenhagen, Abel secured a letter of introduction to August Crelle, from a Danish mathematician. Going to Berlin turned out to be the best thing he could have done, for he managed to see his epic paper on the impossibility of solution of the quintic safely into Crelle's trusted hands, earmarked for Crelle's Journal, and also six other papers.

CURTAIN RISES
[*A Secretary shows the nervous Abel into the office of August Crelle*]

SECRETARY: Excuse me, Herr Crelle, that Norwegian mathematician has arrived to see you.

CRELLE: Ah, Herr … er … Abel … Niels Abel? Is that how I pronounce it? I do not often meet mathematicians from Norway! Welcome to Berlin!

ABEL: *Guten Morgen, mein Herr*! I am deeply honoured that you have consented to see me.

CRELLE: *Ach*, young man, your letter of introduction from the Danish mathematician was glowing, to say the least; but I suspected him of going over the top! However, once I had spent some time with the packet of papers you delivered last week, I had no doubt at all that I wished to meet the author! I did not expect him to be so youthful! I can see that already you are able to penetrate to the very foundations of problems, attacking them with extraordinary energy. You seem to see things in new and higher ways that your seniors have not seen.

ABEL: I am deeply grateful, Herr Crelle, for your encouragement –

CRELLE: *Nein*, I suspect the benefit will be mutual, my friend, for your work might well make my new journal famous! Come, let us discuss what needs to be done to get your work into a form I can publish...

[*Abel sits down at Crelle's invitation, and they pore over the manuscript*]

CURTAIN FALLS
EMMY: Did Abel stay long in Berlin? He might have been able to meet other important mathematicians.

---

8 From the MacTutor archive, quoting E T Bell, *Men of Mathematics* (New York, 1986), 307-326.

MARIA: It was probably very exciting but quite stressful having to win his way into the corridors of mathematical power, and survive in a big German city. He was certainly inspired, but longed to have peaceful thinking time! From Berlin, he writes a letter home in March:

[*Abel sits at a café table in Berlin, pen in hand, reading his scribbled letter*]

ABEL: I look very much forward to my return home to obtain the possibility to work in tranquillity. I have material enough for many years, and just now so many ideas are wandering around in my head.[9]

EMMY: He must have been starting to do that great work in analysis for which we celebrate him today in a number of theorems! He was a pioneer of true rigour in analysis, wasn't he?

ABEL: All my efforts I will now apply to bring more light into the tremendous obscurity which undoubtedly there is in the Analysis.
[*shakes his head in amazement*] It is quite amazing that it can be studied by so many, and yet is not at all strictly treated. There are exceedingly few theorems in the Higher Analysis which are being demonstrated with convincing rigour. [*sighs*] Everywhere one finds this unfortunate habit of reasoning from the particular to the general.

EMMY: Yes! He really shared the vision of Cauchy!

MARIA: And, like Cauchy, he was appalled at the careless manner in which mathematicians had been treating infinite series. He wrote to his old teacher and friend, Bernt Holmboe:

ABEL: My eyes have been opened in the most surprising manner. If you disregard the very simplest cases, there is in all of mathematics not a single infinite series whose sum had been rigorously determined! In other words, the *most important parts of mathematics* stand without foundation. It is true that most of it is valid, but that is very surprising. I struggle to find a reason for it, an exceedingly interesting problem …

[*Abel gets up from chair and makes EXIT*]

MARIA: Before going back to Norway to work out all the ideas beginning to simmer in his mind, Abel continued to travel through Europe, intending to meet and talk with some of the mathematicians whose works he had read. He arrived in Paris in July, hoping especially to meet Augustin-Louis Cauchy, the great proponent of rigour in analysis.

From Paris, in August, he writes to Professor Hansteen:

---

[9] This and the following speeches to Holmboe from Berlin are from Abel's biography in *Encyclopaedia Britannica.*

[*Abel sits at a café table in Paris, reading his scribbled letter*]

ABEL: Finally I have arrived at the Focus of all my mathematical wishes, to Paris! [...] Above all I would like to have my memoir completed [...] to be presented to the Institute. I have the hope that the Academy will print it.
[*EXIT*]

MARIA: Young and unknown, poor Abel was too optimistic.

CURTAIN RISES
[*Abel excitedly runs after Cauchy as he walks by, and attempts to speak to him*]

ABEL: Euh, *excusez moi*, Professeur Docteur Cauchy!

CAUCHY: [*brusquely, over his shoulder*] Oui, monsieur, qu'est-ce que tu veux?

ABEL: [*trying to keep up with Cauchy*] Bonjour, Monsieur Professeur! I would be greatly honoured if I could have a talk about mathematics with you at your convenience.

CAUCHY: [*stops*] I am very busy, young man! Do you not purchase my *Exercises des Mathematiques*?

ABEL: *Oui*, Monsieur, I am very impressed, but I was hoping to discuss some work of my own –

CAUCHY: [*glares at Abel*] Your work? With me? Indeed! Young man, I have far too much of great importance to do, I cannot spare time to listen to your speculations.

ABEL: [*thrusting a packet at Cauchy*] But, Monsieur Cauchy, may I give you my paper to read? I can come back and discuss it later.

CAUCHY: [*hesitating, and then reluctantly taking the packet and glancing at the contents*] Hmm, *oui*, if I have time perhaps I may look at it. But no – if you are serious, you should submit it through the proper channels to the Académie. [*hands the pile of papers back and walks off*] Adieu, Monsieur.

ABEL: Adieu, Professeur, merci!
[*sighs, stuffs the papers back in the packet, then turns and walks away, looking sadly over his shoulder as the great Cauchy strides off*]

CURTAIN FALLS
MARIA: Two months later Bernt Holmboe receives a letter:

[*cameo appearance of Holmboe*]

HOLMBOE: [*holding an envelope, looking excited*] A letter – from Paris! It must be from young Niels!

[*taking out and reading letter – which can be this script*] ... Hmmm ... The French are much more reserved with strangers than the Germans. It is extremely difficult to gain their intimacy, and I do not dare to urge my pretensions as far as that; any beginner struggles to get noticed.
[*aside to audience*] Ah, poor Niels – a little fish in a big pond! But what a very talented fish! I wonder if he has managed to meet the biggest fish of all – Monsieur Cauchy...

ABEL: [*voice off-stage, while Holmboe reads on silently, pulling appropriate faces*] I have just finished an extensive treatise on a certain class of transcendental functions to present it to the Institute which will be done next Monday. I showed it to Mr Cauchy, but he scarcely deigned to glance at it. Without bragging I dare say it is good. I am curious to hear the verdict of the Institute. Cauchy is '*fou*' [*proud, aloof*], and he is unapproachable. But he is the mathematician who these days best knows how to present mathematics. He is now publishing a series of memoirs entitled '*Exercises des Mathematiques'*. I buy them and read them diligently.[10]

HOLMBOE: [*shaking his head*] Poor, poor, Niels! How did he find the money to buy those pamphlets? I expect he is starving himself...
[*EXIT*]

MARIA: This paper that the great Cauchy would scarcely glance at is known to historians today simply as Abel's Paris Memoir. Dealing with the transcendental functions, this was one of the most important mathematical achievements of the early 19th century. Cauchy was supposed to referee it, but *[speaking with sarcasm]* it presumably stayed buried on his desk, while poor Abel worried about it for the rest of his short life. In the winter of the year 1826, Abel returned home to Oslo.

EMMY: His head must have been buzzing with new ideas! Did his home University give him a post?

MARIA: He was desperately hoping so, but the one available post went to his old teacher, Bernt Holmboe, who would be famous for writing many of the first Norwegian mathematics textbooks. The University did give Abel a small grant, but he had to scratch a living as a supply-teacher and tutor.

He was deeply in love with a girl called Christine Kemp. They became engaged and hoped to be married as soon as he secured some means of supporting a family. Meanwhile he researched and wrote with all the energy he could muster, while battling recurrent illness. Much of his time was spent trying to unwrap the mysteries of the general theory of solvability of equations:

[*cameo appearance of Abel, writing at table in between bouts of coughing*]

[10] Partially in MacTutor archive and in E. T. Bell, *Men of Mathematics* (New York, 1986), 307-326.

ABEL: I aim to investigate the following problems: One – To find all equations of any given degree that are algebraically solvable. Two – To decide whether a given equation is algebraically solvable or not.

MARIA: These questions would not be fully resolved in his short life. What he did show, and published in Crelle's Journal, in Berlin in 1829, is this result:

ABEL: Gauss's method for finding the roots of the equation $x^n - 1 = 0$ can be generalised – I have to find a criterion for algebraic solvability of a general equation. His method depends on the fact that there is one of the *n* roots such that each root is a power of that one.

[*slide*]

Gauss's method for: $x^n - 1 = 0$,
depending on the fact that there is one of the *n* roots
such that each root is a power of that one,
can be generalised to find a criterion
for algebraic solvability of a general equation.

ABEL: Now, at last, I believe I have proved a wonderful theorem: If the roots of an equation ... [*scribbles while theorem is displayed on another slide*]

If the roots of an equation of any degree are related so that
all of them are rationally expressible in terms of one, designated as x,
and if, furthermore, for any two of the roots, θx and $\theta_1 x$,
where θ and θ_1 are rational functions, we have

$$\theta\,\theta_1 x = \theta_1\,\theta x,$$

then the equation is algebraically solvable.

In an **Abelian group**, composition is **commutative**: for any elements g, g_1 of the group

$$gg_1 = g_1 g.$$

[*EXIT*]

MARIA: Today Abel's name is celebrated in the name 'Abelian group' – a group in which the composition operation is commutative: multiplying any two elements of the group can be done in any order, the answer will be the same. You can see the connection with his theorem, so it is a fitting way to remember him.

EMMY: Whatever became of that paper Abel left with the Paris Académie?

MARIA: He wrote a two-page note to Crelle with a brief summary, which was published in Crelle's Journal. The original manuscript was only found by Cauchy after a search

prompted by Jacobi! It was finally published in 1841. That ill-fated original manuscript was mysteriously lost again, but over a century later there was great excitement among historians when it was discovered in Florence, in 1953.[11]
But let's go back to September 1828 ….[12]

CURTAIN RISES
[*Abel lying on a couch holding manuscript and pen; he coughs periodically. His* fiancée *Christine Kemp is seated beside him*]

ABEL: Ah, Christine, you must help me reply to Herr Crelle! It's over two weeks since I received his kind letter.

CHRISTINE: I remember how happy you were when you received it!

ABEL: [*smiling*] Yes, I had not expected such a quick answer. And he also kindly took the trouble to copy and send me excerpts from letters of Jacobi and Legendre! I really must thank him. I should have replied sooner, but I wanted to finish this short manuscript on elliptic functions and send it too.

CHRISTINE: It's wonderful, isn't it, that he holds out good hope of an appointment for you in Berlin? Maybe we can get married at last and live in Berlin!

ABEL: I will implore him to let me know as soon as possible when anything has been decided, favourable or not, for if it does not work out as I hope, I must be prepared to improve on my conditions here. I can do nothing until I know what will happen.

CHRISTINE: He will certainly grant you this request – he knows your abilities, Niels!

ABEL: You must help me draft this letter! I want to say how much pleasure I have taken in the remarks of Legendre and Jacobi! I can see that Jacobi has arrived at my transformation theory for elliptic functions by taking a different path, and I am very curious to learn about his method.

CHRISTINE: Are you going to ask him to publish in the journal this new paper of yours?

ABEL: Yes, perhaps he will have a few pages to spare in the upcoming fourth issue. I want to ask him to print this paper first. Then the paper on equations can follow afterwards.

CHRISTINE: Why? He is very keen on the equations paper isn't he?

ABEL: I know, but I believe that the elliptic functions will be of greater interest to mathematicians. And also, you know, my health will hardly permit me to occupy myself with the equations for a while – it's exceedingly difficult stuff, and will take too much

[11] See full story in (Del Centina, 2002).
[12] The following scene is based on a letter from Abel to Crelle (Abel, 1828).

effort. Shall I confide in him that I have been ill for a considerable period of time, and compelled to stay in bed?

CHRISTINE: Oh, Niels, you must be careful! [*she puts arms around him*] Even if you are now recovered, the physician has warned you against any strong exertion!

CURTAIN FALLS

MARIA: Christine's fears were justified. Six months later, Niels Abel died, on the 6th of April 1829.

[*cameo appearance of Christine, sobbing. A maid brings a letter in on a tray*]

CHRISTINE: Oh! A letter ... from Berlin! It must be from Herr Crelle ... *[tears the envelope open, then gasps]* Niels has been appointed to a post in Berlin! ... Too late, too late, oh my poor Niels, two days ago you would have been so, so proud! ...

[*she breaks down, crumpling the letter and hurling it on the floor, then she runs off stage covering her face*]

MARIA: Niels Abel's body was buried in the churchyard at Froland. August Crelle writes a glowing tribute to his young friend, in his journal:

[*cameo appearance of Crelle with pen in hand, writing obituary, and reading from his draft*]

CRELLE: All of Abel's works carry the imprint of an *ingenuity* and *force of thought* which is unusual and sometimes amazing, even if the youth of the author is not taken into consideration! … [*reflective pause*]

One may say that he was able to penetrate *all* obstacles down to the very foundations, with a force which appeared irresistible. He attacked the problems with an extraordinary energy. He regarded them from above and was able to soar so high over their present state that all the difficulties seemed to vanish. … [*pause*]

But it was not only his great talent which created the respect for Abel and made his loss *infinitely* regrettable. He distinguished himself equally by the *purity* and *nobility* of his character … and by a rare modesty equal by his genius – what a combination!

[*walks off slowly, speaking to himself and looking very sad*]

Those of us who knew him cherished him. If only he'd lived and come to Berlin …

[*EXIT*]

EMMY: Looking back, it seems that in spite of the efforts on his behalf by Crelle, Abel was relatively unknown when he died – so sad!

MARIA: Well, both Legendre and Jacobi recognised his brilliance. But Cauchy mislaid his monumental Paris manuscript, and Gauss seems never to have read the copy Abel sent him of his 1826 paper on the quintic!

EMMY: Maybe because Gauss still believed the thing was impossible! [*snorts*]

MARIA: But maybe because Gauss minimised the problem, imagining he himself had more or less solved it earlier. Nobody can be sure with Gauss! Some say Gauss was loathe to give the glory to another. It has even been suggested that, because Abel omitted to slit the pages before posting it, Gauss couldn't be *bothered* to cut it open and read it.

EMMY: [*groaning, and covering her face with her hands*] Oh, no! I feel like I could slit his throat!

THE END

Figure 3.2: Abel monument, Oslo, designed by Gustav Vigeland (1908)

4 Review, critique & reflection

Following the play, open discussion took place, with guiding questions: What were the best moments? What didn't work well? What improvements (for this script) can you suggest, in staging, action, emotion, voice projection? How could you re-script Abel's story for different groups, levels of communication, different balances of the personal, the historical, and the mathematical? In particular, how much mathematics to include? Did the nature and magnitude of Abel's mathematical achievements get through? The problem of authenticity versus accessibility – how close should such a play stay to the primary source material? Can any of you share experiences of using theatre?

In the glow of achievement, few participants seemed in the mood to criticise, and the discussion was largely positive, with different people particularly liking different moments of the play. There was little experience of using anything similar in teaching. Some feedback is collected in the final section below.

During the discussion some hardcopy materials were circulated, and then participants were invited to form smaller groups, and reflect on how they might go about weaving highlights from such primary sources and biographical materials into dialogues suitable for classroom use at various levels. The materials were selected and extracted according to a number of criteria: relevance to Abel's life and work, reasonably short, potential for dramatization, easy availability (to promote the idea that source material for creating reasonably authentic dialogues is not hard to find), illustration of the richness of further resources, insight into the historiographical challenges of achieving authentic dialogues. The materials included extracts from the *MacTutor History of Mathematics archive*; one of Abel's letters in full; biographical material on Holmboe and Hansteen from the *Dictionary of Scientific Biography* and *Encyclopaedia Britannica*; a collection of quotes from Abel; a paper on the fortunes of Abel's famous 'Paris Memoir' which features in the play; and references for further reading and possible play-writing.

5 General feedback & conclusions

The workshop concluded with a short plenary discussion, inviting feedback from the small groups. This section collects some responses and retrospective feedback.

Length of play

The scheduling was generally agreed to be fine for such a workshop, but would naturally have to be adjusted for a particular classroom at a particular level. Some felt plays of 30 minutes would be the maximum to aim for. "This workshop worked well as people were there because they wanted to attend. In a classroom there are often students present who would rather be elsewhere, so in my opinion shorter plays may be a way forward." "It was amazing how well the actors played their role to bring the persons to life with such short notice." "I think it is a good idea to write such pieces for schools. They would have to be short like this one or even shorter, but they can awaken feeling for the human side of these scientific persons."

Practicalities for classroom use

It was important for this workshop to get the casting and rehearsing done in a very short time, and to demonstrate how to achieve this. Useful strategies were the use of large colour-coded name-labels for the characters, casting directors to lead the rehearsing of mini-scenes, highlighting the relevant lines in everybody's script, encouraging a relaxed attitude to reading the lines, and keeping the staging simple. The enthusiasm and enjoyment shown by the participants was palpable. "What made this work well, was the limited time we had to prepare ourselves. It would not have worked if we had a longer time because some people would not have been engaged." "I was surprised how well the event ran practically without any preparation." "Having a large cast was important: everyone had a part to play." "Everybody was required to participate and it was extremely enjoyable." "It is good to give opportunity for everyone to participate in some way, but to balance this with each person having space to watch some of the action." "It helped not to have to worry about memorizing lines. This freed up some people to improvise."

Coverage and general content

While the play was felt to do justice to the main events of Abel's life, some discussion took place on omitted episodes with dramatic potential, e.g. the story of how Holmboe became Abel's teacher when the previous teacher was removed from his position after beating a student who died of the injuries. In any attempt to portray in a classroom some facets and moments of a mathematician's life, there will be a rich menu of possibilities, and selection will depend on the class and level. It is not necessary to be comprehensive; one small dramatic episode can spark interest in the person's life and give colour to the mathematics. Humour was seen to be an important ingredient.

The mathematics

In this workshop "there was a good balance between personal and mathematical", "good balance between maths and biographical information", but the question of how much mathematics to bring into a play, and how to frame it, is heavily dependent on the class and the purpose. This play presents just three glimpses of mathematics (on slides). Equations should be kept to a minimum, adjusted according to the background of the participants. At higher levels it may be worth weaving in some information on the development of the formula for third and fourth degree equations. To avoid the mathematics entirely is to miss the chance to signal that such apparently dry symbols are tied up with human drama. At appropriate levels teachers could follow up with work in the classroom. "… it engages students who might not have taken an active participation in mathematics in the past as well as introducing them to their heritage." "[While the] level of maths in the play is too advanced for high school, yet because it involves solving of equations, which is a very familiar subject to students, it shouldn't be a problem [to interest them]"; "[Such theatre] enriches the mathematics curriculum and gives students an appreciation of the thoughts behind the evolution of mathematics … it shows them that everyone has struggled with new concepts, so they should not feel bad about themselves when they do not understand the mathematics they are being taught the first time they meet it." "In the average classroom, there will be a few students who are mathematically talented, others who have dramatic talents, and such a joint activity can open up lines of peer communication."

Can such theatrical experience enliven mathematics education?

Participants commented as follows: "Some people who were generally shy and quiet came out of themselves and so seemed to grow in stature." "In general, it was a fun experience and very memorable." "It was great! I really enjoyed to be part of the play." "What made this memorable for me was working with such a lovely group of people and discussing how we would present it. It threw light on Abel and Cauchy as people and portrayed events in a vivid way that was not possible reading about them in a book." "The play brings you very close to a different perspective of mathematics, the appreciation of mathematics as a cultural-human endeavour. I hope one day I will incorporate it in one of my classes." "This play is a great way to expose students to lesser-known aspects of the human side of Niels Abel." "It was the first time I had attended a theatrical workshop based on mathematics, so I didn't know what to expect. I came because I'm interested in exploring different ways to present the mathematical concepts to students. I found it interesting, amusing and original." "I love the concept of a play as a means to incorporate history of maths in education."

Future use and new directions

We close with some ideas from participants: "… we were presented with the text as a finished article, but for it to be an enriching experience for the students; they will need to be … part of the creative process." "It would be worthwhile to look at the social context of the time, the geographical location and other events of the time leading up to the development of the concepts involved.” “Teachers should try something different and outside their comfort zone sometimes!” “[When I incorporate this in one of my classes] I will first assign the roles to my students, then give them time to practice on their own and then play together. It will be very similar to what we did in Oslo, but … I will probably not do it in one class period. I would want them to have more time to think about this play and the role Niels Abel played in the history of mathematics.” “I intend to use the play on either Abel or Cauchy/Gauss at a conference in 2020; have to translate it, to make it really work with my students (Dutch teacher training, ages 17-20) … I will find a student to write a play in Dutch or on Dutch mathematicians.”

REFERENCES

Online biographical resources

MacTutor archive. http://www-history.mcs.st-andrews.ac.uk/Biographies/Abel.html. Accessed: 13 Nov 2018.

Encyclopaedia Britannica. http://www.britannica.com/biography/Niels-Henrik-Abel. Accessed: 13 Nov 2018.

Dictionary of Scientific Biography. New York, 1970-1990.

http://www.encyclopedia.com/doc/1G2-2830900018.html. Accessed: 13 Nov 2018.

Articles and books

Abel, N. H. (1824). *Mémoire sur les équations algébriques, où on démontre l'impossibilité de la résolution de l'équation générale du cinquième degré (Memoir on Algebraic Equations in Which There is Demonstrated the Impossibility of Solving a General equation of Fifth Degree).* Pamphlet printed in Christiana (Oslo); in: Abel, 1881, *Oeuvres Complétes,* vol. 1, 7. http://www.abelprize.no/c54178/binfil/download.php?tid=53608

Abel, N. H. (1826). Beweis der Unmöglichkeit algebraische Gleichungen von höheren Graden als dem vierten allgemein aufzulösen (Proof of the Impossibility of Solving Algebraically General Equations of Degree Higher Than the Fourth). *Journal für die reine und angewandte Mathematik (Crelle's Journal),* 1, 65–84. Also in: Abel, 1839, *Oeuvres Complètes,* vol. 1, 66–87.

Abel, N. H. (1828). Sur la résolution algébriques des équations, in: *Oeuvres Complètes de Niels Henrik Abel*, 1881, vol. 2, 217–243.

Abel, N. H. 1828. Letter to August Crelle, 25 September 1828, from Christiania; in MacTutor archive.

Abel, N. H. (1829). Mémoire sur une classe particuliére équations résolubles algébriquement. *Journal für die reine und angewandte Mathematik (Crelle's Journal), 4,* 131–156. Also in: Abel, 1881, *Oeuvres Complètes*, vol. 1, 478–507.

Abel, N. H. (1829). Démonstration d'une propriété générale d'une certaine classe de fonctions transcendantes, *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, *4,* 200–201. Also in: Abel, 1881, *Oeuvres complètes,* vol.1, 515–517.

Abel N. H. 1841. Mémoire sur une propriété générale d'une classe très étendue de fonctions transcendantes, *Mémoires présentés par divers savants à l'Académie Royale des Sciences de l'Institut de France*, t.VII, Paris, 1841, 176-264. Also in: Abel, 1881, *Oeuvres complètes*, vol. 1, 145–211.

Abel, N. H. (1839, 1881). *Oeuvres Complètes de N. H. Abel, mathématicien, avec des notes et développements*. B. M. Holmboe (Ed.). 2 vols. Christiania: Grøndahl, 1839. Second edition: *Oeuvres Complètes de Niels Henrik Abel,* S. Lie & L. Sylow (Eds.). 2 vols, Christiania: Grøndahl, 1881. Reprinted, New York: Johnson Reprint, 1973.

Crelle, A. (1829). Obituary for Niels Abel. *Journal für die reine und angewandte Mathematik (Crelle's Journal), 4,* 402. Quoted in translation in: Swetz et al, 1995, 297–298.

Del Centina, A. (2002). The manuscript of Abel's Parisian memoir found in its entirety. Historia Mathematica, *29*, 65–69.

Del Centina, A. (2003). Corrigendum to `The manuscript of Abel's Parisian memoir found in its entirety‘. [

Historia Mathematica. *29*, 65–69.]'. *Historia Mathematica*, *30*.1, 94–95.

Euler, L. (1770, 1822, 1984, 2006). Vollstandige Anleitung zur Algebra. Elements of Algebra, 3rd English edition, trans. F. Horner & J. Hewlett, Longman (1822). Reprinted New York: Springer-Verlag (1984); Part I re-printed Tarquin Pub. (2006), ed. C. J. Sangwin. In Euler Archives E387/8 http://www.math.dartmouth.edu.

Gauss, C. F. (1801, 1863, 1966, 1986). *Disquisitiones Arithmeticae*. Leipzig. Also in: Gauss, C. F. (1863), *Werke*, vol. 1. English trans. A. A. Clarke, SJ. New Haven: Yale University Press (1966); reprinted New York: Springer-Verlag (1986).

Holst, E., Størmer, C., Sylow, L., Bjørnson, B. (1902). *Niels Henrik Abel; memorial publié à l'occasion du centenaire de sa naissance*. Oslo: Kristiania, J. Dybwad. This contains Abel's extant letters.

Katz, V. J. (2009). *A History of Mathematics: An Introduction*. Third edition, Boston: Addison-Wesley.

Lagrange, J. L. (1770–1771). Réflexions sur la théorie algébraique des équations, *Nouveaux mémoires de l'Académie Royale des Sciences et Belles Lettres de Berlin*, *1* (1770), 134–215, and *2* (1771), 138–253. Also in: Lagrange, J. L. (1867–92), *Oeuvres de Lagrange,* J. A. Serret, & G. Darboux (Eds.), Paris, vol. III, 203–421. English extract trans. in: Stedall, J. (2008). *Mathematics Emerging: A Sourcebook 1540–1900*. Oxford: Oxford University Press, p. 348.

Ore, O. (1974). *Niels Henrik Abel: Mathematician Extraordinary*. New York: Chelsea.

Sørensen, H. K. (2002). Abel and His Mathematics in Contexts. NTM: *International Journal of History and Ethics of Natural Sciences, Technology, and Medicine, 10*, 137–155.

Swetz, F., Fauvel, J., Bekken, O., Johansson, B., & Katz, V. (Eds.). (1995). *Learn from the masters!* Washington, DC: The Mathematical Association of America.

Wilson, G. (1799). Essay on the Resolution of Algebraic Equations: attempting to distinguish particularly, the Real Principle of every Method, and the true Causes of the Limitations to which it is subject. *Philosophical Transactions of the Royal Society of London*, *69*, 265–306.

WHAT CAN ART TEACH US ABOUT MATHEMATICS?

Snezana LAWRENCE
Middlesex University, School of Design Engineering and Mathematics, The Burroughs, Hendon, London NW4 4BT, UK
snezana@mathsisgoodforyou.com

ABSTRACT

This workshop will build on the plenary lecture that the author will present on *The art and architecture of mathematics education – a study in metaphors*. The workshop will offer opportunities for participants to trace the images of and about mathematics and mathematicians, and play with metaphors related to the aims and the nature of the discipline, from different periods and geographical areas. The participants will be encouraged to work on creating an ideal 'exhibit' to represent one such collection of images or objects, as if it were to be displayed in an environment such as a museum dedicated to mathematics, be it a real building or a virtual one. The purpose of such work would be to analyse, discuss, and decide which precise aspect of mathematics the group or individual participants find most fascinating, and how that could be communicated to an imagined audience.

The workshop would aim to further inspire participants to take active part in constructing images and narratives of and about mathematics, to illustrate in different ways what mathematicians have done or do. The resulting images or ideas of the work from the workshop will be shared with all the participants after the workshop.

HPM AND IN-SERVICE MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT IN CHINA

Xiaoqin WANG[1,2], Jiachen ZOU[2,3], Zengcheng YUE[2,4], Zhongyu SHEN[2,5]

[1]College of Teacher Education, East China Normal University, Shanghai, China
[2]Department of Mathematics, East Normal University, Shanghai, China

[1]xqwang@math.ecnu.edu.cn, [3]jczou@math.ecnu.edu.cn, [4]yuezengcheng@126.com, [5]344080566@qq.com

ABSTRACT

Conceptualization of professional development have moved from "deficit" and "workshop or training" models to models of "professional growth" where teachers engage actively in collaborative inquiry into their own practice to enhance their knowledge of pedagogy, and students (Widjaja et al. 2017). However, through researching papers from Proceedings of HPM Satellite Meeting, ESU, CERME, ICME, in books and journals in the 21st century, we find that the studies on HPM and mathematics teachers' change or professional development mainly happen in universities, are mainly for pre-service teachers and the direct approach in this context is to give graduate courses (Barbin and Tzanakis, 2014).

In Mainland China, the situation is somewhat different. As a teaching research system has been practiced nationally since the 1950s (Wang, 2009), we have combined the teaching research system – mainly a Lesson Study (LS) – with HPM, which we call HPM Lesson Development (HPMLD). In the course of a LS, the teacher who conducts the LS would follow a procedure as shown in Fig.1 (Wang, Qi and Wang, 2017) with support from professional learning community (PLC), which is made up of a school-based group, an HPM research group, and a teaching expert group. Each group in the community has its own expertise, which is the reason why they get together, and without collaboration, it may be impossible to develop a sufficiently complete HPM Lesson. Through HPMLD, the teachers' knowledge, beliefs, attitude, and even instructional competencies would improve (Wang, 2013; Yue and Wang, 2016).

However, the influence of HPM has to be extended. So we publish the developed lessons, and make them open to all teachers. Some teachers interested in HPM adopt the instructional designs of the developed lessons in their own teaching, which is called HPM Lesson Sharing (HPMLS). On the other hand, we use the developed HPM Lessons to teach Pre- and In-service mathematics teachers, which is called HPM Lessons-based Teaching (HPMLbT), as one way of spreading the conception of HPM. What is the procedure of HPMLD, HPMLS and HPMLbT? What are the effects of HPMLD?

In this workshop, we organized a LS according to Fig. 1. Firstly we provide text from a textbook and curriculum's requirements in China, the link with learned knowledge and knowledge learned later, and a historical resource about the studied topic. Participants in small groups design a lesson based on the above resource, the knowledge of the students they would teach, and the format of the design in China (Fig.2), under the guidance of the workshop organizers. Secondly each group presents its design. Then we play a video of an exemplary lesson, and participants watch and assess the lesson based on our assessment worksheet. Thirdly, we conduct a post-lesson debriefing, and participants revise their

design. Lastly, we use a case study to introduce teachers' professional development during the HPMLD and the procedure of HPMLS and HPMLBT, and construct a preliminary framework of HPM and mathematics teachers' professional development (shown as Fig.3).

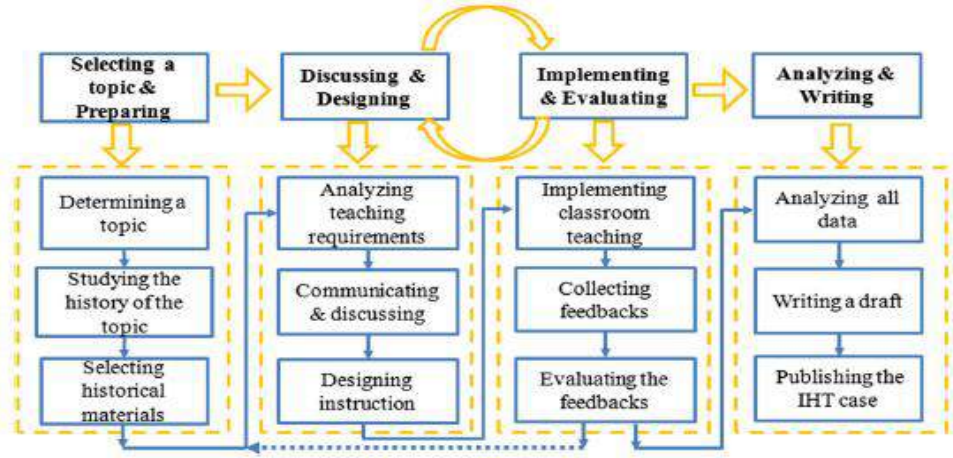
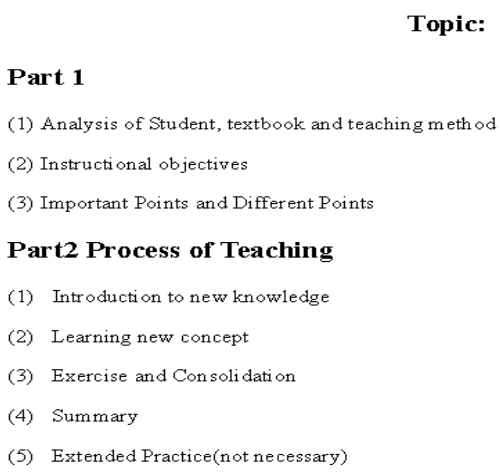


Figure1: The procedure of the HPMLD (Wang, Qi &Wang,2017)



Topic:

Part 1

(1) Analysis of Student, textbook and teaching method

(2) Instructional objectives

(3) Important Points and Different Points

Part2 Process of Teaching

(1) Introduction to new knowledge

(2) Learning new concept

(3) Exercise and Consolidation

(4) Summary

(5) Extended Practice(not necessary)

Figure 2

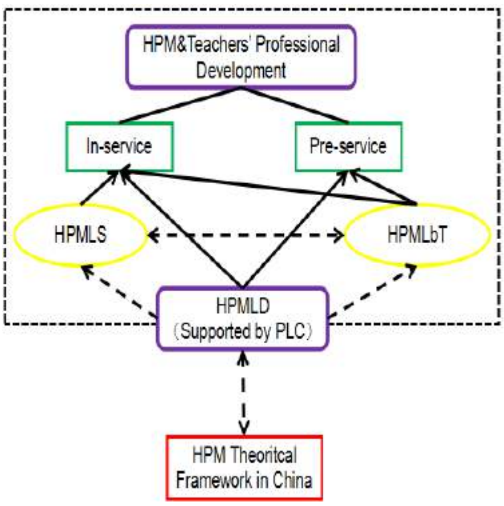


Figure 3

REFERENCES

Barbin, É., & Tzanakis, C. (2014). History of mathematics and education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 255–260). New York: Springer.

Wang, J. (2009). *Mathematics education in China: Tradition and reality*. Jiangshu, China: Jiangshu Education Press (in Chinese).

Wang, X. (2013). Teacher’s Professional Development Promoted by HPM——the Case of a Junior High School Mathematics Teacher in Shanghai. *Journal of mathematics education*, *22*(1), 18-21(in Chinese).

Wang, X., Qi, Ch., & Wang, K. (2017). A categorization model for educational values of the history of mathematics. *Science & Education*, *26*(7-9), 1029-1052.

Widjaja, W., Vale, C., Groves, S., & Doig, B. (2017). Teachers' professional growth through engagement with Lesson Study. *Journal of Mathematics Teacher Education*, *20*(4), 357-383.

Yue, Z., & Wang, X. (2017). Teacher’s professional development driven by HPM Lessons. *Journal of Schooling Studies*, *14*(2), 96-103+112 (in Chinese).

USING ANCIENT INSTRUMENTS IN THE TEACHING OF GEOMETRY WITH BACHELARD'S PHENOMENO-TECHNOLOGY

Évelyne BARBIN
Laboratory LMJL UMR 6629 & IREM, University of Nantes
Evelyne.barbin@wanadoo.fr

ABSTRACT

In this paper we explore Gaston Bachelard's notion of phenomeno-technique to explain how a geometrical instrument can be conceived as a "connaissance-en-action" (knowledge-in-action) in the construction of geometry. In particular, we analyze relations between a field of notions to teach and a field of problems to solve. So, firstly, we examine a hierarchy of instruments (Gerbert d'Aurillac, Jean Errard, Oronce Fine) that can be used to construct notions in an elementary geometrical teaching. They illustrate two kind of genesis of instruments defined by Pierre Rabardel as instrumentalization and instrumentation. Secondly, we analyze how the hierarchy of problems in *Dioptra* of Hero of Alexandria leads to a global notion of similitude in a geometrical space. As conclusion, we present what we call "an instrumental approach" in teaching of geometry.

1 Introduction: Gaston Bachelard's phenomeno-technology

Since around 25 years, the role of instruments of mathematics for teaching had been the subject of meetings and works in French IREMs (Hébert 1994, Johan 1996). More recently, several experiments using ancient instruments had been proposed in geometrical teaching, for students aged 11 until 14 years, in the same time of a revival of this teaching (Barbin 2014, Barbin et al. 2018). Often, these works propose to use an instrument only and the main didactical purpose is to give to students interesting applications of knowledge learned in classroom before. While in our purpose, linked to Bachelard's phenomeno-technology, instruments are sawn as knowledge-in-action and they correspond to notions or theorems that can be introduced and explored in teaching in the same time than the instruments are used.

Gaston Bachelard is a famous French philosopher of sciences, who wrote on physical sciences principally. His historical epistemology had been a basis of works in French IREMs (Institute for Research in Mathematical Education). He introduced the notion of phenomeno-technology in *Le nouvel esprit scientifique* (1934). He began to explain that scientific observation is neither a naked situation: "scientific observation is always polemical; it either confirms or denies a prior analysis, a pre-existing model, an observational protocol" (Bachelard 1984, p.12). Then he continued with experimentation (Bachelard 1984, p. 13):

> "And once the step is taken from observation to experimentation, the polemical character of knowledge stands out even more sharply. Now phenomena must be selected filtered, purified, shaped by instruments; indeed, it may well be the instruments that produce the phenomenon in the first place. And instruments are nothing but theories materialized. The phenomena they produce bear the stamp of theory throughout."

And he concluded:

“A truly scientific phenomenology is therefore a phenomeno-technology. Its purpose is to amplify what is revealed beyond appearance. It takes its instruction from construction.”

We will specially focus on this Bachelard’ conception: “instruments are nothing but theories materialized”. Precisely, we would like to show that instruments are “knowledge-in-action” (Barbin 2004, Barbin 2016).

For this purpose, we can quote two other authors: Gilbert Simondon is an important philosopher of techniques and Pierre Rabardel is a researcher on psychology and ergonomics. The first one wrote that “object that comes out of technical invention takes with it something from the human being who produced it […]; we could say that there is human nature into technical being” (Simondon 1969, p. 248). For the second one, “instrument is a means of capitalizing on accumulated experience (some authors say cristallized experience). In this way, any instrument is knowledge.” (Rabardel 1975, p. 73).

In this paper, we will examine phenomeno-technology about geometrical instruments only and its consequence for implanting an instrumental approach in teaching of geometry. We will begin with the role of instruments for construction of a geometrical world, then we will analyze links between genesis of instruments and construction of knowledge, and then we will examine the invention of a geometrical space with the dioptre of Hero of Alexandria. We will conclude to precise what we name “an instrumental approach in teaching of geometry”.

2 Instrument and invention of a geometrical world: the quadrant of Ionians

Historians of Greek mathematics explained that, in VIth century BC, Ionians used a quadrant to measure the distance of a boat in sea, which is an inaccessible distance by land surveying. A quadrant is made with a quarter of circle and a rod that turns and with which we can make sights (fig. 2.1).

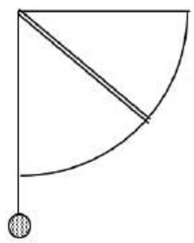


Figure 2.1: Quadrant of the Ionians

I let you imagine the real situation with sea, boat and us on the beach. Now, suppose that somebody asks me the question: how do you use the instrument? It will be more comprehensive to draw a layout. With this layout, I shall explain that I can climb on the top of a tower, make a first sight in direction of the boat, then return myself in direction of the ground and make a second sight (fig. 2.2).

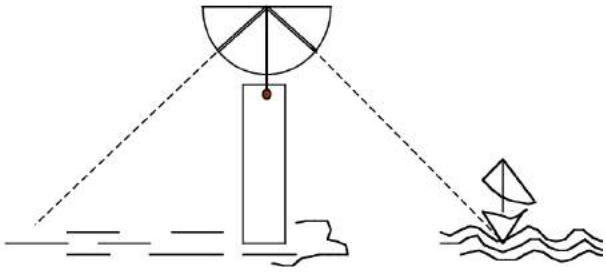


Figure 2.2: Distance of a boat: layout

Then suppose that somebody asks me: how do you find the distance of the boat? It will be better that we go together on the beach and that I draw a diagram on the sand to explain that “an equality of sights implies an equality of distances”, and it is a first rational discourse (fig. 2.3).

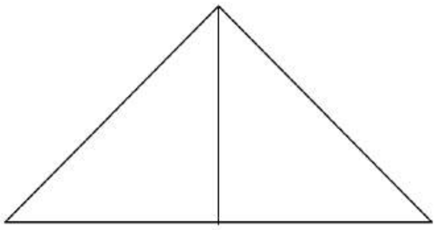


Figure 2.3: Distance of a boat: diagram

More generally, we can precise the different roles of layout and diagram. The layout permits a description of doings to answer to the question: what do you do with the instrument? While diagram is the basis of a first rational discourse to answer to the question: how do you find the result? Moreover, a diagram coordinates elements of a particular configuration, which can be activated, transformed or generalized by its re-cognition in various situations.

Now, suppose that somebody asks me: how do you know that your discourse is true? I shall add letters to the diagram to specify some of its parts called angles and lengths and I shall obtain a figure that permits a theoretical discourse on magnitudes, which is “if angles *BAD* and *DAC* are equal then lengths *BD* and *DC* are equal” (fig. 2.4). From a problem of inaccessible distance, Ionians invented a geometrical world made of figures and of relation between figures. The quadrant can be considered as “knowledge-in-action”, because it contains in itself a knowledge associating angles and distances.

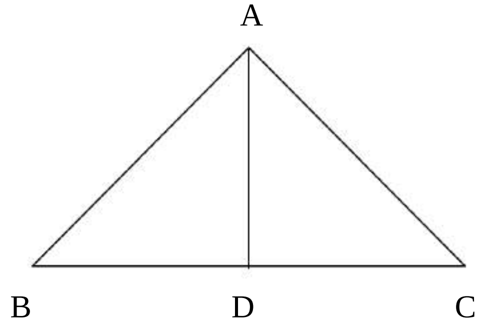


Figure 2.4: Distance of a boat: figure

3 Genesis of instruments and construction of knowledge: instrumentalization and instrumentation

In this second part, we will examine genesis of instruments; that is, birth or invention or creation of instruments, if you prefer. To understand relations between knowing subjects and instruments, we will go on by using history because, as Simondon wrote, “we have to grasp the historicity on how instruments become through how human being become” (Simondon 1969, pp. 107-109).

Rabardel distinguished two types of process in genesis of instruments (Rabardel 1995, p. 109). In the first type, there is an “enrichment” of an instrument (as artefact) by the subject without modification of the underlying diagram. He called it an “instrumentalization”. In the second type, there is a change of diagram by the human being with a modification of the instrument. He called it “instrumentation” (fig. 3.1). We are going to illustrate these two processes with ancient instruments taken in history.

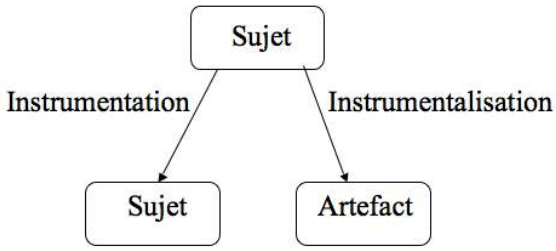


Figure 3.1: Instrumentation and instrumentalization according to Rabardel

3.1 Instrumentalization: from Gerbert’s stick to Errard’s instrument

The stick of Gerbert d’Aurillac is more a tool that an instrument, in the sense that it does not contains knowledge. Indeed, for our purpose, that is to analyze an instrument as a “connaissance-en-action” we have to distinguish an “instrument“ from a “tool” (Barbin 1994). The word “instrument” (in French linked with the verb “instruire”), is taking by us as an object whose conception integrates a knowledge. It is not the case for a simple stick. This distinction is useful in this paper later to compare the knowledge integrated in different instruments.

Let examine how Gerbert of Aurillac solved a problem of inaccessible distance, which is the width of a river, in his *Isagoge Geometriæ* (around 1000). Gerbert d’Aurillac wrote that a geometer has to have a stick with him always. How to do with the stick? Let see the layout and the figure (fig. 3.2). Similarity of triangles permits to write a proportion, in modern writing, the ratio *BD* : *CD* is equal to the ratio *BP* : *OP*. As *BP* is equal to the sum of the lengths *BD* and *DP*, we can calculate *BD* from accessible measures.

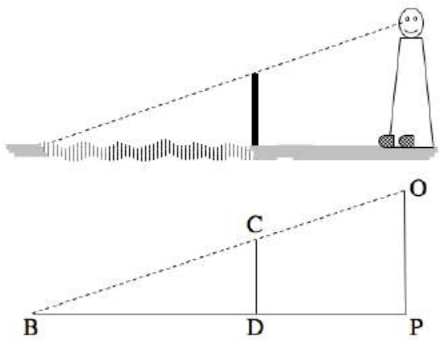


Figure 3.2: Width of a river with Gerbert’s stick

Now, to give an example of instrumentalization, we will go from the stick to Jean Errard's instrument, presented in his *La géométrie et pratique générale d'icelle* (2d ed., 1602). The instrument is more sophisticated, since it is composed of three rulers: *AB* is horizontal, *AC* can turn around *A* and *EF* can glide along *AB* (fig. 3.3).

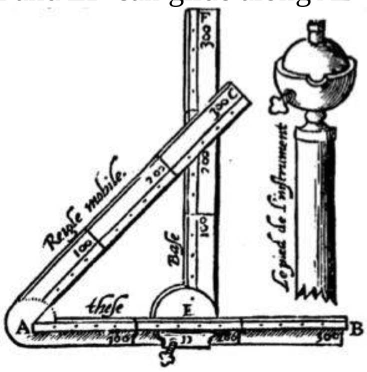


Figure 3.3: Errard's instrument (Errard 1602, p. 18)

The book contains two layouts corresponding to two problems of inaccessible distances: width of a river and height of a tower (fig. 3.4).

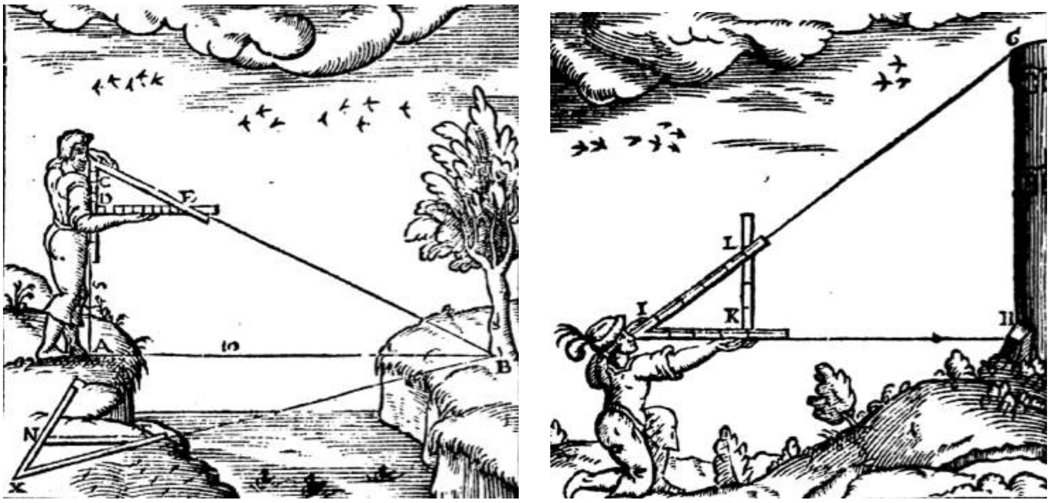


Figure 3.4: Two inaccessible distances with Errard's instrument (Errard 1602, pp. 21-22)

We have to remark that the underlying diagram is the same for these two problems, and also the same than for Gerbert's situation (fig. 3.5). But now, we can say that the diagram is incorporated into the instrument and that this instrument is a "knowledge-in-action" (Barbin 2004, Barbin 2016). The knowledge corresponds to proposition 4 of Euclid's Book VI (Euclid, pp. 200-202). We have an instrumentalization: this instrument is an enrichment of the stick that does not engage new diagram.

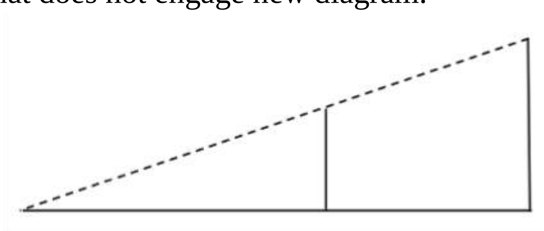


Figure 3.5: Diagram for inaccessible distances

3.2 Instrumentation: Gerbert's instrument and Oronce Fine's articulated set square

We go on with a process of instrumentation that engages new diagram. Gerbert d'Aurillac proposed an instrument more elaborated than a simple stick. A layout with letters permits to show how to use this instrument for measuring an inaccessible height of a tower (fig. 3.6).

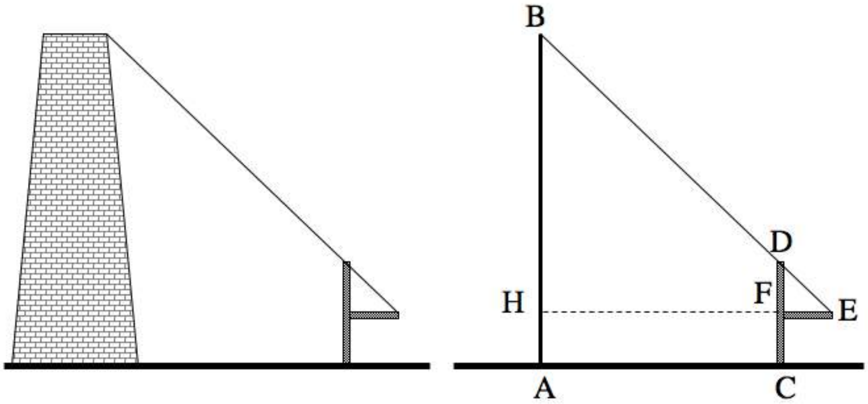


Figure 3.6: Height of a tower with Gerbert's instrument: layout and figure

The instrument is composed of two perpendicular sticks *FE* and *DC*, such that *DF*, *FE* and *FC* are equals. Here, we have a new diagram and we can explain on the figure how to determine the height. By equalities of angles, the triangle *BHE* is isosceles. So *AB* is equal to the sum of the lengths *HE* and *FC*. Now, the user doesn't need to calculate ratios to obtain the result. But the instrument incorporates a new knowledge, which concerns isosceles triangles.

Another example of instrumentation is given with the instrument of Oronce Fine presented in his *Protomathesis* (1532). It is composed of a stick and of a square set turning around the bottom of the stick. The layout shows how to use the instrument to measure the width of a river (fig. 3.7). The instrument is down on a bank of the river, and an alidade is aligned with the other bank.

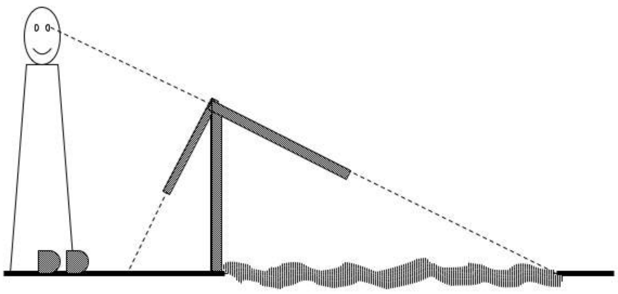


Figure 3.7: Width of a river with Oronce Fine's instrument: layout

How to obtain the width of the river? We have a new diagram and a figure on which we can state the theorem on the height of a rectangular triangle, that is theorem of the height of a rectangular triangle, in modern writing: $AH^2 = BH \times HC$ (fig. 3.8). If we take AH equal to 1 then HC is equal to $1/BH$. The distance is very easy to calculate, but the instrument incorporates a strong theorem, that is proven by Euclid two times, as a consequence of Pythagoras theorem in Book II, and as a consequence of the theorem on similar triangles in Book VI of his *Elements*.

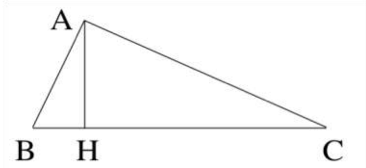


Figure 3.8: Weight of a river with Oronce Fine: figure and theorem

We can conclude with three comments on process of instrumentation concerning teaching. Firstly, in the two chosen examples of instrumentation, the inaccessible distance can be easily obtained from the accessible. Secondly, in each process of instrumentation, a new knowledge is incorporated into the instrument and the underlying diagram changes: isosceles triangle for Errard's instrument, theorem on height in a rectangular triangle for Oronce'Fine's instrument. Thirdly, we can write that more the instrument is "instructed" by theory, then less the user has to be instructed.

3.3 Instrumentalization and instrumentation: the place of the subject

The two processes has to intervene in teaching because they engage activities of the learner. Rabardel emphasized the two places of the subject in the two processes by writing:

> "These two types of processes are the fact of the subject […]. What distinguishes them is the orientation of this activity. In the process of instrumentation, the activity is turned towards the subject himself, while in the correlative process of instrumentalization, the activity is oriented towards the component artefact of the instrument" (Rabardel, 1995, pp. 111-112).

We can complete the scheme above by indicating the three instruments that can illustrate the two processes (fig. 3.9).

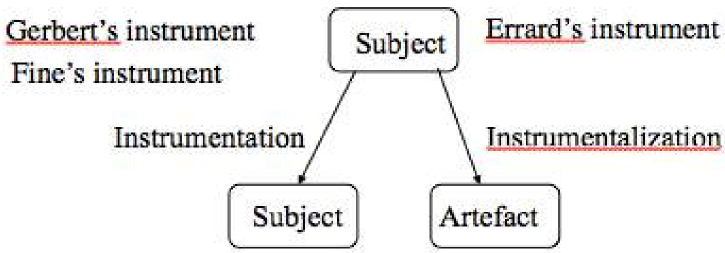


Figure 3.9: Instrumentation and instrumentalization accordingly with Rabardel

In a previous paper (Barbin, 2016), I remarked the difference between this scheme and the one given by Luc Trouche (fig. 3.10). This last scheme corresponds to what Trouche wrote: "Rabardel distinguishes, in the genesis of an instrument, two crossed processes, instrumentation and instrumentalization: the instrumentalization concerns the personalization of the artefact by the subject, the instrumentation concerns the apparition of schemes into the subject (that is to say the manner with which the artefact contributes to pre-structure the action of subject for carrying out the task in question)" (Trouche, 2015, p. 267).

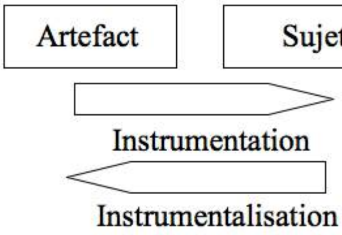


Figure 3.10: Instrumentation and instrumentalization with Trouche

In our epistemological and didactical purpose, we do not see the instrument as a way to pre-structure the action of the subject. But, like in Bachelard's phenomeno-technology, we rather consider that the instrument is structured by the subject's knowledge.

4 Instrument and invention of a geometrical space: on Hero of Alexandria's dioptre

In his *Dioptra* (IIIrd century), Hero of Alexandria considers one instrument only: the dioptre. It is not a complicated instrument. There is an upper part which can only turn around or to be inclined on a leg (fig. 4.1). Moreover, we can make sights through two holes of the upper part, like it is shown on the layout by the discontinued line. Moreover, technical means permit to obtain the horizontal position when it needs for the upper part and the vertical position for the leg of the instrument.

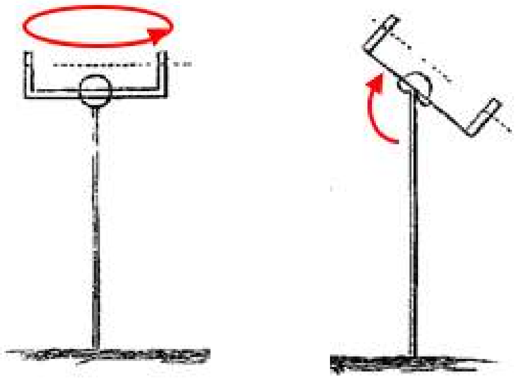


Figure 4.1: Hero of Alexandria's dioptre

Hero wrote: "in general, dioptre is used for measuring distances of any kind, when this operation only can be made by far" (Hero 1858, p. 177). Indeed, the book is composed by a series of problems of inaccessible distances ranged in a "deductive" order, in the sense that one problem is solved by using solving of previous problems. Reading Hero is very interesting because he explained steps by steps how to solve problems. Let read the second problem: "Problem 2. Two points *A* and *B* are given such that it is not possible to see one of them from the other: to join them by a straight line". Hero explained how to bypass the obstacle, which prevents to see one point from the other point, And, as for each problem, we have to make a diagram. Hero wrote: "while making these operations, we write them on a paper, that is we represent the layout, with indicating the summits of the broken line and the lengths of its several parts" (fig. 4.2). We have to put the dioptre in *A* and to mark *AG* on the ground, a straight line with an arbitrary length. Then we have to put the dioptre in *G* and to mark *GD* perpendicular to *AG*, with an arbitrary length. Then we have to put the dioptre in *D* and to mark *DE*, etc The points *G*, *D*, *E*, etc. are accessible points and they have been supposed to be in the same plane.

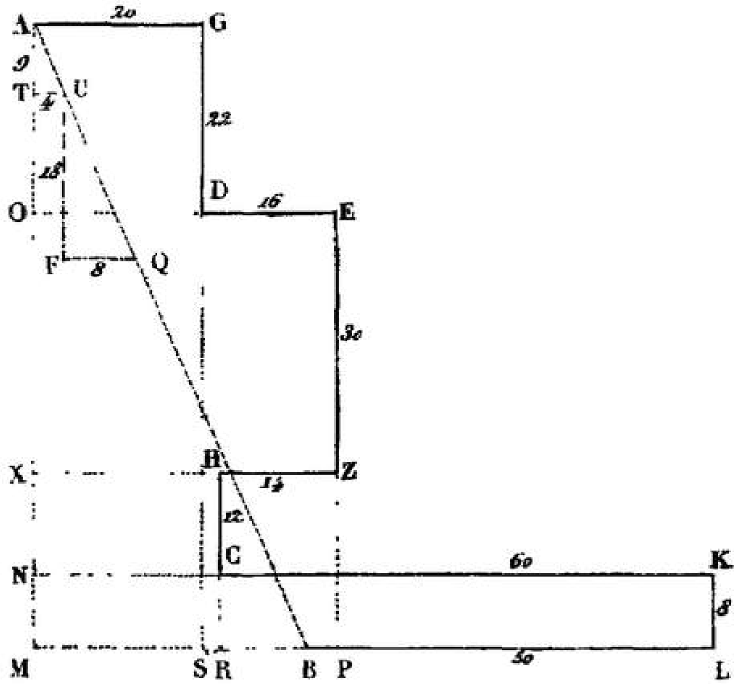


Figure 4.2: Problem of inaccessible distance in Hero: diagram and figure

After which, to obtain different points belonging to the straight line AB, we have to draw a figure with two steps. Firstly, we have to draw “real” lines that are on the ground with their measures. Secondly, like Hero wrote, we have to imagine and to draw other lines, the “imaginary” lines *AM*, *MB*, *AT*, etc., on which the reasoning will be explained. Their drawings are discontinued lines (fig. 4.2). We can calculate *AM* and *MB* from the “real lines” and their ratio. In Hero’s example, we obtain 72 and 32, and the ratio *AM*: *MB* is equal to 72 : 32. Now, let take (for example) *AT* equal to 9 on *AM* and *TU* perpendicular to *AT*. Hero explained that the ratio 72: 32 is equal to the ratio 9: *TU* and so *TU* equal 4 . In the same manner, we can obtain the point *U* of *AB*, etc. Hero wrote: “by observing the same ratio always”. That means there is the conception of a “global similarity” between several rectangular triangles, so, this similarity operates in a geometrical space. So, we can consider that dioptre is a simple instrument that needs an instructed user.

This “global similarity” is different of the notion of similarity in Euclid’s *Elements* (Barbin PUR). Indeed, Euclid’s geometry is a study of figures without introduction of space. In Book VI, Euclid only proved similarity between two geometrical forms of same type (two triangles, two rectangles, two polygons, etc). Hero did not state Euclid’s fundamental theorems on similar forms. But he represented figures accordingly to an implicit “scale”, where an explicit ratio operates on forms of a geometrical space by a “global similarity”.

Let consider two other figures corresponding to Hero’s problems: the problem of the tunnel of Samos (fig.4.3) and the problem of the depth of a well (fig. 4.4). To solve these problems, Hero introduced rectangular triangles that can be deduced thanks the calculation of one ratio only. These figures are linked by a “global similarity”.

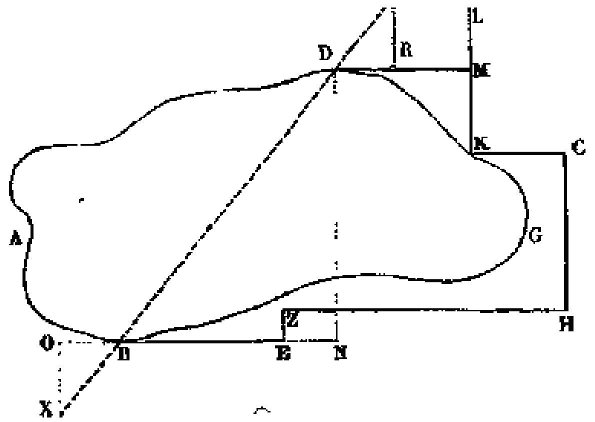


Figure 4.3: Problem of the Tunnel of Samos: diagram and figure

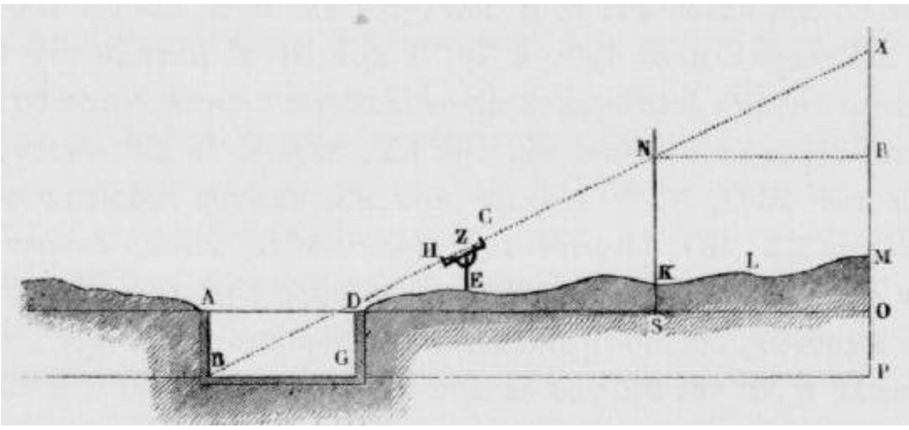


Figure 4.4: Problem of the depth of a well: layout, diagram and figure

5 An instrumental approach in teaching of geometry with phenomeno-technology

As we remarked above, several works on using instruments in geometrical teaching propose to use an instrument only to solve one problem only, and the main didactical purpose is to give to students interesting applications of knowledge learned in classroom before. Also, it is chosen instruments, different of those presented in this chapter, because these instruments permit to read measures of angles by numbers. Thus, these instruments lead to trigonometry rapidly (on instruments and angles see Chatelon & Troudet 2018, Mercier 2018, Guichard 2018).

While in our purpose, linked to phenomeno-technology, instruments are sawn as knowledge-in-action and they correspond to notions or theorems that can be introduced and explored in teaching in the same time than the instruments are used. They are simple, in the sense that they did not include a way to read angles. Here, an "instrumental approach" means a teaching where students use instruments, not in a disparate manner, but use a set of instruments following a cognitive and mathematical order that can permit to construct a geometrical knowledge. Also, following Simondon, it is a teaching where historical elements are integrated thanks to the use of ancient instruments (Barbin et al., 2018).

We can characterize an instrumental approach of teaching by three features. Firstly it is a teaching by problem-solving, where problems of inaccessible distances play a major role to introduce objects and theorems as tools to solve problems. Secondly, it is teaching beginning with spatial situations and going on with two explicit types of activities for learners. There are at first activities of drawings with scales representing situations on the ground (layouts and diagrams), and then activities of enouncing rational discourses on figures with letters, like in Hero's *Dioptra*. For this purpose, an instrument can be used to solve a hierarchy of problems, organized in a deductive manner. Thirdly, it is a teaching where a hierarchical field of instruments and an ordered field of knowledge are constructed in a mutual "enrichment" (Barbin, 2016).

We propose that learners themselves distinguish different steps in activities of problem-solving, corresponding to steps in learning (on activities on the ground and drawings, see Chatelon & Troudet 2014). It is interesting to differentiate the roles of drawing layouts, diagrams, figures with real and imaginary lines, because, in this instrumental approach, an instrument is considered as a knowledge-in-action corresponding to an underlying geometrical diagram. For this purpose, we stress on the mutual "enrichment" between constructions of new geometrical knowledge and of new instruments introduced in teaching. It is important to well conceive and to use the two possibilities in teaching: we can introduce the same knowledge to conceive several instruments and we also can conceive an instrument corresponding to a new geometrical knowledge.

REFERENCES

Bachelard, G. (1984). *The New Scientific Spirit*, trad. A . Goldhammer. Boston : Beacon Press.

Barbin, É. (1994). L'invention des théorèmes et des instruments. In É. Hébert (Ed.), *Instruments scientifiques à travers l'histoire* (pp. 7-12). Paris: Ellipses.

Barbin, É. (2004). L'outil technique comme théorème en acte. In *Ces instruments qui font la science* (pp. 26-28). Paris: Sciences et avenir.

Barbin, É. (Ed.) (2014). *Les constructions mathématiques avec des instruments et des gestes*. Paris: Ellipses.

Barbin É., Bénard D. & Moussard G. (Ed.) (2018). *Les mathématiques et le réel: expériences, instruments, investigations.* Rennes: PUR.

Barbin, É. (2016), L'instrument mathématique comme invention et comme connaissance-en-acte, *Menon*, *2*, 9-30.

Barbin, É. (2018). La Dioptre d'Héron d'Alexandrie : investigations pratiques et théoriques. In É. Barbin, D. Bénard & G. Moussard (Ed.). *Les mathématiques et le réel : expériences, instruments, investigations* (p. 33-48). Rennes: PUR.

Bénard, D. (2014). Agrandir, réduire, cartographier, mesurer l'inaccessible. In É. Barbin (Ed.). *Les constructions mathématiques avec des instruments et des gestes* (pp. 27-56). Paris: Ellipses.

Chatelon, D., & Troudet, M. (2014). Levé de plan au graphomètre dans l'enseignement : de la cour à la feuille de papier ou à l'écran d'ordinateur, *Repères IREM*, *96*, 35-51.

Chatelon, D., & Troudet, M. (2018). Expériences de géométrie pratique avec graphomètre en classe. In É. Barbin, D. Bénard & G. Moussard (Ed.). *Les mathématiques et le réel: expériences, instruments, investigations* (pp. 49-62). Rennes: PUR.

Errard, Jean (1602). *La géométrie et pratique générale d'icelle* (2d ed.). Paris: Auvray.

Euclid (1956). *The thirteen books of Euclid's Elements*, (Vol. II). Trans. T. L. Heath. New-York: Dover Publications.

Fine, O. (1532). *Protomathesis*. Paris: Impensis Gerard Morrhij et Ioannis Petri.

Guichard, Jean-Paul (2018), Angles in secondary school: surveing and navigation. In É. Barbin, et al, *Let history into the Mathematics Classroom*, (pp. 1-16). New-York: Springer.

Hébert, É. (Ed.) (1994). *Instruments scientifiques à travers l'histoire*. Paris: Ellipses.

Héron (1858). De la dioptre, trans. A. J. H. Vincent. In *Notices et extraits des manuscrits de la Bibliothèque impériale*, tome XIX (pp. 174-337). Paris: Imprimerie impériale.

Johan, Patrice (1996), Pratiquer en CM une géométrie de “terrain“ inspirée des méthodes de l'Antiquité et du Moyen-äge, *Repères IREM*, *23*, 31-42.

Mercier, J.-P. (2018). Mesureurs d'angles du XXe au XVIe siècle : recherches en filiation (2017). In É. Barbin, D. Bénard & G. Moussard (Ed.). *Les mathématiques et le réel: expériences, instruments, investigations*. Rennes: PUR

Rabardel, P. (1995). *Les hommes et les technologies: approche cognitive des instruments contemporains*. Paris: Armand Colin.

Simondon, G. (1969). *Du mode d'existence des objets techniques*. Paris: Aubier-Montaigne.

Trouche, Luc (2005). Des artefacts aux instruments, une approche pour guider et intégrer les usages des outils de calcul dans l'enseignement des mathématiques. In *Actes de l'université d'été de Saint-Flour*, 265-276.

USING ARTIFACTS AND DYNAMIC GEOMETRY SOFTWARE IN PRIMARY SCHOOL INSPIRED BY MONTESSORI METHOD

Roberto CAPONE[1,3], Maria Rosaria DEL SORBO[2,4], Virginia NINNI[1,5]

[1]University of Salerno, Fisciano (SA), Italy,
[2]IS "Leonardo da Vinci", Poggiomarino (NA), Italy,
[3]rcapone@unisa.it, [4]marodel@gmail.com, [5]vninni@gmail.com

ABSTRACT

Latest interactive and dynamic geometry software introduces new practices in Mathematics education, adding virtual 2D or 3D animated graphs to traditional artifacts and compass-and-straightedge constructions. This work presents the results of a teaching experiment in primary school, realized using Montessori method and setting math education through perceptual-sensory inputs.

Cheap material was exploited to introduce pupils to one of the most important deductive process: the geometric proof. An all brand-new artifact was realized in order to explain Pythagorean Theorem using a hydro-mechanical system. The proof was performed starting from practice, direct observation by real and virtual tools. Pupils used GeoGebra, a dynamic geometry freeware: they built shapes, determined areas and verified the equivalence. Pupils repeated the proof using crayons, producing pictures full of creativity. Finally, we show a comparison between new technological tools and traditional real tools. Teacher's role was to involve students in an educational project with real and virtual tools.

1 Introduction

Traditional mathematics education throughout the world, especially in the early historical phases, was based on a set of tools borrowed from the world of practical experience (Lave, 1988; Gravemeijer et al, 2013). More so nowadays, in the era of fastest and most pervasive technological advance, we can observe more and more complex relationships between mathematics as a purely conceptual discipline and mathematics as an applicative activity (Blum and Niss, 1991). This ambiguity emerges in expressions like "mathematical models", referred to physical models of mathematical concepts, e.g., reproductions of plane and solid geometric figures made of cardboard, wood, plastic and models of surfaces of higher order that make it possible to materialize abstract concepts of mathematics (Ahmed ed al. 2004; Bartolini Bussi, 1996; Hershkowitz et al., 1990).At the beginning of the last century this trend in mathematics education was well in tune with the development of active methods in education, supported in the same period by John Dewey (Dewey, 1938). Afterwards, the active involvement of students was supported within a laboratory setting, with hands-on approach and exploiting Information and Communications Technology (ICT). This experimental approach, where exploration plays a major role, seems appealing for students who quite often find the evidence offered by experiments much more compelling than a rigorous proof and are bored by the request to produce mathematical arguments. On one side, the experimental approach is suspected to obstacle the development of mathematical styles of reasoning. On the other side, the use of technology suggests many motivations to sustain the experimental approach, as seen in many experiences throughout history. Moreover, there are many researches producing formal results inspired by trialing, conjectures suggested by experiments, descriptions of algorithms, and software for mathematical exploration (Laborde et al. 2006; Capone et al.

2018).

Latest interactive and dynamic geometry software are nowadays introducing new practices in math's education, supporting traditional artifacts (Verillon and Rabardell, 1995) and compass-and-straightedge construction with virtual but very effective 2D or 3D graph, geometric shapes and animations (Arzarello et al., 2014). Nevertheless, as a result of recent researches in cognitive neurosciences (Kosslyn, 1991; Cheng and Newcombe, 2005), learning processes of mathematics and geometry happen in the same brain regions and circuits which people exploit for intuitions about space, time, approximate amount and number awareness. Language is ordinarily used while learning mathematics, but mathematical reasoning itself seems to happen in its own parts of the brain (Dehaene, 2011). On the other side, among the main features of Montessori method, there are respect for individuality of each child, hands-on experiences, skills in cooperative and manual work. Therefore, this method is one of the possible instances of Pedagogical Activism: each pupil, if properly taught, may become an active constructor of his own knowledge.

In our work, we show the results of a teaching experiment in a primary school, realized using Montessori method and setting math education through perceptual-sensory inputs, especially by touch sense and visual representations. Everyday and cheap material, e.g. colored frames, tiles, golden pearls, and many other classical tools, as "regoli", have been exploited to create geometric shapes and to introduce pupils to one of the most important deductive process: the geometric proof.

In order to achieve this goal, an artifact was realized in order to explain Pythagorean Theorem using a hydro-mechanical system, relying on a fluid contained in three squared boxes, two built on the two perpendicular of a right-angled triangle and the third built on the hypotenuse. In this way, the artifact gives a visible proof of the equivalence of the areas. Therefore, a backtrack path was treading: the theorem was not demonstrated as in traditional proofs but starting from the opposite point of view - practice, direct observation by real and virtual tools, then, only at the end, thorough formulation of the theorem.

In addition, a specific user friendly and intuitive dynamic geometry freeware, GeoGebra, was used, eliciting enthusiasm and curiosity in the pupils: they built the shapes, determined the areas and so it was very straightforward to verify the equivalence between the squares. Furthermore, they repeated the proof of the theorem using real graphics tools, producing pictures full of creativity. Finally, we present a short comparison between technologically innovative virtual tools and traditional real tools, such as ruler and compass. Teacher's task was to motivate the students, involving them in an education project that refers to an effective integration of both real and virtual tools.

2 Principles of Montessori's Method

Maria Montessori was an Italian physician and educator who lived in Italy at the turn of the 19th and 20th century. She was a great innovator: she founded a new pedagogical thinking about children with mental illness and disability, up to the formulation of a method suitable for all children. Montessori's approach is founded on original principles: teachers must start from "things", concrete representations of geometrical objects. But, above all, teachers are facilitators and they must let the things themselves speak to the students.

Essential Montessori education principles include teaching child-centered, early scholarization, pupil's freedom, centrality of environment, senses education through material objects, the school is considered as a small society and teachers play the role of guides.

According to the principles of Pedagogical Activism, the synthesis of the Montessori method is the importance attributed to direct experiences. There is a revaluation of the realistic exercise, concreteness, primacy of activity and learning by doing: practice and technology are media and not targets. There is respect for individuality, hands-on experiences and skills in cooperation are considered very relevant. Each pupil, if properly taught, may become an active constructor of his own knowledge.

3 The Pythagorean Theorem

As in precious Montessori's book "Psicogeometria" (Montessori, 1934/2012), the target of our experiment was to study the potential benefit of artifacts and dynamic geometry software to get to the concept of one of the best-known topics in elementary Geometry, the Pythagorean theorem.

3.1 Project outlines

Our experiment was set in a primary school, for fifth year's pupils and in the lower level of a secondary school. Teachers worked, at first, in homogeneous groups, with parallel classes, with the same objectives. Subsequently, teachers of primary and secondary school worked in heterogeneous groups, to try to build vertical curricula.

The organization of the vertical curriculum has stimulated innovations both on the methodological and managing level of the disciplines also to facilitate connections, relationships and awareness.

3.2 Lab activities

The terms of the so-called Pythagorean Theorem were already known by Babylonian a thousand years before Pythagoras, but the famous philosopher was the first to prove it. Over 371 different proofs of the Pythagorean theorem were found and proposed in the history of Mathematics, as collected in a book in 1927 (Loomis, 1927). Our project is aimed to develop a practical approach to this fundamental theorem to investigate the impact on learning of this methodology, inspired by Montessori's ideas: it is fundamental to teach mathematics first of all through perceptive-sensorial stimuli, especially through the hands, because the cerebral areas that allow us movements are very close to those that make us perceive the geometric shapes and the approximate quantities.

The explanation of the Pythagorean theorem usually starts with the statement, continues with the demonstration and then the applications follow. Montessori's concept, on the other hand, is different and very simple: we need to start from "things", that is, from the concrete representations of geometrical objects. Here is her quote: "Was it not from the things that the first surveyors drew their knowledge? Were not correspondences and relations between things, which stimulated some active and interested mind to formulate axioms and therefore theorems?" "The way a concept has been understood for the first time by human beings is the natural way to present that concept to children".

In the following subsections we will describe 3 lab activities, through which we conducted our experiment in Montessori's spirit.

3.2.1 The artifact

Theoretical reflections just described inspired our educational experimentation in primary school, in a fifth-grade class, aiming to study the potential benefits of artifacts, in particular a hydraulic device, for helping students to conceptualize Pythagorean Theorem through water containers properly shaped.

The artifact is realized starting from the construction of a box shaped as a right triangle, on whose sides squares were built; the square built on the hypotenuse has a hole on the outer side, to fill the device with a liquid. All the boxes are communicating with each other through pipes, so that the liquid can flow from the big square to the small ones at the same time and vice versa (figure 3.1). The entire device is built on a circular plane with a pin that allows the rotation of the artifact. The smallest perpendicular side measures 15 cm, the biggest one 20 cm and the hypotenuse 25 cm. Then, it is possible to show how all the liquid contained in the square box built on the hypotenuse can be contained exactly in the square boxes built on the perpendicular sides.

Figure 3.1: Hydraulic artifact to prove the Pythagorean Theorem

Artifact can help children to have a different view of mathematics, which is often distressing and negative; so, mathematics appears "colored" and touchable, improving their attitude towards mathematics and their mathematical skills. In this case all the channels are involved to receive the information: you learn exploiting visual memory, (therefore non-verbal visual channel), by listening (auditory channel), by reading (verbal channel) and by doing (kinesthetic channel). While using an artifact, a teacher observes how students use it and their cognitive patterns, as well as their special ways of thinking/knowing. Here is the importance of mental images in mathematics: mental images are not only passive figures inside the head, but productive mental representations that allow us to imagine something, even in the absence of perceptive stimuli and which therefore allow us to construct forms of creative thought in order to realize new forms of knowledge.

3.2.2 Crayons

After formalizing the theorem, children reproduced the construction in the maths notebook, using the squares as minimum units. This activity allowed them to have an immediate, visual and practical approach to the equivalence between the squares, as in figure 3.2.

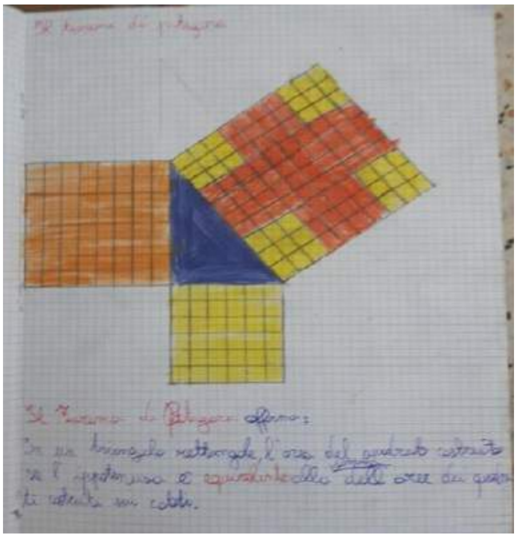


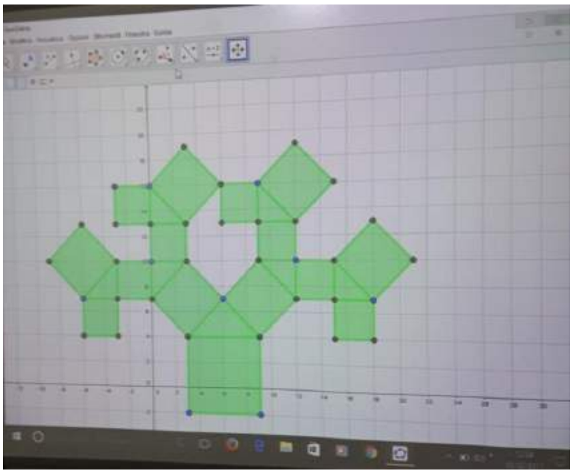
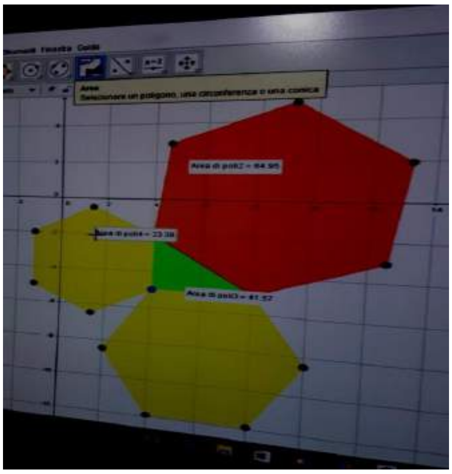
Figure 3.2: Construction of squares on the sides of a right triangle with crayons

The exercise was carried out with the construction of squares; we can certainlysay that there were no difficulties in the construction of the squares on the perpendicular sides, but it was different for the construction of the square on the hypotenuse. The students showed uncertainties about the correct position of the lines; in this sense, there was the necessity to talk about technical expertise that they will have to acquire in the following of their learning process. Thus, once the sketch was drawn and colored, the children counted the unit squares and verified the Pythagorean theorem. Moreover, this activity further simplified and facilitated the understanding of the concept of area as the measure of a surface.

3.2.3 Geogebra

Pupils, after few and short preliminary lessons inspired by practical learning by doing education methodology, got mastery of basic features and functions of GeoGebra (Zenging, 2018) so that they could begin to draw geometric shapes and right triangles and squares. Subsequently, the theorem was depicted, calculating the areas of the surfaces of the squares by GeoGebra's specific command and directly proving the evidence. In addition, they spontaneously tried to extend the Pythagorean theorem by iterating the construction indefinitely, giving life to creative images and to apply Pythagoras's theorem also to polygons with 3, 5 or more sides, as shown in figures 3.3 and 3.4.

Nevertheless, it was very interesting to let pupils experiment the difference between GeoGebra and compass-and-straightedge construction of geometric figures and besides, to exploit GeoGebra's toolbox and functions for step by step constructions, which melts old- and new-fashioned geometry learning methods.

Figures 3.3 and 3.4: Example of development of ideas in Geogebra activity

3.3 Rethink and discovery

After GeoGebra experience, we draw many considerations: for example, we can assert that we detected an increasing students' attention about the possibilities of exploiting this software to design, analyze, calculate, proof. We tried to summarize and list our observations, classifying them in pros and cons.

3.3.1 Pros of Virtual

We registered a significant reduction of teaching time, because, first, pupils are very much attracted by computer's features and quickly and intuitively become used to employ active software graphical instruments. This educational process helped students to achieve a better awareness of the demonstration in much shorter time and an improvement of learning effort perception.

Thus, there was also an optimal integration between class teaching and preparation of lessons and materials: teacher essentially played the role of a guide, but pupils impressed their personal fingerprint in the development of the lessons and the results were finally original and sometimes unexpected.

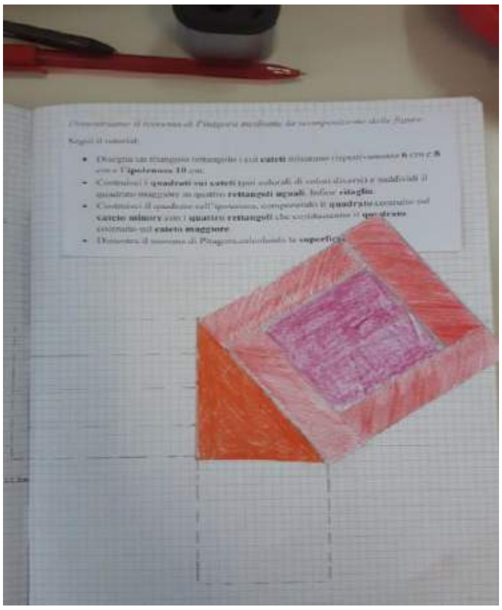
Lastly, GeoGebra's dynamical geometry tools, with its easy way to draw lines and shapes, which does not require manual ability to get perfect graphical outcomes, helped students to directly concentrate on the process and then they had the opportunity to better understand the deep meaning of geometrical constructions and relationships.

3.3.2 Cons of Virtual

Even though the overall outcomes of virtual applications were positive, as described above, students' trend is towards action without pre-thinking: it does not happen rarely that pupils fail to understand the reason for some results, typically those that are an exception to the rules, while they tend to accept uncritically the output produced by the app they are using. "Even if confidence and motivation could be enhanced by using software, the question still remains if this translates into better overall performance in the classroom. This may be a question as to how the software is implemented as to how well it enhances or deteriorates a student's learning." (Formaneck, 2013). Generally speaking, software use could generate a superficial attitude in the learner, whose confidence with virtual tools risks to make him to surrogate the necessity of learning with the ease of quickly getting a result, without minimally submitting it to an aware checking process.

3.4 Evaluation

In order to test the effectiveness of our teaching activities, we administered to pupils several worksheets, basically centered on two different strategic approaches: equidecomposability and application of inverse formula, as respectively shown in figures 3.5 and 3.6.



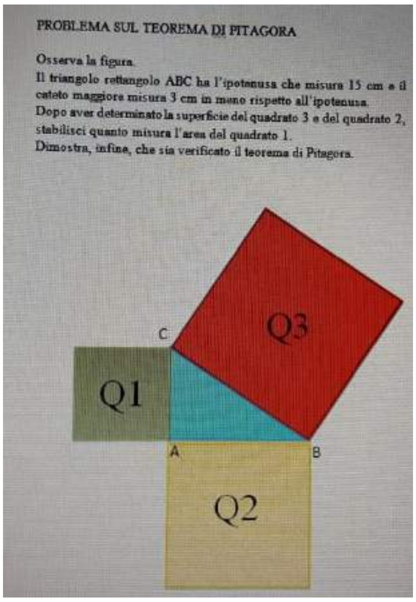


Figure 3.5 and 3.6: two examples of worksheet administered to test pupils after activities about Pythagorean theorem.

The text in figure 3.5 is: Let's demonstrate Pythagorean theorem using decomposition. Follow the tutorial: Draw a right triangle of dimensions 6, 8, 10 cm. Build the squares on the perpendicular sides and colour them with two different colours and then divide the greater sqaure in four equal rectangles. Finally cut these shapes. Build the square on the hypotenuse, composing the shapes just cut. Calculate the areas as a proof of Pytagorean theorem.

The text in figure 3.6: Problem on Pythagorean theorem. Observe the figure. The hypotenuse in a right triangle ABC is 15 cm and the greater perpendicular side is 3 cm less than hypotenuse. After have calculated the area of Q3 and Q2, calculate the area of Q1. Finally, confirm Pythagorean theorem.

The first worksheet aims to prove the Pythagorean theorem by decomposing figures. The worksheet allows the student to reflect on the statement of the theorem, to demonstrate it practically by decomposing the squares built on the perpendicular sides in two different ways, and then constructing the square on the hypotenuse using the pieces obtained cutting the smaller squares on the perpendicular sides. Despite all previous multiple activities, yet some pupils hesitated about how to combine the four rectangles and the small square (see figure 3.5). The second worksheets a more traditional geometry problem with an illustration and then it also *required reading, reasoning and calculation skills, as well as logic and the capability to find an inverse formula (figure 3.6). Often pupils read the text and the questions too quickly and with a low attention level; this is often the origin of multiple errors. However, in our strategy, pupils were pulled to analyze the text and also reasoned about the inverse formulas and the triangles with sides multiple of 3, 4 and 5 were used only for simplicity's sake, to facilitate the task of decomposing and reasoning at the beginning. Later pupils were made aware of the general case, with right triangle with sides measures different from Pythagorean triples, more difficult to draw but equally compliant the theorem.*

The evaluation table, depicted in figure 3.7, is related to the activities carried out in the learning unit and it was directly conceived according to new teaching guidelines, oriented towards an accurate evaluation of the competences achieved by the pupils. As in figure 3.7, there are descriptors that explain the different levels of competence achieved in the different tasks.

Dimensions	**Startinglevel**	**Basic level**	**Intermediate level**	**Expert level**
To know how to calculate area of the main polygons	Students don't know formulas and cannot calculate areas	Students know formula but they don't know how to apply it	Students can calculate areas of polygons	Students can calculate areas even in different situations
To know how to apply Pytagorean theorem	Students cannot apply the theorem	Students apply the theorem only if they are assisted	Students can apply the theorem autonomously	Students can apply theorem even using inverse formulas
To know how to use geometrical software	Students cannot use geometrical software	Students can use geometrical software with the help of a tutorial	Students can use the software with mastery	Students can use the software also to experiment in new contexts

Figure 3.7: Evaluation table

4 Conclusions and future work

At the end of this experiment, we stress our final observations about multiple ways to represent and prove the Pythagorean theorem using artifacts and dynamic geometry software. It was possible, first, to realize a peer education experience, with much enthusiasm and curiosity in learners. The pupils ran across arithmetic and geometry paths at the same time, enriching their knowledge and skills through a variety of activities and methodologies.

The artifact allowed them to get an immediate understanding of the theorem, thanks to the visual and original aspect given by the presence of the water. Alongside the purely practical activities, experiments have been carried out with the GeoGebra software to develop technological skill as in National Guidelines.

Students' motivation led to new discoveries, as with GeoGebra it was shown that the theorem is valid for all regular polygons and in a iterative application; specifically, this work has come across the demonstration of the validity of the theorem also with equilateral triangles and hexagons. Besides, simulations technologies were considered with new awareness, moving forward to more critical and conscious use of virtual tools. Test worksheets were indeed a way to compare all the activities done previously and draw indications about the effectiveness of teaching and learning.

Though, the time available did not allow us to investigate other important aspects, such as a study of a wider range of topics and the evolution in time and the impact on long term memorization of the concepts learned by this approach; in this regard, we hope that this work can be a starting point for future reflections and studies.

From the results achieved up to now, we had the opportunity to verify how the proposed method led to a concrete and more aware learning, to be regarded as acquisition of skills by primary and secondary students, along with the construction of new knowledge through repeated real and virtual lab experiences.

REFERENCES

Ahmed, A., Clark-Jeavons, A., & Oldknow, A. (2004). How can teaching aids improve the quality of mathematics education. *Educational Studies in Mathematics*, *56*(2-3), 313-328.

Arzarello, F., Bairral, M. A., & Danè, C. (2014). Moving from dragging to touchscreen: geometrical learning with geometric dynamic software. *Teaching Mathematics and Its Applications: International Journal of the IMA*, *33*(1), 39-51.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to other subjects—State, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*(1), 37-68.

Bussi, M. G. B. (1996). Mathematical discussion and perspective drawing in primary school. *Educational studies in mathematics*, *31*(1-2), 11-41.

Capone, R., Del Regno, F., & Tortoriello, F. (2018). E-Teaching in Mathematics Education: The Teacher's Role in Online Discussion. *Journal of e-Learning and Knowledge Society*, *14*(3).

Cheng, K., & Newcombe, N. S. (2005). Is there a geometric module for spatial orientation? Squaring theory and evidence. *Psychonomic bulletin & review*, *12*(1), 1-23.

Construction (Measurement and Geometry: Module 13) For teachers of Primary and Secondary Mathematics - http://www.amsi.org.au/teacher_modules/pdfs/Construction.pdf

Dehaene, S. (2011). *The number sense: How the mind creates mathematics*. New York: Oxford University Press.

Dewey J.,(1938). *Experience and Education*, Kappa Delta Pi.

Formaneck, S.D. (2013) Math Software in the Classroom: Pros, Cons and Tips for Implementation, *International Journal of Pedagogical Innovations*, *209*(1178), 1-4.

Geogebra https://www.geogebra.org/

Gravemeijer, K. P., Lehrer, R., van Oers, H. J., & Verschaffel, L. (Eds.) (2013). *Symbolizing, modeling and tool use in mathematics education*. Vol. 30. Springer Science & Business Media.

Hershkowitz R., Ben-Chaim, D., Lappan, C., Mithelmore, M., & Vinner, S. (1990). Psychological aspects of learning geometry. In P. Nesher and J. Kilpatrick (Eds.), *Mathematics and Cognition* (pp. 70–95). Cambridge: Cambridge University Press.

Kosslyn, S. M. (1991). A cognitive neuroscience of visual cognition: Further developments. In *Advances in psychology* (Vol. 80, pp. 351-381). Amsterdam:North-Holland.

Laborde, C., Kynigos, C., Hollebrands, K., & Strässer, R. (2006). Teaching and learning geometry with technology. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future*, 275-304. Rotterdam: Brill Sense

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge: Cambridge University Press.

Montessori, M. (1934/2012). *Psicogeometria.* Roma: Edizioni Opera Nazionale Montessori.

Loomis E. S. (1927). *The Pythagorean Proposition*. Washington DC: The National Council of Teachers of

Mathematics.
Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education*, *10*(1), 77-10.
Zengin, Y. (2018). Incorporating the dynamic mathematics software GeoGebra into a history of mathematics course. *International Journal of Mathematical Education in Science and Technology*, *49*(7), 1083-1098.

BEING IN RESEARCH AND DOING RESEARCH ON HISTORY AND MATHEMATICS EDUCATION IN A DIALOGICAL PERSPECTIVE

David GUILLEMETTE
Université du Québec à Montréal, Département de mathématiques, C. P 8888, succursale Centre-Ville, Montréal (Québec), H3C 3P8, Canada
guillemette.david@uqam.ca

ABSTRACT

At the beginning of the 20th century, Mikhail Bakhtin developed a critical mode of literary analysis based on the dialogic principle. This dialogical principle emphasizes that each utterance of a discourse is necessarily "in dialogue" in a given sphere of speech communication. For Bakhtin, these utterances are following one another according to laws of appreciative convergence in a close dependence on historical conditions. Bakhtin's philosophy has inspired few researchers to establish or develop theoretical foundations in mathematics education, but also, closer to us, in history and mathematics education.In this paper, we will try to show how Bakhtin's thought can support reflection, not on history, mathematics or mathematics education *per se,* as it has often been discussed, but on our ways of being in research and doing research within or field.

1 Introduction

This paper is a development of the oral communication that took place during ESU-8. The objective, here, is to introduce some elements of the philosophy of Mikhail Bakhtin (1895-1975)in order to bring new ways of thinking about being and doing research within the field of research on history of mathematics in mathematics education.

As discussed by Fried, Guillemette and Jahnke (2016), the field has moved, for the past decade, toward a reflection on didactical and pedagogical foundations concerning the role of history of mathematics in teaching and learning, as well as the development of theoretical and conceptual frameworks. They argue that theoretical or conceptual element that could support research concerning history of mathematics in mathematics education should at least address the question of why we ought to learn history or historical elements related to mathematics? What part of pupils and students intellectual lives is touched by history? The historical nature of mathematics must at least be put on the table, as something to question. In other words, the nature of mathematics itself must be problematized. Our own view of mathematics of the past should also be problematized, asking what it means to stand facing the past? Our own posture towards the past should be explored.

In this quest, this paper tries to draw attention on the dialogical principle that has been developed by Bakhtin in order to bring theoretical and conceptual elements to support teachers' and researchers' reflection. We would like to argue, in this paper, that elements of Bakhtin's philosophy could support reflection, not only upon history, mathematics or mathematics education *per se,* as it has sometimes been discussed, but also upon our ways of being in research and doing research within or field. The main argument that will be discussed is that this perspective could help particularly the researchers by making possible and visible the dialogical interactions between researcher, participants and history of mathematics, but also the teachers trying to deepen their comprehension ofthe relation between their interventions, the learners, the mathematics and the history of mathematics in their classrooms.

2 A Bakhtinian perspective

In this section, we will try to introduce key elements of the Bakhtinian philosophy (Bakhtin 1929/1977, 1979/1982, 1978/1997, 1963/1998). This quick introduction will focus on two major elements: the *dialogical principle* and the concept of *polyphony*.

The philosophical works of Mikhail Bakhtin, born in Russia, can be included in the dialectical tradition inaugurated by Hegel, developed by Marx and continued among Russian thinkers such as Ilienkov, Mikhailov and Vygotsky. At the beginning of the 20th century, after the Russian revolution, Marx's philosophy and the development of phenomenology in Europe have motivated and influenced a group of thinkers called *The Bakhtin Circle* constituted for instance by the linguist Valentin Voloshinov (1895-1936), the literary scholar Pavel Medvedev (1891-1938) and the philosopher Matvei Kagan (1889-1937). The circle was animated by profound insights around literature, linguistic, psychology and philosophy, at a time where Marx was discussed in its very philosophical potential. It was a very productive Marxist intellectual group. Unfortunately, the group didn't survive to the Stalinian purges.

2.1 The dialogical principle

From his analyses of Freud, de Saussure and linguistic theories conducted within the group, Bakhtin has developed a critical mode of analysis based on what he called the dialogical principle. He then applied it in different areas such as literary analysis and more generally to the analysis of ideology (see Sabo and Nielsen, 1984). This said, this reflection goes far beyond literary criticism, to the point of disrupting the foundations of the human sciences.

Succinctly, the dialogical principle emphasizes that each utterance of a speech is necessarily an answer to another utterance in a given sphere of speech communication. This concept of sphere of speech communication is important because it determines the very condition of the dialogue in which the speech inscribes itself inherently. Indeed, for Bakhtin, at every moment of the dialogical analysis, one must understand and utterance as an answer of another utterance, but, ate the same, time as a condition to other utterance to emerge. In other words, Bakhtin would say that any movement of consciousness is itself dialogical, penetrated by and in dialogues with other movements of consciousness, and thus, cannot be approached without consideration for other movements of consciousness to which it answers, and which it allows as an answer.

A speech is then perceived here as a dialogue, but this dialogue is not understood as a simple sequence of statements constituting a form of exchange or conversation. These aspects would be only the superficial manifestation of dialogism which "goes far beyond the relations between the replicas of a formally constructed dialogue, because it is almost universal and crosses the whole human discourse [...] in a general way, all that has meaning and value" (Bakhtin, 1963/1998, p. 77, our translation).

Indeed, for Bakhtin, there is an ongoing dialogue both at the level of language and at the level of ideas, and these levels are intimately linked. He emphasises that our inner or outer speeches are constituted by whole monologues, analogous to paragraphs, or by whole utterances. But these monologues and dialogue hardly lends itself to an analysis of grammatical constituents. In this sense, Bakhtin would say that: “These units of speech, which could be called the global impression of enunciation, are related to each other and succeed one another not according to the rules of logic or grammar, but according to laws of appreciative convergence, of dialogical concatenation, and in a close dependence on the historical conditions of the social situation and the whole pragmatic course of existence. There is between the *psyche* and ideology an indissoluble dialectical interaction: the *psyche* dismantle itself, destroy itself to become ideology, and reciprocally” (*id*., pp. 63-64, our translation).

But what are those “laws of appreciative convergence” and “of dialogical concatenation”? What is happening concretely? For Bakhtin, “when the listener perceives and understands the meaning of speech, he simultaneously takes an active-responsive attitude toward it. Either he agrees or he disagrees with it (completely or partially), augments it, applies it, prepares for its execution, and so on. And the listener adopts this responsive attitude for the entire duration of the process of listening and understanding, from the very beginning - sometimes literally from the speaker’s first word” (1986, p. 68). On the other side, the speaker is also expecting such an active-responsive relation. Bakhtin add that “[The speaker] does not expect passive understanding that, so to speak, only duplicates his or her own idea in someone else’s mind. Rather, the speaker talks with an expectation of a response, agreement, sympathy, objection, execution, and so forth... Therefore, each kind of utterance is filled with various kinds of responsive-reactions to other utterances of the given sphere of speech” (*id*., p. 91). This attitude of the speaker and the listener as active-responsive communication lead inherently to the observation of different speech genres regarding the given sphere of speech communication. Indeed, the speaker will adapt is speech and perform a certain style when situated himself in relation with the listener, in the ways he perceived the listener, in a close relation to history, ideology, social determinations, but also to direct relations such as subordinated, fraternal or enmity relations for instance.

To sum up, Bakhtin suggests understanding the utterance as something produced within the dialectical relation between the self and the ideology. In other words, he simply means that the words that I used are necessarily the words of someone else. This simple observation will lead to deep insights regarding human sciences, making the possibility to bring new ways of thinking human subjectivity and human cultural production.

Indeed, for Bakthin, the dualism between the world of objects and the I who thinks it, isolated atom without history, immutability in prey to the laws of the phenomenon, is everywhere present in the philosophy of his time. In contrast, Bakhtin (1986/2003) argues that the impoverishing dualism of Cartesian rationalism must be fought by repudiating the abstractions of idealist philosophy, in order to better grasp the nature of the concrete action or “act” constituting the “value-center” of human existence. The ego is here a dynamic, corporeal, creative and moving entity. Bakhtin strives to formulate a phenomenology of “the practical act”, a phenomenology that focuses on our activities as bodily beings in a world that pre-exists abstract constructions.

As we will see history of mathematics and mathematics education do not escape from the scope of this dialogical principle.

2.2 The concept of polyphony

Taking through the lens of this dialogical principle, a scientific, literary or philosophical work is called "polyphonic" when, setting the scene in a given sphere of communication; it offers a large plurality of discourses and understandings of the world.

For Bakhtin, Dostoyevsky's novel *The Brothers Karamazov* would be the archetype of the polyphonic work. Indeed, the author depicts many characters inhabited by singular personalities who have finely established roles (the bourgeois, the liberal, the scientist, the atheist, etc.). These characters act as "spokespersons of worldviews" (Sabo and Nielsen, 1984, p. 80), and Dostoevsky strives to make them talk together. The confrontation of these individuals, endowed with a strong ipseity, highlights the existential, ideological and socio-historical texture of that time.

For Bakhtin, the polyphonic aspect of the novel of Dostoyevsky is the more objective and effective way to describe the reality of the author, in this case, Russia after the reforms of 1860. In the dialogical perspective, the work of the author is to perceive "great ideas" and "representations of men who speak of their ideological universe" (Bakhtin, 1978/1997, p. 182). The concept of polyphony is central to the philosophy of Bakhtin. Indeed, for Bakhtin, it is the meeting of discourses, "acts" and social horizons that allows us to say something about reality. It is in the illustration of tensions and reconciliations between different positions that brings clarifications.

2.3 A critical standpoint

By promoting the polyphonic aspect of his work, the author ensures that "reality loses its statism and naturalism [...] the future begins to penetrate in the form of trends, possibilities, anticipations" (Bakhtin, 1970/1982, p. 129, our translation). Such cultural production, for Bakhtin, "has essential views on freedom, overcomes determinism and strict mechanisms" (*ibid*.). Indeed, for Bakhtin and is collaborators, there is an important link between human struggle for social justice and more broadly with the struggle for meaning that occurs within all sectors of human cultural production, such as sciences, literature, art and philosophy.

3 Bakhtin and mathematics education

Bakhtin's philosophy has often been summoned in human sciences. For instance, it has inspired many researchers in mathematics education to support theoretical and empirical research. We will present in this section some examples (of course, we will refer to work to whom we are familiar and will inevitably omit number of cases), and we hope that these examples could help to understand more precisely the Bakhtinian perspective and how it could be convoked in research.

In the emergent historico-cultural perspective on mathematics education (Radford, 2011, 2018; Roth and Radford, 2011), element of the Bakhtinian philosophy are discussed in order to think about the subjectivity of the learners and the very concept of sociability within the classroom. Inspired by Vygotsky, this point of view pleads for a non-mentalist conception of the mind. Opposed to rationalism and idealism, it proposes a sensitive and historical conception. On the one hand, it is sensitive, rooted in the body, senses and affect. Body, perception, gestures and signs are considered as constituent parts of the mind. On the other hand, it is historical, rooted in culture, history and language. We shall then speak about the mind as a *praxis cogitans* (Radford, 2011).

Regarding mathematical objects, this perspective suggests that objects are "historically generated during the mathematical activity of individuals" and constitute "fixed patterns of reflective activities anchored in the changing world of social practice" (*id*., p. 7, free translation). In other words, individual activities constitute the genetic root of the abstract object, which contains varied expressive dimensions; rational, aesthetic and functional aspects related to culture.

From this perspective, learning cannot be understood as merely a personal process of knowledge construction or reconstruction. Rather, learning results from our contact with our environment's cultural artefacts and social interactions. It is "the perceived which comes to light in the intention, which expresses itself in the sign or in the action mediated by the artefact during the sensorial practical activity […] something likely to be converted in a reproducible action, which meaning aims at this cultural eidetic pattern which is the abstract object itself" (*ibid*.).

Thus, learning mathematics, as it could be called cultural objectivation, is not simply learning "to do" mathematics (even less to solve mathematical problems), but rather "to be-in-mathematics", the mathematical activity being nothing else than a way "to be-with-others" (Radford, 2012). This is where the important ethic thematic of the theory settles down. As such, Radford (2008, 2012), through Bakhtin (and Levinas), insists on the fact that subjectivity "begins" in the ethical relation to the Other, "is" as a responsibility to the Other. Ethic here is not taken in as "satellite" elements of human existence; it is rather the central and the determinant field of reflections. As Bakhtin would say, "extracted from the interactive context which puts in relation the I, the Other, and the World, the subject succumbs to solipsism. At this moment, the subject loses its grip, becomes empty, arrogant, degenerates and dies" (1978/1997, p. 40).

From this perspective, history of mathematics in the context of mathematics education can take special meanings (see Guillemette (2015) for more elements of reflection about that). Indeed, history of mathematics offers meeting opportunities with ways to do and to be radically different in mathematics. Articulating the problem of learning with the question of Alterity, History could bring particular experiences of otherness. Thus, attention is not on an individual with personal possibilities of emancipation, but rather on the possibility for learners to discover new ways of being-in-mathematics, to open, with others, the realm of possibilities in mathematics.

Sfard's sociocultural approach in mathematics education and the concept of "commognition" (2001, 2008) also convoke elements of Bakhtin's philosophy. The idea here is to define "communication" which will become the central core of this perspective. For instance, Sfard and Keiren (2001) argue, with references to Bakhtin, that communication takes place between people, but it can also be an interaction between a person and herself, more often than not our thoughts take the form of an inner dialogue. This perspective claim, drawing on Bakhtin that communication is a process in which "any particular action always means addressing somebody or reacting to somebody's former utterances, or both" (*id*., p. 58).

The authors speak about reactive and proactive (response-inviting) utterances, thus distinguishing between the two types of speaker's meta-discursive intentions: the wish to react to a previous speech or the wish to evoke a response in another interlocutor.

Therefore, when performing, for instance, data analyses, the idea is to perceive consecutive utterances in a discourse as endowed with invisible arrows that relate them to other utterances—those which have already been pronounced and those which are yet to come. As Sfard and Keiren explain, “these arrows are our metaphor for a speaker’s meta-level intentions, communicated indirectly. By addressing her partners, the speaker lets them understand that she is interested in an interaction. The organization of these invisible arrows in a conversation often reveals certain regularities […] interaction analysis is performed with the help of a diagram in which the imaginary arrows […] are made visible” (*ibid*.).

In order to think about the process of learning mathematics, this discursive approach claim that there is a dialogical aspect of the mind, and that, at both individual and society levels, thinking, like a conversation between two people, “involves turn-taking, asking questions and giving answers, and building each new utterance—whether audible or silent, whether in words or in other symbols—on previous ones in such a manner that all are interconnected in an essential way” (Sfard, 2000, p. 299). These are the basic arguments deployed here to understand “thinking as communicating” (Sfard, 2008).

A third example can be found with the work of Barwell (2014, 2016) who has convoked a Bakhtinian perspective when analysing multilinguistic classrooms in mathematics. Attention to the dialogical interactions within the classroom has permitted to observe different speeches genre when teachers engage learners in mathematical activities. For instance, Barwell show importance of what he calls formal and informal speeches within this context, and how teachers answer (or sometimes didn’t notice) to informal demands perceived from a dialogical perspective. These informal demands have, particularly here within the context multilinguistic classroom, important consequence concerning the pedagogical intervention of the teachers and classroom interactions.

A last example is the work of Gerovsky (2010, 2012) who tackles gender issues and problems related to social justice in mathematics education. Thinking with Bakthin about speech genre in the classroom, she observes that there are dominant voices and also marginal voices that have a hard time to be heard. The perspective developed by Gerovsky carries critical aspects by bringing into focus fragile, marginal or in-minority ways of being-in-mathematics, often suggesting social and political demands. This perspective also suggests that there is no ideologically neutral knowledge and that all acts of knowing are embedded in an ethical problem for which we need to develop our sensitivity.

4 Bakhtin and the field of history of mathematics in mathematics education

Closer to our concerns, Bakhtin has also been summoned more precisely in our field of research. Specifically, a Bakhtinian perspective has helped to think about what it means for the students or the pupils to meet history of mathematics or elements related to history of mathematics, but also helped to understand history of mathematics itself by giving insights about what to look for and to discuss about texts within history and to develop way to discuss about a historical text with the learners. Two examples will be quickly discussed in this section.

For Radford, Furinghetti and Katz (2007), the particular meaning attributed to mathematical objects is circumscribed to the limits of our own experience. This limit can only be crossed by the encounter with a foreign form of understanding, as Bakhtin would say “A meaning only reveals its depths once it has encountered and come into contact with another, foreign meaning: they engage in a kind of dialogue, which surmounts the closeness and one-sidedness of these particular meanings” (Bakhtin, 1986, cited in Radford, Furinghetti and Katz, 2007, p. 108). In this sense, history of mathematics is a possible place where it is possible to overcome the peculiarity of our own understanding of mathematical objects limited to our personal experiences. It “history erects itself as the place where we can surmount the one-sidedness of our particular meanings; it is a place to enter into a dialogue with others, and with the historical conceptual products produced by the cognitive activity of those who have preceded us in the always-changing life of cultures.” (*ibid.*, p. 109).

The story here appears as the background or a place that could help to stimulate introspection, or what Bakhtin would call our inner dialogue, and also to bring confrontation and critical reflection around our own conceptions and knowledge in mathematics. In this sense, Radford et al. (2000) pointed out that the history of mathematics is “a wonderful place where it is possible to reconstruct and reinterpret the past in order to open up new possibilities for future teachers” (p. 165).

It should be noted that the focus here is not on an individual experiencing personal emancipation possibilities, in a more or less sustained movement of self-reliance and self-reference, but towards the possibility for the learners to discover new ways-of-being-in-mathematics, to open, with others, the space of possibilities of the mathematical activity. Indeed, the mathematics class is perceived here as a community space, political and ethical, open to novelty and subversion (see Radford, 2006, 2008, 2011).

A second example can be found in the work of Barbin, who thinks both history itself and the work of historians in terms of a Bakthinian perspective (see Barbin, 2014), but also the interaction between voices from history of mathematics and the mathematics classroom (see Barbin, 2011).

For Barbin, history has a certain subversive potential that can be linked to the Bakhtinian critical standpoint mentioned above. One of the major roles that history could play in the scientific and educational world is that it can dethrone styles and worldviews that enjoy a traditional or official status. The idea here is to look at the place of the Other within historical text and dialogical interaction in the writings of mathematicians from the past. She emphasises the necessity to read the author as somebody explaining something to somebody else, and that both are holding a specific and to-be-described position in the ongoing dialogue.

In this sense, she argues that: “An original source has to be read as a rejoinder in a dialogue. What dialogue? Firstly, it is a dialogue between author and his or her contemporaries. To take dialogism into account is a good means for pupils to understand that mathematics is not a ‘long quiet way’, but that mathematics is a struggle for spirit. We have to read the author as somebody explaining something to somebody else. So, it is also a means to establish a second dialogue, a dialogue between the teacher and his or her students. In this case, an original source could be filled also with utterances between the teacher and the students” (Barbin, 2011, p. 15).

The purpose, in the perspective developed by Barbin, is especially not to separate the dialogue taken place within history and the one taken place within the mathematics classroom. On the contrary, the idea is to put forward the passage between the two spheres of speech communication. This, in order to understand better the role and the implication of introducing history in the classroom, but also, for the classroom itself, to maintain the polyphonic aspect of the interactions that are taken place in order to achieve pedagogical goals such as reorientation (*dépaysement* in French) (see Barbin, 1997).

5 Bakhtin and research itself within the field of history of mathematics in mathematics education

In this section, we will try to show how Bakhtin's thought can support reflection, not on history, mathematics or mathematics education *per se*, as it has been discussed above, but on our very ways of being in research and doing research within or field.

Through our reading of Bakhtin's works, we will propose new ways of thinking about the role and position of the researcher and the participants, but also that of history of mathematics, in such a context of research. This will include ways to appreciate and account for the dialogical interaction between researchers, participants (teachers and students) and history of mathematics (understood as a third-party interlocutor) that this perspective suggests. To support our point, and to reach more "applied" or "practical" issues, some examples of interaction between researchers and participants from our own research concerning mathematics teachers' education will be discussed within this dialogical perspective.

5.1 A reflection around research and history

Research in our field, or a certain important part of it, is characterised by the objective of understanding better the role and the potential of history of mathematics within mathematics education.

What could it mean to think about research itself in a dialogical way? The idea is to think about research itself as an opportunity to get ourselves in a dialogue with the participants and elements of the history of mathematics. In this sense, researchers are creators of events, of course, related to specific objects of research. The objective here is to give back to the community the voices of the participants that are confronted, and in dialogue, with elements of history of mathematics.

This said, for Bakhtin, any utterance of a speech cannot be understood without considering the sphere of communication in which it inscribes itself. It means that each voice that is presented and analysed in research cannot be presented and analysed without any reference to order voices that have made it possible and to other voices that are possible because of it. This is why a proper Bakhtinian perspective in research carries this injunction to give not only the voices to participants, but to present these voices in their dialogical interaction.

This said, what is interesting in the Bakhtinian perspective regarding our own field of research is that this perspective can include voices that cannot manifest themselves, voices from the past. Indeed, theses voices coming from the history of mathematics cannot manifest themselves in the classroom, but they can be summoned by the means of interpretation. Interpretation is from the Latin *interpretatio*; itself built on *inter*, which is between, and *pretare,* which is close, *praesto,* what is present. Here, voices from the past have to join voices in the present by the means of something in-between. But, again, the movements of consciousness that bring these voices in the present have themselves to be understood here in their dialogical interaction. This is a very delicate aspect to take in consideration, because the pedagogical implications are very important.

As Barbin put it many times, having in head what has been brought in section 4 above), history could bring a “culture shock” in “immediately immersing the history of mathematics in history itself” (2012, p. 552, our translation). Therefore, the objective is not to read historical texts simply related to our (modern) knowledge, but rather in the context of the one who wrote them. This is where history becomes a source of “epistemological astonishment” by questioning knowledge and procedures typically taken as “self-evident” (*ibid*.).

We join here the position of Jahnke (1994; 2014) around the idea of learning-to-listen with the history of mathematics. For Jahnke, starting from a hermeneutic approach, the reading of a historical text in mathematics brings two interrelated forms of reflections. Firstly, there is the experience of “dissonance” or “alienation”, just like the feeling of being in a foreign country. The students learn something about their own mathematics by experiencing and “reflecting on the contrast between modern concepts and their historical counterparts” (Fried et al., p. 218). This reflection goes in both directions, so that the students deepen both their understanding of history and of their own set of modern conceptualizations regarding mathematics and mathematical objects. Secondly, the task is now to think about the situation of the mathematicians living in the past. This task requires being able to argue from the assumptions of these persons, to use their symbols and methods. This poses completely new demands on the students’ abilities in their mathematical activities.

According to the hermeneutic perspective, a text consists in the merging of different horizons, the horizon of the reader and the horizon of the author. This means, of courses, that different readers embedded in their different backgrounds arrive at different interpretations. As Jahnke would say, the texts and historical artefacts are here the problems and the things that students are confronted with.

Yet, we would like to say with Bakhtin that these interpretations arise in the classroom within an already ongoing dialogue on mathematics, and that this dialogue could include the teacher and also the researcher. The concepts of dialogism and polyphony, borrowed from Bakhtin, can here provide the necessary means to think about the elaboration of a description of what happened objectively and that includes us as a teacher/researcher in its description, and also voices from the past.

5.2 The example of *Direct cinema*

Within the oral communication that took place during ESU-8, we brought the example of *Direct Cinema* in order to give some insights about our understanding of this Bakhtinian perspective on research. This example seems for us particularly relevant as one can retrieve an investigation that is going on, researchers (here the filmmakers) that try to understand a community (here the protagonists) trying to make sense of their past. In this context, the dialogical principle and the concept of polyphony can be illustrated concretely.

In every encyclopaedia related to cinema, there is a small part of it dedicated to *Direct Cinema*, a way of doing documentaries that arise during the1960s in the province of Québec in Canada. Filmmakers from this movement had the idea to produce a very special kind of documentary. They were searching to present some kind of fiction related to real events that could include themselves. Paradoxically, they were looking for a more authentic way of investigating phenomena with this medium. More concretely, the idea was about to go within the community that is concerned with the phenomena that interest the filmmakers. The film proposes to show this meeting and how the interaction with the members of the community has created a certain event that could have brought new ways of thinking about the phenomena. There is no "God's view" or contemplative perspective, but a real engagement by the filmmakers in their object of investigation.

The movie *Pour la suite du monde* by Michel Brault and Pierre Perrault (1962) is a perfect example of a movie from this *Direct Cinema* movement. In this film, the filmmakers are interested in a community that are living on an isolated island on the St-Laurent river in the province of Québec in Canada, called *Ile aux Coudres*. This more or less isolated community has conserved a kind of ancient ways of living and traditions from the people of Québec and are confronted, by that time, with modern ways of living. They went there with their camera and tried to create something with the community. As the spectator understands it when seeing the movie, the filmmakers proposed to the inhabitants of the island to go fishing for the beluga whale. At that time, this particular fishing activity has been abandoned 40 years ago. The camera then follows the members of the community in his quest to retrieve the way to organise this fishing for the beluga whale that requires special technique and competencies.

What the film provides is a series of conversation between the members of the community, as well as decisions and actions that they are taking and doing together. The camera is present during these truly realised events, but the filmmakers do not interact explicitly. The result is that we have a tissue of dialogical interactions that include the members of the community, the filmmakers (that implicitly influencing the events and by controlling the camera) and the voices from the past that arisen from artefacts related to the ancient fishing activity. Soon, in the film, tensions emerged from these interactions, a very polyphonic dimension appears as progressive, conservative or pragmatic, for instance, speeches and ways-of-being reveals themselves. It is by revealing these tensions and by finding a way to make accessible the dialogical interaction that the filmmakers succeeded in their quest to describe the reality of this special community.

5.3 To be and to do research in a Bakhtinian perspective

This little example of *Direct Cinema* can help to understand more concretely the Bakhtinian perspective. We will now describe how the perspective was deployed our research recently. From this description, we will try to highlights some final reflections related to the pertinence of it in our field and ways to pragmatically conduct research in this sense.

In this study (see Guillemette 2017, 2018), we were searching to describe the *dépaysement épistémologique* lived by prospective teachers engaged in the reading of historical texts during a history of mathematics courses. Six participants were recruited in this study. Seven activities consisting in the readings of historical texts were experienced:

- A'hmosè: Rhind Papyrus, problem 24
- Euclid: Elements, proposition 14, book 2
- Archimedes: The Quadrature of the Parabola
- Al-Khwarizmi: The Compendious Book on Calculation by Completion and Balancing (Al-kitāb al-mukhtaṣarfīḥisāb al-ğabrwa'l-muqābala), types 4-5

- Chuquet: Tripartys en sciences des nombres, problem 166
- Roberval: Observations sur la composition des mouvements et sur le moyen de trouver les touchantes des lignes courbes, problem 1
- Fermat: Méthode pour la recherche du minimum et du maximum, problems 1 to 5

Phenomenology, the dialogical principle and the concept of polyphony help us to develop our methodological framework. Inhabited by the comprehensive and critical perspective that carried these elements in human sciences, the study proposes a description of the lived experience of *dépaysement épistémologique* that takes the form of a polyphonic narration.

These reading activities were conducted following Fried's (2007, 2008) recommendations. For this author, just like many others, the reading of historical texts appears to be the preferred approach when using history of mathematics in order to create this *dépaysement épistémologique*, the very meeting with mathematicians from the past.

Video recordings of classroom activities, individual interviews and a group interview were conducted and provide the data. For video recordings, analysis allowed us to describe the learning process that took place in the classroom. The individual interviews dealt with the experience of the course, the experience of the readings and the experience of *dépaysement épistémologique*.

The polyphonic novel was then constructed from extracts of the interview group and enhanced by video recordings and individual interviews previous analysis phases.

More precisely, in order to obtain this polyphonic novel, the first step was to construct the transcript of the group interview with care. Then, several attentive readings of the transcript were made. These readings have revealed some extracts of dialogue containing rich and profound reflections in relation to the lived experience of the participants. Twelve extracts of the transcript were selected. Thereafter, a careful reading of each of these extracts was made again and a list of various topics, thematics, reflections or statements were created for each of these extracts. The twelve extracts were then systematically treated individually. For each of them, four writing phases succeeded each other.

The first step of writing was to rework the raw extract from the transcription of the dialogue. The dialogue was then shaped so as to make it more readable with the addition of paragraphs and spacing.

The second writing step was to complete the extract, with the addition of information on the participants. These additions allowed to "defend" each participant in the dialogue and to refine and highlight their thoughts and appreciative orientations. Taking the form of paragraphs inserted into the dialogue, these additions allow us to position ourselves author/researcher as the agent of the participants, as their spokesman. These intercessions were both fuelled and justified by the descriptions of reading activities and the specific descriptions of the experience of the participants obtained during previous phases of analysis.

In the third step of writing, personal reflections were added. It was to be heard more as an author/researcher in the narrative. Usually at the beginning of the extract, one or more paragraphs were added. These provided space to express our thoughts that were emerging at the time of writing.

The fourth and final step of writing was to refine the narrative by emphasizing the theme of the extract and the polyphonic style exercised.

These four writing steps were repeated for each of the twelve extracts released initially. These were then combined to form the final polyphonic novel describing the *dépaysement épistémologique* experienced by future teachers of mathematics. This narration of the collective experience takes its density from fine description of each character/participants from previous analyses. It has led to the emergence of tensions, viewpoints moving away and approaching each other, viewpoints that overlap and influence each other.

The description provides multiple looks, which, in tension, carries fruitful discourses on the lived experience of participants. As Bakhtin put it in its dialogical critic explained above, it is in the tension between discourses coming from different spheres of communication, different ideological horizons and different aesthetic spaces that one could grasp the reality of human life.

Globally, the form of a polyphonic narration for this description is a methodological response to an epistemological challenge that underpinned this research. Indeed, this discursive strategy allows the production of a description that, first, can respect the phenomenological requirement and stringency to keep alive the subjectivity of the participants without objectivizing it in any manner and, second, embrace a conception of teaching and learning in mathematics education that claims that learning is necessarily "learning-with-others" (Radford 2011, 2013).

Yet, this study cannot provide any clue concerning the way one could provoke "systematically" *dépaysement épistémologique* in his classroom, and above all, in the same way that happened in this particular study or in any "positive" way. This study had much humbler objectives, by trying, from an empirical position, not to "confirm" or "infirm" theoretical considerations around the introduction of history of mathematics in the classroom, but to enrich and deepen them by a reflection that is emerging from the contact with the participants.

In the next excerpts from video analysis, we can see how participants' mathematical activity interacts with Fermat's minima and maxima method, and how it is interpreted. The excerpt concern Fermat's general description of his method and the first example given. He finds the maximum or minimum of a given term $f(x)$ by "adequating" (which means approximately equal) the two expressions $f(x)$ and $f(x + e)$, reducing and clearing remaining "e-terms". The first example (divide a line AC at a point E such that rectangle ACE area is maximized) involve a term in the form of $f(x) = bx - x^2$.

A team of three students (Martha, Aliocha and Ninotchka) are engaged in this reading:

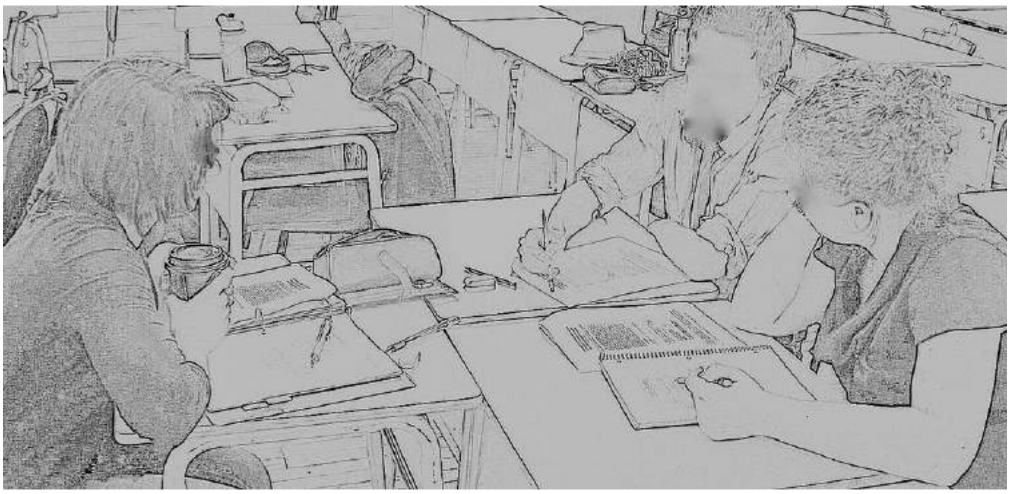


Fig. 5.1: Martha, Aliocha and Ninotchka reading Fermat.

Martha is saying that e is a very small value.

Aliocha is trying to reconcile Fermat's method and the basic elements of modern calculus. He asks, "if adequating means to subtract the terms?". Ninotchka answers that "adequating means simply to equalise".

Aliocha then asks how she relates Fermat to modern calculus. Ninotchka shows his calculations and Aliocha concludes that their reasoning is equivalent.

After few moments, Martha points out that Fermat removes e. Aliocha indicates that "e is almost 0, so the multiplication by e also gives almost 0". Martha asks herself whether the reader should "decide on the value of e". Aliocha replied that yes. Martha emphasizes that there is something missing in the reasoning. Aliocha asks why Fermat is using symbol of inequality, and concludes that adequating means to reduce to the minimum.

With Fermat's method, which participates of the beginnings of calculus formalization, participants are confronted with an exploratory reasoning showing genuine and foreign way of dealing with these mathematical objects and procedures. The encounter with this fragmented and emerging mathematical discourse brings an impression of distance to the participants. They cannot do nothing else but to convoke their own modern modalities of expression (especially here representation of algebraic quantities) in order to enter in dialogue with Fermat's utterances (reflecting on geometrical magnitudes), responding themselves to other utterances (especially here to Diophantus around the notion of *adaequalitas*). This distance, which is here a temporal one, as well as the polyphonic aspect of the text itself and of the classroom organisation, emphasis for the participants how individual activities, mediated by the sociocultural context, constitute the genetic root of the mathematical activity, containing rational, aesthetic and functional expressive dimensions. In our reading, this is where history of mathematics, with the experience that it provides, and the dialogue that it forces, seems to bring the most for pre-service teacher's reflection on mathematics and mathematics education.

The methodological framework inspired here by elements of Bakhtin's philosophy has helped us to get in a dialogue with the participants and the mathematicians of the past. It has also helped to describe how together the researcher, the participants, the voices from the past and the actual mathematical culture come into dialogical interactions.

On the one hand, from a research perspective, the idea is to give access to this share meaning, to grasp the world in common that emerges from the introduction of history into the mathematics classroom. From this perspective there is a need to report the multiplicity of experiences. This doesn't mean to report, side-by-side, each of the participants' experiences, but to really provide the "common world". This common world has nothing to do with the consensus that could emerge around a certain understanding of history or mathematics or their relations, but is constituted of tensions emerging from dialogical interactions between participants researchers and voices from the history of mathematics.

It is by revealing these tensions and by finding a way to make accessible this dialogical interaction that a study could eventually succeed in its objective to describe what it means for the students or the pupils to meet history of mathematics or elements related to history of mathematics. Again, the challenge here is to find ways to "write" and "present" a description that could introduce these dialogical interactions to the research community.

On the other hand, from a more pedagogical perspective, the idea could be to think about the sphere of speech communication related to Fermat and his contemporaries. Without referring explicitly to Bakhtin with the students (but why not?), it could be to interesting to investigate more profoundly the link between the participants of this sphere of speech communication in which Fermat is inscribed. Asking to whom Fermat is writing? In response to what? How Fermat's methods, concepts and idea have been received? Which attitude does he enact and which attitudes are expected by Fermat from his reader? The reading of a historical text such as Fermat's text on minima and maxima method could then become an exploratory and interpretative activity within mathematics that could generate element of a debate (such as Barbin 2011 suggests) both at the level of Fermat and his contemporaries, but also at the level of the classroom when students enter in a dialogue with Fermat having themselves an active-responsive attitude related to the mathematics that are proposed by Fermat. Indeed, from a Bakhtinian perspective, students have expectations from a mathematical text, could be in terms of rigor, organisation, tone, clarity, utility, generalizability, notation, etc. and those expectation could lead to disillusion, agreement, disagreement, consideration, reconsideration, etc.

From this Bakhtinian perspective on the mathematics classroom and on this particular type of activity referring to history, one could perceive an ongoing dialogue in the classroom. Pedagogical goals could emerge from this perspective and be pursued. For instance, reflection on meta-issues in mathematics such as the historicity of concepts, methods, definitions and notion, historicity of notation and rigor, mechanisms underlying the discovery or development of mathematical objects or procedures, intrinsic and extrinsic forces that drive mathematicians, links between the development of these concepts and the development of societies and cultures, etc. Moreover, these pedagogical goals could be pursued by emphasising, promoting and maintaining the polyphonic aspect of the context in which the students are subsumed, both at the level of history and at the level of the classroom.

6 Conclusion

We hope that his paper could help researchers and teachers to reflect upon ways of conducting research in our field and to think about the very meaning of introducing history in the mathematics' classroom. In our quest to develop reflection around conceptual and theoretical elements related to history in mathematics education, this paper has tried to draw attention on the dialogical principle and the concept of polyphony that has been developed by Bakhtin and his collaborators. We have argued that elements of Bakhtin philosophy could support reflection, not only on history, mathematics or mathematics education, but on our very ways of being in research and doing research within or field and to think about the classrooms in such context.

We believe that the Bakhtinian perspective deserves to be explored and discussed within our field. The work of Bakhtin presents an important radicalness regarding its way of thinking about the human subjectivity and the way in which it is inserted in the historical, social and cultural world. We think that the elucidation and the development of these positions could, in many ways, help investigations and intervention related to the introduction of history of mathematics in mathematics education.

REFERENCES

Bakhtin, M. (1977) [Published by the name of V. N. Volochinov]. *Le marxisme et la philosophie du langage* [Marxism and the philosophy of language]. Paris: Minuit. (Originally published in 1929)

Bakhtin, M. (1982). *L'œuvre de François Rabelais et la culture populaire au Moyen-Âge et sous la Renaissance* [Rabelais and his world]. Paris: Gallimard. (Originally published in 1970)

Bakhtin, M. (1986). *Speech genres and other late essays*. Austin: University of Texas Press.

Bakhtin, M. (1997). *Esthétique et théorie du roman* [Questions of literature and aesthetics]. Paris: Gallimard. (Originally published in 1978)

Bakhtin, M. (1998). *La poétique de Dostoïevski* [Problems of Dostoevsky's poetics]. Paris: Seuil. (Originally published in 1963)

Bakhtin, M. (2003). *Pour unephilosophie de l'acte* [Toward a philosophy of the act]. Lausanne: L'Âge d'Homme. (Originally published in 1986)

Barbin, É. (1997). Histoire et enseignement des mathématiques: pourquoi? comment? *Bulletin de l'AMQ (Association Mathématique du Québec), 37*(1) , 20-25.

Barbin, E. (2011). Dialogism in mathematical writing: historical, philosophical and pedagogical issues. In V. Katz & C. Tzanakis (Eds.), *MAA notes, vol. 78* (pp. 9-16). Washington: MAA.

Barbin, É. (2012). L'histoire des mathématiques dans la formation: une perspective historique. In J.-L. Dorier & S. Coutat (Eds.), *Acte du congrès de l'EMF 2012* (pp. 546-554). Geneva: EMF.

Barbin, É. (2014). Une approche bakhtinienne des textes d'histoire des sciences. In A. L. Rey (Ed.), *Méthode et histoire: Quelle histoire font les historiens des sciences et des techniques ?* (pp. 217-232). Paris: Classiques Garnier.

Barwell, R. (2014) Centripetal and centrifugal language forces in one elementary school second language mathematics classroom. *ZDM*, *46*(6) 911-922

Barwell, R. (2016). Formal and informal mathematical discourses: Bakhtin and Vygotsky, dialogue and dialectic. *Educational Studies in Mathematics*, *92*(3), 331-345.

Brault, M., & Perrault, P. (1962). *Pour la suite du monde*. Canada: Office National du Film du Canada. [available on line at: http://www.onf.ca/film/pour_la_suite_du_monde/]

Fried, M. N., Guillemette, D., & Jahnke, H. N. (2016). Theoretical and/or conceptual frameworks for integrating history in mathematics education. In L.Radford, F. Furinghetti & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (pp. 211-230), Montpellier: IREM de Montpellier.

Gerofsky, S. (2010). The impossibility of "real-life" word problems (according to Bakhtin, Lacan, Zizek and Baudrillard). *Discourse*, *31*(1), 61-74.

Gerofsky, S. (2012). Democratizing "big ideas" of mathematics through multimodality: using gesture, movement, sound and narrative as non-algebraic modalities for learning about functions. *Hellenic Mathematical Society: International Journal for Mathematics in Education*, *4*, 145-150.

Guillemette, D. (2015). Rôle de l'histoire des mathématiques dans l'enseignement-apprentissage des mathématiques : le point de vue socioculturel. In C. Sabena & B. Di Paola (Eds.), *Actes de la 67[e] réunion de la Commission internationale pour l'étude et l'amélioration de l'enseignement des mathématiques (CIEAEM 67) - Enseigner et apprendre les mathématiques :ressources et obstacles* (pp. 607-618), Aosta, Italie: CIEAEM.

Guillemette, D. (2017). History of mathematics in secondary school teachers' training: towards a nonviolent mathematics education. *Educational Studies in Mathematics*, *96*(3), 349-365.

Jahnke H. N. (1994). The historical dimension of mathematical understanding: Objectifying the Subjective. In J. P. da Ponte & J. F. Mato (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education (vol. 1)* (pp. 139-156). Lisbon: University of Lisbon.

Jahnke H. N. (2014). History in mathematics education: a hermeneutic approach. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics & mathematics education: searching for common ground* (pp. 75-88), Dordrecht: Springer.

Radford, L. (2006). Communication, apprentissage et formation du je-communautaire. In B. D'Amore & S. Sbaragli (Eds.), *Proceedings of the 20th National Italien Conference "Incontri con la Matematica"* (pp. 65-72). Bologna: University of Bologna.

Radford, L. (2008). The ethics of being and knowing: towards a cultural theory of learning. In L. Radford, G. Schubring& F. Seeger (Eds.), *Semiotics in mathematics education: epistemology, history, classroom and culture* (pp. 215-234). Rotterdam: Sense Publishers.

Radford, L. (2011). Vers une théorie socioculturelle de l'enseignement-apprentissage: la théorie de l'Objectivation. *Éléments*, *1*, 1-27.

Radford, L., Bartolini Bussi, M. G., Bekken, O., Boero, P., Dorier, J.-L., Katz, V. & Vasco, C. (2000). Historical formation and student understanding of mathematics. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: the ICMI study* (pp. 143-167). Dordrecht: Kluwer Academic Publishers.

Radford, L., Furinghetti, F., & Katz, V. (2007). Introduction: the topos of meaning or the encounter between past and present. *Educational Studies in Mathematics*, *66*(1), 107-110.

Roth, W.-M., & Radford, L. (2011). *A cultural historical perspective on teaching and learning*. Rotterdam: Sense.

Sabo, K., & Nielsen, G. M. (1984). Critique dialogique et postmodernisme. *Études françaises*, *20*(1), 74-86.

Sfard, A. (2000). Steering (dis)course between metaphors and rigor: using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education*, *31*(3), 296-327.

Sfard, A. (2001). There is more to discourse than meets the ears: looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, *46*(1-3), 13-47.

Sfard, A. (2008). *Thinking as Communicating*. Cambridge: Cambridge University Press.

Sfard, A., & Kieran, C. (2001). Cognition as communication: rethinking learning-by-talking through multi-faceted analysis of students' mathematical interaction. *Mind, culture, and activity*, *8*(1), 42-76.

DIGITAL TECHNOLOGIES AS A WAY OF MAKING ORIGINAL SOURCES ACCESSIBLE TO STUDENTS

Uffe Thomas JANKVIST[1,3], Eirini GERANIOU[2,4]
[1]Danish School of Education, Aarhus University, Tuborgvej 164, 2400 Kbh. NV, Denmark
[2]UCL Institute of Education, 20 Bedford Way, London, WC1H 0AL, UK
[3]utj@edu.au.dk, [4]e.geraniou@ucl.ac.uk

ABSTRACT

In this paper, we argue for the use of digital technologies in making original sources more accessible to students. We present a teaching scenario outlining a use of GeoGebra to 'unpack' a selected proposition from Euclid's Elements. We discuss potential benefits of applying digital technologies through Duval's (2006) framework of semiotic registers, through Trouche's (2005) framework of instrumental genesis, and a use of Barnett and colleagues' (2014) approach of guided reading. The combination of original sources and use of digital technologies appears to be a somewhat overlooked area in the HPM research, not least in terms of empirical investigations. Yet, in this paper we lay down the theoretical bricks for such further investigations.

Keywords: Original sources, digital technologies, semiotic registers, instrumental genesis, guided reading.

1 Introduction

As often pointed out in the HPM literature,[1] use of original sources is one of the most rewarding but also one of the most challenging endeavours in the teaching and learning of mathematics (e.g. Jahnke et al., 2000). One challenging aspect for the students of course concerns situating the source in a historical context often rather different from that of the present. Another challenging aspect concerns the fact that original sources often are quite inaccessible to the students, e.g. because the language of the source is different from that in their usual textbooks, the mathematical notation is different, mathematical concepts are defined differently or even called something different. Hence, in the HPM literature it is often debated how to make the original sources more accessible to the students (e.g. see Jankvist, 2014). Several approaches have been suggested, developed and refined over the years, e.g. the hermeneutic approach (Jahnke et al., 2000; Glaubitz, 2010), guided reading (Barnett, Lodder & Pengelley, 2014), comparative readings (e.g. Siu, 2011). (For further discussion of these and other approaches, see Jankvist, 2014).

In this paper, we argue for, if not an actual approach, then yet a different way of making original sources more accessible to students, namely by having students use digital technologies, e.g. Dynamic Geometry Software (DGS) or Computer Algebra Systems (CAS) or other digital technologies, to 'open up' the original source material. The free dynamic geometry tool GeoGebra (www.geogebra.org) combines geometry and algebra as well as spreadsheets on one interface and can offer dynamic visualisations of mathematical concepts, facts, statements, axioms, etc., and potentially motivate students in accessing original sources and learning mathematics. We are not the first to suggest a combined use of historical sources and digital technologies. Oftentimes students are quite

[1] HPM is the ICMI affiliated *International Study Group on the Relations Between History and Pedagogy of Mathematics*.

familiar with one or several such digital technologies – e.g. in Danish upper secondary school CAS is mandatory in the mathematics program – and may thus be able to use their competencies within these as a way of ‘unpacking’ an original source. In the terminology of Jahnke (2019), we could consider the digital technologies as making up another hermeneutic circle for the students’ work. If relying on Sfard’s (2008) notion of discourses in relation to mathematics, we might argue that the digital technologies makes up a familiar discourse for the students, and that this familiarity can be profited from in relation to using original sources. Or if applying Duval’s (2006) notion of register, we may argue that the digital technologies offer a more familiar ‘register’ to the students. But whether choosing one or the other theoretical basis, the fact remains that there seems to be an unresolved potential in relation to using digital technologies when teaching with original sources in mathematics – a potential which the HPM literature largely seems to have missed out on – with a few exceptions[2] – and this despite the fact that attention was already drawn to such potentials in the ICMI Study (Fauvel & van Maanen, 2000).

Through a number of examples, Isoda (2000a) argued how digital technologies – e.g. DGS, CAS, spreadsheets – could contribute to students’ mathematical inquiry and reflective thinking by providing multiple representations. For example, in relation to a use of DGS in the reading of Descartes’ *Geométrie*: “One of the major pedagogical concerns for many years has been that students have lost the opportunity to experience classical geometrical intuitions, which are not replaced by a haze of algebraic symbols; DGS begins to offer a chance to re-experience some age-old intuitions.” (Isoda, 2000a, p. 354). Further argumentation and illustration are to be found in the HPM proceedings (Isoda, 2000b; 2004), the ESU proceedings (Aguilar & Zavaleta, 2015; Bruneau, 2011; Chorlay, 2015; Hong & Wang, 2015; Jankvist, Misfeldt & Aguilar, 2019), and in miscellaneous other channels (Baki & Guven, 2009; Burke & Burroughs, 2009; Caglayan, 2016; Erbas, 2009; Kidron, 2004; Olsen & Thomsen, 2019; Papadopoulos, 2014; Zengin, 2018). Yet, only about a handful of these may be considered as actual *empirical research studies*.[3] And equally important, only very few of these studies make use of the extensive mathematics education literature on digital technologies in the teaching and learning of mathematics.

This paper aims to address the above claimed potentially fruitful interplay between the history of mathematics and the use of digital technologies. We do so by outlining mathematics education theoretical constructs, also related to digital technologies, which when combined may play a role in structuring such an interplay, and by providing an example of a teaching scenario on Euclid’s proposition 22 from the *Elements* Book I and the use of GeoGebra.

[2] As part of a recent review in his master’s thesis at the Danish School of Education, Balsløv (2018) identified only some fifteen incidents in decades of HPM-related literature that specifically address the combination of using history of mathematics in combination with digital technologies. All of them are published from year 2000 and on.

[3] Jankvist, Clark and Mosvold (in review) provided an empirical example of how a Danish School of Education graduate student on her own initiative used DGS to further her understanding of Vieté’s geometrically-inspired method to solve two third degree equations. In addition, two master’s theses from the Danish School of Education provide empirical evidence concerning a use of Fermat’s method for evaluation of maxima and minima in relation to a use of CAS in upper secondary school (Balsløv, 2018), and a use of Euclid’s proposition on the construction of equilateral triangles and a use of DGS in primary school (Olsen & Thomsen, 2017).

2 Representations and semiotic registers

As occasionally pointed out in the available HPM literature[4], digital technologies hold a large potential in terms of multiple representations. To this end, Duval's theory of semiotic registers seems suitable to articulate key aspects of this potential.

The outset for Duval (2006) is that in mathematics one cannot directly access the mathematical objects, as one for instance can in physics through various measuring instruments, etc. In principle, one can only access mathematical objects through semiotic representations, which makes students' work with semiotic representations all-important in their mathematical activities. In particular, it is the transformations between the different semiotic registers that are of importance rather than the representations in themselves. Duval points out that the role played by signs in this regard is not to be placeholders for the mathematical objects, but for other signs. This is to say, from this perspective, signs and transformations between different semiotic representations are the kernel of mathematical activity – as opposed to what is going on in other scientific disciplines. From a mathematics education point of view, one question of course becomes: If one can use different forms of semiotic representations for every mathematical object, how can students then recognize the same represented object through different semiotic representations, which are produced within different systems of representations?

The possibilities of substituting one semiotic representation with another depends on the semiotic system, and every system offers specific possibilities. The capacity of a given representation does not depend on the individual symbol (or sign), but on the semiotic system of which is a part. Natural distinctions are for example language (natural and symbolic) versus images (figures, graphs, etc.). But according to Duval such distinctions are too general and causes us to overlook an important point; namely that some semiotic systems may only be used to perform mathematical processes, while others possess a larger variety of functions. Duval (2006) suggests distinguishing between *monofunctional* and *multifunctional semiotic* registers: "Some semiotic systems can be used for only one cognitive function: mathematical processing… within a monofunctional semiotic system most processes take the form of algorithms" (p. 109). A multifunctional semiotic system "can fulfil a large range of cognitive functions: communication, information processing, awareness, imagination, etc." and "within a multifunctional semiotic system the processes can never be converted into algorithms" (p. 109).

Next, Duval distinguishes between a *treatment*, which takes place within one semiotic register, and *conversions* which happen between registers. An example of a conversion might be the mathematization of the equation story 'Aya is 3 years older than her brother Ali. Together they are 23 years old. How old are they?' into the equation $x+(x+3) = 23$, which takes place between a multifunctional (natural language) register and a monofunctional (symbolic system) register. Solving the resulting equation step by step, however, to reveal that $x=10$, is a treatment, since this takes place within the same register, i.e. the symbolic system. So, for conversions, source register and target register are different, whereas for treatments they are the same. Duval also notes that there are two different types of conversions. A *congruent* conversion is a straightforward translation – or coding – e.g. as the mathematization of the equation story into a symbolic expression

[4] HPM is the ICMI affiliated *International Study Group on the Relations Between History and Pedagogy of Mathematics*.

above. A *non-congruent* conversion is, however, much more complicated. For example, this could be doing the opposite translation in our example, i.e. going from the symbolic expression, $2x+3 = 23$, to an equation story, since there are infinitely many stories to be told based on this equation.

Duval also distinguishes between *discursive representations* resulting from one of the three kinds of discursive operations:

1. Denotation of objects (names, marks...)
2. Statement of relations of properties
3. Inference (deduction, computation...)

and *non-discursive representation* which consist of

1. Shape configurations (1D/2D, 2D/2D, 3D/2D)

Discourse here refers to something like 'articulation'. It is d*ifficult to articulate geometrical representations and transformations through words, hence these ar*e non-discursive. Together with the distinction between monofunctional and multifunctional, this results in four types of semiotic system representation registers.

	Discursive registers	**Non-discursive registers**
Multifunctional registers (non-algorithms)	Basically the use of natural language, spoken or written	Typically depicting drawings, sketches, figures, patterns
Monofunctional registers (algorithms)	Refers to symbol containing and symbol using systems	Diagrams, graphs, etc. subject to rules of construction

Figure 2.1: Duval's classification of registers mobilized through mathematical processes.

In relation to the students' difficulties with mathematical proof and proving, Duval offers the following insights:

> Now we can only mention the important case of language in geometry. We can observe a big gap between a valid deductive reasoning using theorems and the common use of arguments. The two are quite opposite treatments, even though at a surface level the linguistic formulations seem very similar. A valid deductive reasoning runs like a verbal computation of propositions while the use of arguments in order to convince other people runs like the progressive description of a set of beliefs, facts and contradictions. Students can only understand what is a proof when they begin to differentiate these two kinds of reasoning in natural language. In order to make them get to this level, the use of transitional representation activity, such as construction of propositional graphs, is needed. (Duval, 2006, p. 120)

It appears obvious that digital technologies, in the sense of DGS, has a role to play in this respect as well.

3 Instrumental genesis, instrumental orchestration and instrumental distance

Instrumental genesis involves the process of transforming artefacts, such as digital tools, into mathematical instruments (Guin and Trouche, 1999). These instruments then become part of a student's cognitive scheme (Vergnaud, 2009) and can be used epistemically to support their learning of mathematical concepts (Guin & Trouche, 1999; Artigue, 2002). In more detail, a student can internalise an artefact's constraints, resources and procedures in the process of instrumental genesis (Guin & Trouche 1999). There are two processes involved. The process of *instrumentation*, which is how a digital tool shapes and affects the user's thinking, and the process of *instrumentalisation*. During the instrumentalisation process, students may acquire knowledge that may lead to a different use of the artefact, and once achieved, then the student is able to critically reflect upon the activity they are engaged in, potentially reinterpret it but also creatively use these artefacts. We do recognise of course that this process of transforming digital tools into mathematical instruments is lengthy and that instrumental genesis develops over time (Artigue, 2002). Moreover, the interrelation between technical knowledge about the artefact, i.e. knowing how to use and using the artefact, and knowledge of mathematical concepts can prove crucial in succeeding instrumental genesis (Drijvers et al., 2010). Drijvers and Gravemeijer (2005), for example, argued that the apparent technical difficulties students had were maybe due to cognitive difficulties with mathematical concepts. It is quite crucial therefore to consider how best and why to support students during their interactions with a digital tool and achieve the initial step of instrumental genesis (instrumentation). Once this step is achieved, we could say that the digital tool has served its epistemic purpose, i.e. being used to support students' understanding and learning within their cognitive system (Artigue, 2002; Lagrange, 2005; Trouche, 2005). The instrumentalisation process, involving how a student discovers the various functionalities of artefacts and transforms them in their own personal way, can then follow and allow a digital tool to serve its pragmatic purpose too, i.e. being used to create a difference in the world external to the student (ibid.).

Considering though the instrumental approach and how digital tools can be integrated in the mathematics classroom, we should discuss Trouche's (2004) notion of instrumental orchestration. It provides a framework for expressing teachers' work before and during their lessons and interactions with their students. As Drijvers and colleagues (2010) described it with reference to Trouche: "An *instrumental orchestration* is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a – in this case computerised – learning environment in a given mathematical task situation, in order to guide students' instrumental genesis" (pp. 214-215). Trouche (2004) distinguishes three elements of the instrumental orchestration framework, namely a didactical configuration, an exploitation mode and a didactical performance. A *didactical configuration* describes the arrangement of artefacts in a certain environment and the configuration of the teaching setting. For example, in a mathematics classroom, such an arrangement would involve a certain orchestration of mathematical discourse. An *exploitation mode* regards the strategies the teacher uses to exploit a didactical configuration in order to achieve their teaching objectives. For example, a mathematics teacher would need to make decisions on how to introduce and model a mathematical task using an artefact, on the possible roles an artefact they use for their own teaching, but also for students to interact with, can have, and on the schemes and techniques students should develop and establish. Finally, a *didactical performance* involves the decisions a teacher should take during a lesson and how best to perform in their chosen didactic configuration and exploitation mode. For example, a mathematics teacher would consider what the best probing questions to use to develop students' mathematical thinking or understanding of a concept are, how to respond to certain students' comments, shared solutions and answers and their justifications, how to improvise and identify the best approach when an unexpected aspect of the mathematical task or the technological tool surfaces, or any other emerging goals appear in a lesson.

For digital tools to be integrated in the mathematics classroom, besides looking at teachers' instrumental orchestrations, we need to consider a number of factors. Some of these were described by Haspekian's (2005; 2014) research work and the introduction of the notion of *instrumental distance*. Haspekian explains:

> For a given tool, if the distance to the 'current school habits' is too great, this acts as a constraint on its integration [...]. On the other hand, the didactical potential of technology relies on the distance it introduces with regards to paper-pencil mathematics as, for instance, by providing new representations, new problems, increasing calculation possibilities, etc. This is the case for the dynamic figures in geometry softwares, with respect to the static figures in paper-pencil geometry. The didactic potentialities of these dynamic objects and their benefits for students' learning have been evidenced by many research studies [...] (Haspekian, 2014, p.246).

The notion of distance in Haspekian's work refers to the distance between the praxeologies involved in two different environments. There is a distance between praxes involved when interacting with a digital environment and the praxes involved when interacting with a paper and pencil activity for example. There is also a distance between the scope of a digital environment and how it was designed to be used and the culture of a mathematics classroom and the school's policies.

4 An illustrative case based on the 'guided reading' approach

In this section, we offer a potential teaching scenario for introducing Euclid's Proposition 22 with GeoGebra based on the approach of guided reading. The GeoGebra digital tool can be used to present this proposition, but also allow students to explore the idea Euclid shared with the ultimate goal of proving the proposition. We rely on the approach of *guided readings* of original sources developed by Barnett, Lodder and Pengelley (2014), and also used by Jankvist (2013). This approach offers a sensible way of dealing with the occasional inaccessibility of primary original sources. The main idea is to supply or interrupt the reading of an original source by explanatory comments and illustrative tasks along the way as orchestrated by the teacher. One feature of guided reading is, as claimed by Barnett, Lodder and Pengelley (2014) that "the primary source is now being used not just to introduce the mathematics in an authentically motivated context, but also as a text which the student is explicitly challenged to actively "interpret" as part of their personal process of making modern mathematics their own. In alignment with this shift, the tasks we now write for students increasingly adopt a more active "read, reflect, respond" approach to these sources" (p.10).

In our teaching scenario below, we will showcase how we also made use of Barnett and colleagues' so-called *read-reflect-respond* type of tasks. Barnett and colleagues (2014) aimed at students achieving "a deep understanding of both the similarities and differences between past and present mathematics, not merely the past as a convenient or most natural avenue to the present" (p. 23). In our proposed teaching scenario, we focus on two resources (or media for acquiring mathematical knowledge), paper and digital technologies. "Radical engagement with the disparate discourses of original sources selected from various mathematical communities appears to also support student learning by providing the scaffolding necessary to become a participant in a new (e.g., modern) mathematical discourse" (ibid., p. 24). We focus on how to connect and "unpack" past resources using a modern medium, such as the GeoGebra digital tool designed for mathematical learning, a tool that is familiar to the student, as already mentioned in the introduction. Another goal of Barnett and colleagues was to promote students' mathematical reasoning skills and further develop their ability to create valid mathematical arguments, a goal that we did adopt too.

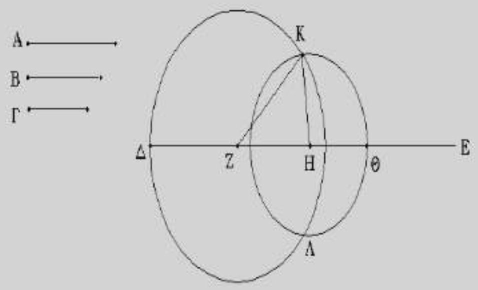
Euclid's Proposition 22 states that "To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one" (source: https://mathcs.clarku.edu/~djoyce/elements/bookI/propI22.html). This is the English translation from the original source in Ancient Greek, which is presented in Figure 4.1. In Figure 4.2, the same proposition is presented in Latin from the 1482 version of Euclid's Book of Elements. Both sources offer a diagrammatic representation for describing Proposition 22. Can you make a comment/argument here about Duval's registers here?

Πρότασις κβ'. [22]

Ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας].

Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν Α, Β τῆς Γ, αἱ δὲ Α, Γ τῆς Β, καὶ ἔτι αἱ Β, Γ τῆς Α· δεῖ δὴ ἐκ τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συστήσασθαι.

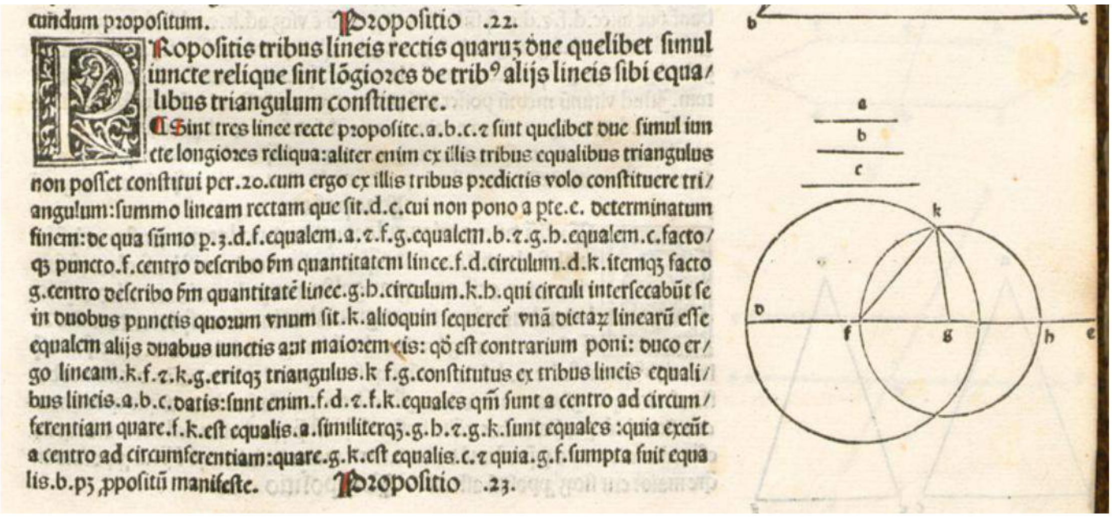
Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΖ, τῇ δὲ Β ἴση ἡ ΖΗ, τῇ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.



Ἐπεὶ γὰρ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου, ἴση ἐστὶν ἡ ΖΔ τῇ ΖΚ· ἀλλὰ ἡ ΖΔ τῇ Α ἐστιν ἴση. καὶ ἡ ΚΖ ἄρα τῇ Α ἐστιν ἴση. πάλιν, ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΛΚΘ κύκλου, ἴση ἐστὶν ἡ ΗΘ τῇ ΗΚ· ἀλλὰ ἡ ΗΘ τῇ Γ ἐστιν ἴση· καὶ ἡ ΚΗ ἄρα τῇ Γ ἐστιν ἴση. ἐστὶ δὲ καὶ ἡ ΖΗ τῇ Β ἴση· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΚΖ, ΖΗ, ΗΚ τρισὶ ταῖς Α, Β, Γ ἴσαι εἰσίν.

Ἐκ τριῶν ἄρα εὐθειῶν τῶν ΚΖ, ΖΗ, ΗΚ, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις εὐθείαις ταῖς Α, Β, Γ, τρίγωνον συνέσταται τὸ ΚΖΗ· ὅπερ ἔδει ποιῆσαι.

Figure 4.1: The Euclid proposition 22 in its original presentation in ancient Greek (source: http://www.physics.ntua.gr/mourmouras/euclid/book1/postulate22.html)



cundum propositum. Propositio .22.

Propositis tribus lineis rectis quarum due quelibet simul iuncte relique sint longiores de trib9 alijs lineis sibi equalibus triangulum constituere.

Sint tres linee recte proposite.a.b.c.z sint quelibet due simul iuncte longiores reliqua:aliter enim ex illis tribus equalibus triangulus non posset constitui per.20.cum ergo ex illis tribus predictis volo constituere triangulum:summo lineam rectam que sit.d.e.cui non pono a pte.e. determinatum finem:de qua sumo p.3.d.f.equalem.a.z.f.g.equalem.b.z.g.h.equalem.c.facto q3 puncto.f.centro describo fm quantitatem linee.f.d.circulum.d.k.itemq3 facto g.centro describo fm quantitaté linee.g.h.circulum.k.h.qui circuli intersecabũt se in duobus punctis quorum vnum sit.k.alioquin sequeretur vnã dictaz linearũ esse equalem alijs duabus iunctis aut maiorem eis: qd est contrarium poni: duco ergo lineam.k.f.z.k.g.eritq3 triangulus.k f.g.constitutus ex tribus lineis equalibus lineis.a.b.c.datis:sunt enim.f.d.z.f.k.equales qm sunt a centro ad circumferentiam quare.f.k.est equalis.a.similiterq3.g.h.z.g.k.sunt equales :quia exeũt a centro ad circumferentiam:quare.g.k.est equalis.c.z quia.g.f.sumpta fuit equalis.b.p3 ppositũ manifeste. Propositio .23.

Figure 4.2: The Euclid proposition 22 in its original presentation in Latin from 1482.

There are attempts at recreating this proposition in GeoGebra. For example, such a resource is: https://www.geogebra.org/m/bp2mpZVz. In this construction, the lengths of the three sides of the constructed triangle are variable and can be dragged, i.e. dynamically moved, while the construction reflects that movement and change of the lengths of the three sides. In Figures 4.3 and 4.4, we present how dragging one line segment, CD, impacts the construction.

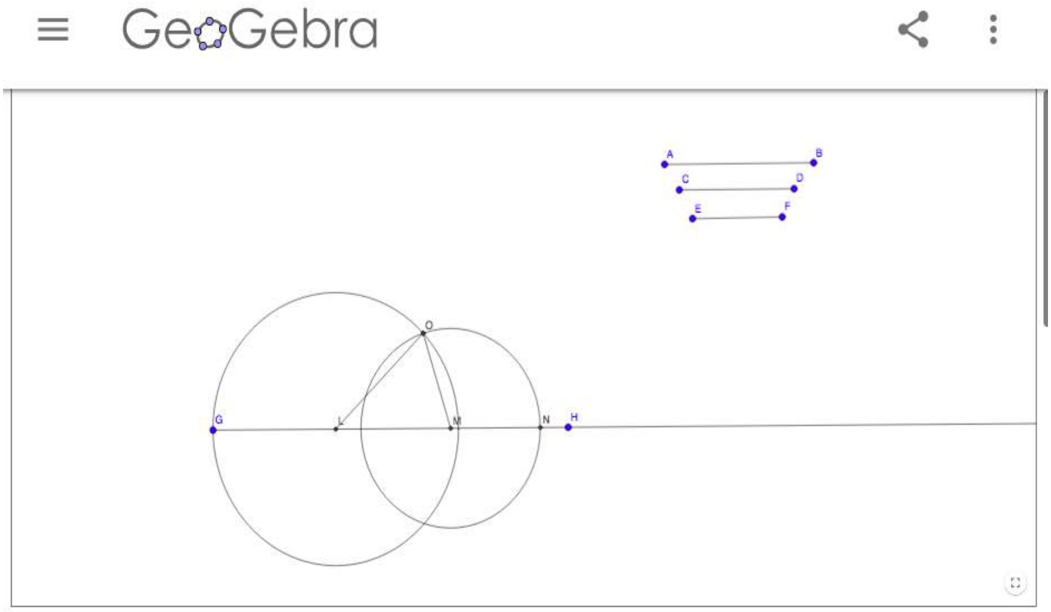


Figure 4.3: GeoGebra construction of Proposition 22
(source: https://www.geogebra.org/m/bp2mpZVz).

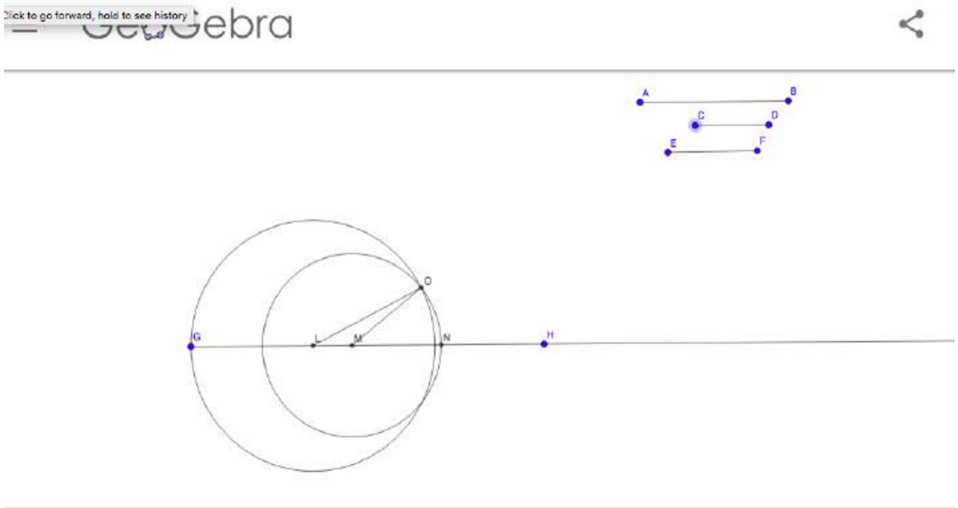


Figure 4.4: GeoGebra construction of Proposition 22, after dragging line segment CD from its original position presented in Figure 4.3 and making its length shorter than the original (source: https://www.geogebra.org/m/bp2mpZVz).

Considering these resources and with the aim of using GeoGebra as a tool for 'unpacking' Proposition 22 and making it more accessible to students, we have designed the following teaching scenario using the guided reading approach, as mentioned earlier. Besides taking into account aspects of Duval's semiotic registers, we considered both the instrumentation and instrumentalisation processes a student is expected to go through in achieving instrumental genesis, but focused mostly on the teacher's potential perspective and in fact a teacher's instrumental orchestration processes in achieving their students' instrumental genesis. We also added the analytical lens of instrumental distance. This considers the distance between the praxeologies of Euclid's Proposition 22 original resource and potentially how it was intended to be used for the teaching and learning of Geometry and the praxes involved when students interact with GeoGebra to interpret Proposition 22 and paper and pencil to record their reflective comments and arguments. We neither intend to discuss the culture of a mathematics classroom nor any school's policies or how this teaching scenario could be carried out considering such factors, as these differ from school to school.

Students are presented with the following **Learning Objectives (LOs)**.

In this learning sequence, you are going to be introduced to some of Euclid's work, and in particular his Proposition 22, as presented in his book of Elements. Euclid was an ancient Greek mathematician, who is often referred to as the "founder of Geometry" (http://www-groups.dcs.st-and.ac.uk/history/Biographies/Euclid.html) and his book, Elements, is one of the most influential work in the history of mathematics which has been used as a core textbook for the teaching and learning of Geometry. "In the *Elements*, Euclid deduced the theorems of what is now called Euclidean geometry from a small set of axioms.". Your Learning Objectives (LOs) are:

LO1. to interpret Euclid's Proposition 22 using paper and pencil and GeoGebra
LO2. to argue about the 'correctness' of this proposition and improve your mathematical reasoning skills.

Using a paper and pencil can support you in the first steps towards interpreting Proposition 22 and potentially creating a diagram. GeoGebra can support you in recreating and analysing Proposition 22 with an interactive and dynamic diagram, as opposed to the static diagram on paper.

The above LOs should set the scene for the activity sequence that supports in bridging the instrumental distance between the paper and pencil medium and GeoGebra. The teacher presents Proposition 22 on the board and printed for students, but without any diagrams. This strategy is used to exploit the didactical configuration of reading the proposition and using a familiar and easy to use medium, that of paper and pencil, as preparation for using the digital medium later on (*exploitation mode*).

TASK 1:

Read the following Proposition 22, as it was presented by Euclid and think for a few minutes on your own about its meaning. What does this proposition state ?

Proposition 22

To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one. I.20

Students are expected to read carefully this sentence and interpret its meaning using any means available to them (e.g., either mentally or by drawing a diagram on paper), before moving on to a paired task, Task 2, (*exploitation mode*). At this stage, GeoGebra is not used as we envisage students need to consider what the proposition states in its paper presentation before interacting with the GeoGebra tool to explore its meaning by carrying out constructions of triangles. This process could be viewed as the first stage of students' *instrumentalisation*, as it is planned with the aim of students' acquiring knowledge that may lead to a different use of the GeoGebra tool, e.g. not just as a tool to construct and explore, but as an analytical tool for reflecting upon their construction through a dynamic exploration (*exploitation mode*).

TASK 2:

In your pairs, share your interpretations of Proposition 22 with each other. Say to your partner what you believe this proposition says. Please use paper and pencil to record your agreed statement of what this Proposition says.

Students are expected to come up with any diagramatic or symbolic representations on paper in their efforts to make sense of the proposition. They are also expected to work as a pair and agree upon a statement for what Proposition 22 states using their own words. Such a preparatory work is aimed at giving meaning to students' interactions with GeoGebra that follow in Task 3. They should have reached a certain level of understanding of what Proposition 22 states and may use GeoGebra as a tool for validating their conjectures by constructing a triangle and dynamically manipulating it (instrumentalisation process), as opposed to carrying out meaningless actions.

Students are then presented with the GeoGebra tool. In this scenario, students are expected to be familiar with GeoGebra and its main functionalities.

TASK 3:

Using GeoGebra, construct a triangle of lengths 3cm, 4cm and 5cm.

Students could potentially use GeoGebra as a drawing tool and create triangles by constructing three line segments of 3cm, 4cm and 5cm respectively, and place them in such a way so that each two segments are connected at a corner. Such an approach cannot be criticised when considering the instrumental distance between the paper medium, which they are used to and are familiar with and their tendency to follow a similar strategy in GeoGebra. It could though be considered as students' initial steps of instrumentation since students 'drawing' with GeoGebra differs from that of drawing on paper and students are 'forced' to use GeoGebra's tools for constructing line segments of given lengths as opposed to a ruler and measuring a line segment in order to draw it. The rationale for this task 3 is to give students the opportunity to consider how to construct a triangle with sides of three different fixed lengths. After a 10-15 minutes exploration of how to construct a triangle in GeoGebra when being given the lengths for its three sides, the teacher runs a class discussion focusing on the strategies students followed to construct their triangles. Such strategies may involve: (a) using the GeoGebra 'polygon' feature, with which students may form a polygon of 3 sides, or (b) using the GeoGebra 'segments with given length' feature, with which students could construct three line segments of 3cm, 4cm and 5cm respectively and join them in such a way so that a triangle is formed. These two didactical configurations of course are not general enough and the students restrict themselves from using a dynamic tool such as GeoGebra to its full potential; (c) Students could construct a 'segment with given length', e.g. AB = 5cm, and then construct two circles. One circle of centre 'A' and radius 3cm and one circle of centre 'B' and radius 4cm. Either of the two points where the circles intersect can be chosen as the third vertex, 'C', of the triangle. This second construction path will be referred to as the 'triangle construction’ for the rest of the paper. It is also worth considering that students may have been taught already how to construct a triangle of given lengths (or not) by using a ruler and compass. Such prior knowledge would certainly influence their instrumentalisation process as they would potentially use similar strategies of constructing line segments and circles to recreate such a construction of a triangle on GeoGebra. On the other hand though, such prior knowledge may act as a bridge for the instrumental distance between the paper and pencil medium and GeoGebra. Students would look for ways to construct line segments and circles in GeoGebra mapping their prior experiences on paper. Hovering over the various GeoGebra features, but also the iconic representation of these features can support, shape, as well as affect their thinking processes (*instrumentation*).

The teacher then refers to the Proposition 22 and asks students to compare their actions in Task 3 to what the Proposition states.

TASK 4:

Using GeoGebra, construct a triangle of lengths 3cm, 4cm and 5cm Consider your constructed triangle of sides 3cm, 4cm and 5cm and read again Proposition 22. Using your constructed triangle as an example, describe what Proposition 22 says. Write down any arguments to support your claims and be ready to share with your peers in a class discussion.

The rationale for this discussion is for students to recognise the condition for being able “To construct a triangle out of three straight lines which equal three given straight lines”, or in other words to construct a triangle when being given three line segments of certain lengths. Interacting with and exploring their constructed triangle of fixed side lengths in GeoGebra, while bearing in mind the two sentences of Proposition 22 and stating these in their own words, should prompt students’ critical reflection on the validity of Proposition 22. Recognising that their triangle is ‘fixed’ in GeoGebra and none of its corners can be dragged in such a way that their triangle changes size and a different triangle is formed, should reinforce the idea that any constructed triangle of given lengths for its three sides is unique. They should start thinking then about the generalisability of the Proposition. Should this proposition be true for any triangle of given lengths for each side? Are there any conditions for the proposition to hold true? Such probing questions may be used by the teacher to support students’ reflections and their efforts in writing down arguments regarding the Proposition. In their written arguments, students are expected to reveal their instrumentalisations, as they may exploit GeoGebra’s features and their interactions to form arguments and support their claims. Moreover, students are expected to start thinking about the value of the second sentence in the Proposition, “thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one”. Why is this a “necessary” condition to construct a triangle? Students’ written arguments will be valuable information as they will reveal the current state of their instrumentation and instrumentalisation processes as well as their mathematical reasoning skills.

Considering Duval’s theory, in Task 1, the representation of the translated to English Proposition 22 on paper serves multiple purposes (*multifunctional semiotic register*). It communicates to students Euclid’s original presentation of the proposition and makes students aware of the condition for a triangle of given lengths for its three sides to exist.

Students are expected to process the shared information and using for example, their imagination, creativity and/or visualisation skills to ‘translate’ or convert this proposition using different registers, such as a diagram and/or mathematical notation and symbols. Such a process can lead to the creation and use of a non-discursive representation and it is expected to happen in Task 2. In pairs, students have to argue about their interpretations and could potentially ‘move’ between different registers. After articulating their thinking using natural language (*multifunctional discursive register*), they could decide to draw a triangle on paper (*multifunctional non-discursive register*) combined with the use of mathematical notation to describe the condition for their triangle to exist (*monofunctional discursive register*). In Task 3, students make shifts between various registers, but these occur between the paper medium and GeoGebra and within GeoGebra. They create a geometrical figure, i.e. a triangle with sides 3cm, 4cm and 5cm, and interpret the icons representing GeoGebra features, which also involve natural language (hovering over them shows what they do) and symbolic language (algebraic notation to represent angles, points, lengths, etc.). Then, in Task 4, students reflect on what the Proposition states once again and are asked to articulate arguments for supporting their claims and convince their peers. In this case, we could argue that students should be ready to share their beliefs, facts and contradictions, but they are not maybe ready to produce a valid deductive reasoning, as per Duval’s (2000) distinction.

TASK 5:

Have a look at the triangle the teacher made in GeoGebra. What happens if you drag corner **A** of the triangle? What happens to the triangle?

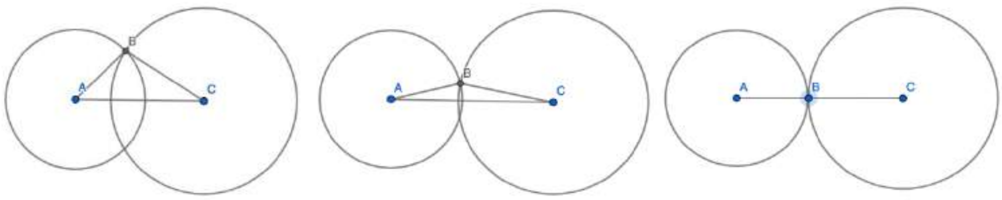


Figure 4.5(a), (b), (c): Students are expected to drag corner A of the triangle in GeoGebra and explore what happens. In (a) the triangle is formed with the longer side as the base. In (b) the corner A is dragged towards the left and the base gets longer, while corner B gets closer to the base. In (c) the corner A is dragged further to the left and corner B ‘lands’ on side AB of the triangle and as a result the triangle ABC ceases to exist and instead a longer line segment AC is created.

The rationale for Task 5 is for students to explore a triangle constructed in GeoGebra with non-fixed lengths for its three sides. This didactical configuration should trigger students’ instrumentalisation of using GeoGebra as a tool for critically reflecting upon the second statement in Proposition 22. Students are expected to recognise that if the corner *A* gets dragged further to the left in such a way that the length *AC* gets longer than the sum of *AB* and *BC*, then the triangle ceases to exist. This realisation is key in understanding the importance of the condition stated in Proposition 22. We could argue for the benefits of using GeoGebra and in fact a triangle of varied lengths for its sides to ‘unpack’ and validate Proposition 22.

Considering a triangle of no given lengths for their three sides in GeoGebra as a didactical configuration can certainly trigger students' critical reflection of Proposition 22, but at the same time some students may still have difficulties thinking in such an abstract way and require working with triangles of specific lengths. The teacher could present students with a number of sets of 3 lengths for constructing a number of triangles. Such a strategy will prompt students to reflect upon sets of lengths that can produce a triangle and others that won't and with some further 'guided reading' and the teacher's intervention, students should recognise the condition of having the sum of two lengths being greater than the remaining length for a triangle to be created, as stated in Proposition 22, and gain a deeper understanding.

TASK 6:

Work in pairs and construct in GeoGebra the following triangles for each set of 3 lengths for the sides:

2, 3, 3
2, 3, 4
2, 3, 5
2, 2, 2
2, 2, 5

What do you notice?

In this process, students are expected to recognise that only for (a), (b) and (d), a triangle can be constructed. The teacher asks for any instances the students couldn't make a triangle, encouraging them to share their reflections and arguments. What does Proposition 22 state in relation to constructing a triangle? Were you all able to construct all triangles? Why? Why not? Anything 'special' about these lengths?

The teacher runs a class discussion aiming at supporting students in reaching a conclusion about the condition for being able to construct a triangle from three given straight lines. Why can't we have a triangle of lengths [2, 2, 5] and [2, 3, 5]? Why can we have a triangle of lengths [2, 2, 2], [2, 3, 4] and [2, 4, 5]? What's the same and what's different between these sets of lengths?

All the above probing questions are aimed at supporting students' development of critical thinking regarding Proposition 22. GeoGebra can support them in constructing quickly and accurately the different triangles and comparing their constructions to reach their conclusions and potentially articulating valid arguments relying upon the specific examples.

TASK 7:

Read Proposition 22 again and focus on the second condition: "thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one". What does this mean? How many sums do you need to find and check if the condition holds? Record your answers and arguments.

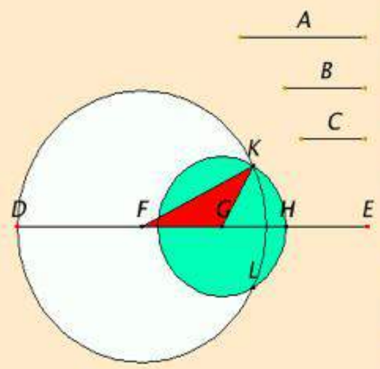
In Task 5, students were presented with the teacher's constructed triangle of varied lengths for its sides, whereas in this Task 7 students are expected to use GeoGebra to test the second condition in Proposition 22 by choosing different lengths for a triangle they construct. GeoGebra then becomes a validation tool for their own constructions, but also a tool that helps them critically reflect upon an original source (*instrumentalisation*).

Considering Duval's (2000) work on registers, working with specific examples could also help them in articulating their arguments regarding Proposition 22 and why and when a triangle can be created when being given a set of lengths for its three sides. The coherency of those arguments though would reveal the type of reasoning students would have achieved at this stage in the learning sequence. Students are prompted to use multifunctional semiotic registers and discursive representations, such as their verbal communications, their strategies for constructing the requested triangles in GeoGebra, and non-discursive representations, such as their constructed triangles in GeoGebra. By the end of Task 7, students should feel confident about what the Proposition 22 states and have a better understanding of the condition. Their arguments may still be based on specific examples and certain cases of lengths for the three sides of a triangle and may not be general enough. Depending on their confidence with mathematical notation, some students may bring algebraic notation into their written statements. In their collaborations, some students may go through a valid deductive reasoning process at this stage, where they argue about the correctness, truthfulness, or in other words, the proof of Proposition 22, using their constructions as specific examples. Students' explorations and arguments most likely based on specific examples, though, on paper and in GeoGebra, cannot be considered as formal proofs. The next step is for students to be exposed to the proof as presented in one of the original sources, but translated into English.

TASK 8:

Read the proof for Proposition 22 presented in the Figure below and follow the steps to construct a triangle of 3 lengths of your choice in GeoGebra.

Let the three given straight lines be *A*, *B*, and *C*, and let the sum of any two of these be greater than the remaining one, namely, *A* plus *B* greater than *C*, *A* plus *C* greater than *B*, and *B* plus *C* greater than *B*.



It is required to construct a triangle out of straight lines equal to *A*, *B*, and *C*.

Set out a straight line *DE*, terminated at *D* but of infinite length in the direction of *E*. Make *DF* equal to A, FG equal to *B*, and *GH* equal to *C*. I.Post.2 I.3

Describe the circle *DKL* with center *F* and radius *FD*. Again, describe the circle *KLH* with center *G* and radius *GH*. Join *KF* and *KG*. I.Post.3 I.Post.1

I say that the triangle *KFG* has been constructed out of three straight lines equal to *A*, *B*, and *C*.

Since the point *F* is the center of the circle *DKL*, therefore *FD* equals *FK*. But *FD* equals *A*, therefore *KF* also equals *A*. I.Def.16 C.N.1

Again, since the point *G* is the center of the circle *LKH*, therefore *GH* equals *GK*. But *GH* equals *C*, therefore *KG* also equals *C*. I.Def.16 C.N.1

And *FG* also equals *B*, therefore the three straight lines *KF*, *FG*, and GK equal the three straight lines *A*, *B*, and C.

Therefore out of the three straight lines *KF*, *FG*, and *GK*, which equal the three given straight lines *A*, *B*, and C, the triangle *KFG* has been constructed.

Q.E.F.

Students are presented with this image taken from https://mathcs.clarku.edu/~djoyce/elements/bookI/propI22.html and are asked to use

GeoGebra to create the presented construction as described in those steps. Students are expected to choose three lengths for their triangle and then recreate the above construction in GeoGebra following the given steps. Depending on students' attainment and other factors, such as time or students' engagement and their characters, these steps could be given as individual separate tasks (presented below) as opposed to being presented in one figure as a lengthy sequence of steps where students are trusted to follow the steps accurately. Such didactical configurations should be decided by the teacher.

TASK 8a:

Construct a triangle in GeoGebra following the following instructions:

It is required to construct a triangle out of straight lines equal to *A*, *B*, and *C*.
Set out a straight line *DE*, terminated at *D* but of infinite length in the direction of *E*. Make *DF* equal to A, FG equal to *B*, and *GH* equal to *C*.
Describe the circle *DKL* with center *F* and radius *FD*. Again, describe the circle *KLH* with center *G* and radius *GH*. Join *KF* and *KG*.
I say that the triangle *KFG* has been constructed out of three straight lines equal to *A*, *B*, and *C*.

By the end of this task, all students should have their triangles constructed provided that they chose lengths for the three sides that comply with the triangle inequality condition (the sum of the lengths of any two sides is bigger than the length of the remaining side). Their construction should be similar to the one they were presented with on paper (see Figure within Task 8a above). Students could potentially have difficulties with understanding what "Set out a straight-line *DE*, terminated at *D*, but of infinite length in the direction of *E*" means. The given diagram can support them with this statement, but also the "Ray" feature within the 'Lines' GeoGebra tools can help them in 'unpacking' this statement and constructing such a line (*instrumentation*).

TASK 8b:

Write down what the next step in the proof states in your own words:

Since the point *F* is the center of the circle *DKL*, therefore *FD* equals *FK*. But *FD* equals *A*, therefore *KF* also equals *A*.

Do you agree with this statement? Why? Why not?
Use GeoGebra to help you recognise whether and why KF = A.

In this step, we should comment on how a length is represented by a capital letter, *A*, whereas usually students are presented with line segments of given lengths, e.g. AB = 3cm. Students are expected to interpret and potentially use mathematical notation, which is a requirement a teacher can share with their students:

$FD = FK$

$FD = A$, so $FK = A$

Their arguments should involve the fact that *FD* and *FK* represent the radius of the circle *DKL* and since the radius of this circle was constructed to be equal to the given length of side *A* of the triangle, then all three lengths are equal, $FD = FK = A$.

Using GeoGebra for this step in the proof, students should validate the equalities as described in the written statement above by potentially dragging certain points in their construction and comparing the lengths of line segments in question. Students could argue about the validity of the statements based on their constructions and in essence argue about the validity of the statements by using ‘proof by construction’.

> **TASK 8c:**
>
> Write down what the next step in the proof states in your own words:
>
> Again, since the point *G* is the center of the circle *LKH*, therefore *GH* equals *GK*. But *GH* equals *C*, therefore *KG* also equals *C*.
>
> Do you agree with this statement? Why? Why not?
> Use GeoGebra to help you recognise whether and why KG = C.

Similarly to Task 8b, students could follow a similar process to derive that:

$GH = GK$

$GH = C$, so $KG = C$

Their arguments should involve the lengths of GH, GK and C, all being equal to the radius of the circle LKH and therefore $GH = GK = C$. Some students may notice in this statement the change in the order of the letters for representing the line segment GK.

As mentioned above regarding Task 8b, GeoGebra can support students’ thinking processes and argumentation in interpreting a mathematical statement presented in an original source. We believe though that using a combination of representations (i.e. the construction in GeoGebra and mathematical notation on paper) is crucial in understanding the statement, but also recognising the presentation of a formal proof. Writing down the actual equalities ($GH = GK$, $GH + C$ and $GH = GK = C$) could be re-enforced by the teacher as a valid strategy in developing students’ mathematical thinking. And even though GeoGebra is designed to bring together different forms of representations, students may not necessarily recognise this feature during their interactions as they are focusing on constructing triangles following the steps in the given proof.

> **TASK 8d:**
>
> Write down what the next step in the proof states in your own words:
>
> And *FG* also equals *B*, therefore the three straight lines *KF*, *FG*, and GK equal the three straight lines *A*, *B*, and C.
>
> Do you agree with this statement? Why? Why not?
> Use GeoGebra to help you recognise whether and why KF = A, FG = B and GK = C.

Students are expected to justify why:

$KF = A$

$FG = B$

$GK = C$.

GeoGebra can certainly help them visualise and test why these 3 sides are equal to the given length. Students could argue that the above mathematical statements are true by construction. But would that be enough? GeoGebra’s power lies in enabling students to vary the lengths of the three sides, and if these sides are linked to the constructed triangle, then students can see the impact of their dragging and changing the given lengths of the triangle. Students could be asked to choose a different set of lengths and go through the same process of constructing a triangle in GeoGebra to reflect upon the condition for a triangle to be formed.

TASK 8e:

Write down what the next step in the proof states in your own words:

Therefore out of the three straight lines *KF*, *FG*, and *GK*, which equal the three given straight lines *A*, *B*, and C, the triangle *KFG* has been constructed.

Q.E.F.

Do you agree with this statement? Why? Why not?

Depending on their constructions and provided that their chosen lengths meet the required condition, students should argue about their beliefs on the validity of this proof. They are prompted to use GeoGebra to argue about the constructions and instantly see the outcome of any action they take, e.g. choosing different lengths for the sides and investigating when a triangle ceases to exist. If they were to choose the three lengths and place them in a straight line (see Figure 4.6 below), then they would receive instant and accurate feedback from GeoGebra showing that the circle with centre *G* and radius *GH* is not big enough to intersect the circle with centre *F* and radius *DF*.

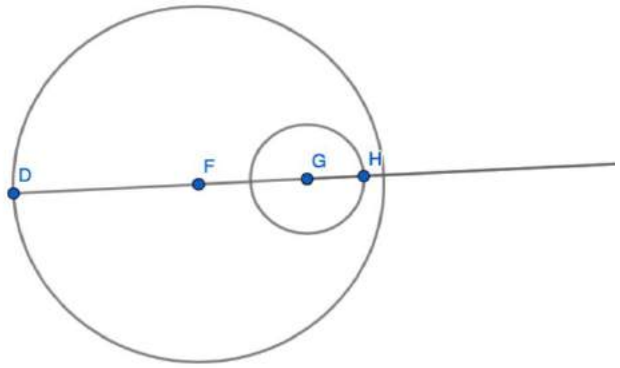


Figure 4.6: An attempt to construct a triangle using GeoGebra, when the sum of the 2 lengths FG and GH is less than the length of DF, i.e. |FG| + |GH| < |DF|.

Considering students' possible GeoGebra construction described earlier, students may find it easier to 'see' why the triangle ceases to exist when the sum of the lengths of the two sides is less than the length of the third side. For example, if they were to construct a triangle *ABC* (see Figure 4.7), where |*AB*| + |*B*'*C*| is less than |*AC*| by constructing the line segment *AC* and then two circles *C*(*A*, *AB*) and *C*(*C*, *B*'*C*), they could see that the two circles never intersect and the third vertex of the triangle cannot be constructed. Comparing these two constructions (the proof construction and the triangle construction) and how they could be presented and explored in GeoGebra by students, we could argue about the different level of difficulty with each one of them and how GeoGebra can make them more accessible to students and support students' valid deductive reasoning, as per Duval's (2000) theory. Considering the triangle construction in GeoGebra, students, for example, may keep the length of the sides *AC* and *B*'*C* constant and increase or decrease the radius *AB*. This exploratory process can help students recognise the importance of the triangle inequality as a condition for forming a triangle. Even though it cannot be considered as a formal proof, GeoGebra certainly supports students in getting a better understanding of Proposition 22 by allowing the creation of interactive diagrams (*multifunctional registers and discursive and non-discursive representations*) and promoting students to analyse, evaluate and confirm (or falsify) the derived results by carrying out various conversions (*discursive representations and multifunctional and monofunctional registers*) (Duval, 2000). Considering the proof construction in GeoGebra, students can compare the lengths while following carefully each step of the proof and convincing themselves of the equalities suggested by the formal proof.

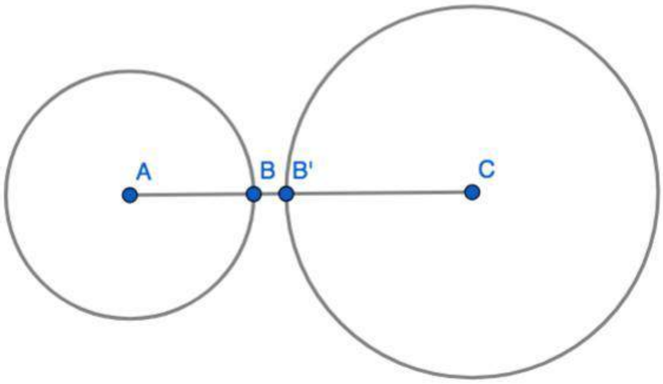


Figure 4.7: When |AB| + |BC| < |AC|, then no triangle could be formed.

Going back to the proof presented in the translation of Proposition 22 (see the presentations of Task 8 and Task 8e), it is worth referring to the three letters on the bottom right, “Q.E.F.”. These represent the Latin phrase “quod erat faciendum”, which means “that which was to be done”. In Euclid’s original source, as presented in Figure 4.1, this is written as “ὅπερ ἔδει ποιεῖσαι”, and which is commonly presented in other original sources as “ὅπερ ἔδει δεῖξαι” and which means “that which was to be shown” and which in essence means “that which was to be proved”. This was used at the end of a proof to indicate that the proof has been completed.

5 Concluding discussion

Through our outlined teaching scenario above, we believe to have shown how a use of digital technologies can assist in ‘unpacking’ an original source, and hence make this more accessible to potential students. Surely, our example of Euclid’s proposition 22 is not a long and comprehensive source, as those described by Barnett and colleagues (2014). Yet, the manageability of this limited excerpt from the Elements seems to serve well as an illustrative case for our line of argument in this particular paper. While our example seconds the claims of the previous studies considering the use of digital technologies in relation to history, as first laid out by Isoda (2000a), e.g. the benefits of multiple representations, support of students’ reflective thinking and mathematical inquiry, or those closer related to students’ concept formation (e.g. Chorlay, 2015), it does so by attempting to ground these arguments in the mathematics education literature. More precisely, we have attempted to articulate the aspects concerning multiple representations, and underlying concept formation, by using Duval’s framework of semiotic representations.

The aspects concerning proofs and proving in relation to students' reflective thinking is also attempted and articulated through a use of Duval, while that of students' mathematical inquiry is addressed through the rich literature on digital technologies in mathematics education. More precisely, we used the theory of instrumental genesis to consider and discuss students' potential instrumentation processes, i.e. how GeoGebra shaped and affected their thinking, and instrumentalisation processes, i.e. how their acquired knowledge through their preparatory work on paper and then their reflective tasks in GeoGebra may lead to a potentially different use of the GeoGebra tool that may allow them to critically reflect upon their own interpretations and understanding of Proposition 22. We used the theory of instrumental orchestration to analyse the teacher's potential intentions and aims for using GeoGebra for 'unpacking' Proposition 22. We discussed mainly the chosen didactical configurations and the rationale for those decisions, but also a teacher's strategies for exploiting the chosen didactical configurations (exploitation mode). Since this work is preparatory in its nature, we did not discuss the 'didactical performance', as this concerns the various decisions a teacher takes throughout a lesson aiming at using the digital tool in question as best and as effectively as possible. Finally, we touched upon the notion of instrumental distance (Haspekian, 2005; 2014) that allowed us to consider how a teacher through carefully designing a task sequence can take into consideration strategies to bridge the gap between students' learning, classroom and culture norms, and the norms involved when interacting with a digital tool, such as GeoGebra.

In terms of the claimed 'unpacking' of the original sources through digital technologies – or making it more accessible – GeoGebra can enable students to (a) explore statements, such as the two sentences in Proposition 22, but also statements shared by their peers and/or their teacher; (b) translate or convert the written statements in geometrical figures using the various GeoGebra features; (c) validate such statements by creating accurate constructions, comparing those constructions and using them as objects to think and test conjectures; (d) critically reflect upon mathematical statements through dynamic interactions with accurate constructions; (e) consider and prove Proposition 22 by construction.

Duval's framework of semiotic representations allowed us to take into account in the design of the above teaching scenario how students reach a good understanding of mathematical ideas, concepts and statements, such as that presented in Proposition 22. Students can be presented with and interact with two media, 'paper and pencil' and GeoGebra, and may be prompted through a task sequence to shift within and between registers. For example, for Task 8, students would have to recreate the static diagram presented in the original source in GeoGebra and therefore create a 'dynamic' construction involving line segments, circles, intersection points and of course a triangle (provided that the condition was met). They need to consider the mathematical statements presented by the original source in combination with the use of some symbolic language (mainly the use of letters to refer to certain parts of the diagram) and the static diagram and convert these to a dynamic geometrical figure constructed in GeoGebra, also annotated by letters. They could also convert from the semi-natural and semi-symbolic language to a series of mathematical statements presented in symbolic language, e.g. $FD = FK$ and $FD = A$, so $FK = A$. Critically reflecting upon these registers and shifting between them could certainly enhance students' understanding of Proposition 22, and support our argument for the value of combining paper and pencil resource with a digital tool to access, explore and even prove mathematical statements presented in an original source.

Although the use of GeoGebra throughout the duration of the teaching scenario serves a number of minor pragmatic purposes, the overall purposes are epistemic ones. In particular, GeoGebra serves the role of letting the students grasp the nature of the construction by dragging the GeoGebra construction (Task 5, Figure 4.5) and by relying on GeoGebra in the proof of the proposition (Task 8, Figures 4.6 and 4.7). The digital technology is used to more than just solve a mathematical task. It is used to deepen the understanding of the mathematical content of the original source. The combination of the original source and the digital tool seem to draw the use of the digital tool in an epistemic direction.

So, while digital technologies assist in making the original source more accessible to the students, the original source seems to ‘enforce’ upon the students an epistemic use of the digital technologies. This indeed appears to be a promising and positive synergy.

6. Future perspectives and questions to answer

As for the use of digital technologies in relation to the work with original sources, there are still many stones that are left unturned. If we agree to the seemingly large potential in this relationship, then several new questions arise: questions which need addressing in order to exploit the fruitful interplay between original sources and digital technologies further. We shall end this paper with outlining such questions or issues that appear central to us.

Firstly, it should be considered which original sources may benefit from which digital technologies. The ‘unpacking potential’ may certainly differ from technology to technology in relation to a given source. In our illustrative case a DGS was of course the obvious choice, since the source concerns plane geometry, whereas, say, a CAS tool would not have provided us with much assistance.

Secondly, we should ask ourselves which mathematics education frameworks may be applicable in our pursuit to describe the interplay between a use of original sources and digital technologies. In our illustrative case, we made use of Duval’s framework of semiotic registers to articulate potential benefits of register shifts, in particular conversions between mono- or multifunctional discursive registers and multifunctional non-discursive representations in the form of geometrical figures. But surely there are other mathematics education theoretical frameworks that would apply better to a different combination of sources and technologies. Say, for instance, we are working with a source where we only operate in the monofunctional discursive register, i.e. symbolic systems – this could be some symbolic proof in algebra, etc. Then Duval’s framework appears not to be the most suitable, since no conversions would take place. Similarly, in relation to the first question above, a DGS tool might not be so suitable with such a source, whereas a CAS tool might be.

Thirdly, as argued above, and as claimed in some of the available studies (Balsløv, 2018; Olsen & Thomsen, 2017), it appears that the combination of original sources not only seems to assist the students in their reading of the source, but that the presence of the source appears to draw the use of the technology in a more epistemic direction.

While the two first questions above require *a priori* theoretical analyses, in line with what has been presented in this paper, the third question or hypothesis calls for empirical investigations. Yet, if the hypothesis holds, i.e. if the study of original sources ‘enforces’ upon the students an epistemic use of the digital technology, then both knowing that this is so and exactly how this is, is not only a result that is of interest to the HPM community. While it appears common knowledge that the use of digital technologies in schools most often serves pragmatic purposes, it is well known (e.g. Artigue, 2010) that any use which is only, or mainly, pragmatic is of little – or even negative – educational value. Hence, if indeed a use of original sources fosters a positive educational effect on the use of digital technologies, then this would not only be a significant contribution to the field of HPM, but to the mathematics education research field at large.

REFERENCES

Aguilar, M. S., & Zavaleta, J. G. M. (2015). The difference as an analysis tool of the change of geometric magnitudes: the case of the circle. In E. Barbin, U. T. Jankvist, and T. H. Kjeldsen (Eds.): *History and Epistemology in Mathematics Education – Proceedings of the Seventh European Summer University* (pp. 391-399). Copenhagen: The Danish School of Education, Aarhus University.

Artigue, M. (2002) Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245–274.

Baki, A., & Guven, B. (2009). Khayyam with Cabri: experiences of pre-service mathematics teachers with Khayyam’s solution of cubic equation in dynamic geometry environment. *Teaching Mathematics and its Applications*, *28*(1), 1-9.

Balsløv, C. U. (2018). *The mutual benefits of using CAS and original sources in the teaching of mathematics*. Master's Thesis. Copenhagen: Danish School of Education, Aarhus University.

Barnett, J. H., Lodder, J., & Pengelley, D. (2014). The pedagogy of primary historical sources in mathematics: classroom practice meets theoretical frameworks. *Science & Education*, *23*(1), 7-28.

Bruneau, O. (2011). ICT and history of mathematics: the case of the pedal curves from the 17th-century to the 19th-century. In E. Barbin, M. Kronfeller, and C. Tzanakis (Eds.) *History and Epistemology in Mathematics Education Proceedings of the 6th European Summer University* (pp. 363–370). Vienna: Holzhausen Publishing Ltd.

Burke, M. J. & Burroughs, E. A (2009). Using CAS to solve classical mathematics problems. *Mathematics Teacher, 102*(9), 672-679.

Caglayan, G. (2016). Exploring the lunes of Hippocrates in a dynamic geometry environment. *Journal of the British Society for the History of Mathematics*, *31*(2), 144-153.

Chorlay, R. (2015). Making (more) sense of the derivative by combining historical sources and ICT. In E. Barbin, U. T. Jankvist, and T. H. Kjeldsen (Eds.): *History and Epistemology in Mathematics Education – Proceedings of the Seventh European Summer University* (pp. 485-498). Copenhagen: The Danish School of Education, Aarhus University.

Drijvers, P., Doorman, M., Boon, P., Reed, H. & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75(2)*, 213-234.

Drijvers, P., & Gravemeijer, K. (2005). Computer algebra as an instrument: Examples of algebraic schemes. In D. Guin, K. Ruthven, & L. Trouche (Eds.), The didactical challenge of symbolic calculators turning a computational device into a mathematical instrument (pp. 174–196). New York, NY: Springer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103-131.

Erbas, A. K. (2009). Reexamining Archimedes’ quadrature of the parabola with dynamic geometry.

Mathematics and Computer Education, *34*(2), 165-180.
Euclid (1956). *The thirteen books of Euclid's Elements*. (translated from the text of Heiberg, with introduction and commentary by Sir Thomas L. Heath. 2d ed., rev. with additions). New York: Dover Publications.
Fauvel, J. & van Maanen, J. (Eds.) (2000). *History in Mathematics Education, The ICMI Study*. Dordrecht: Kluwer Academic Publishers.
Glaubitz, M. R. (2011). The use of original sources in the classroom: empirical research findings. In E. Barbin, M. Kronfellner, and C. Tzanakis (Eds.): *History and Epistemology in Mathematics Education Proceedings of the 6th European Summer University* (pp. 351–362). Vienna: Holzhausen Publishing Ltd.
Guin, D. & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: the case of calculators. *International Journal of Computers for Mathematical Learning*, *3*(3), 195–227.
Haspekian, M. (2005). An "Instrument Approach" to study the integration of a computer tool into mathematics teaching: The case of spreadsheets. *International Journal of Computers for Mathematical Learning, 10(2)*, 109-141.
Haspekian, M. (2014). Teachers' instrumental geneses when integrating spreadsheet software. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), The mathematics teacher in the digital era: an international perspective on technology focused professional development (pp. 241– 276). Dordrecht: Springer.
Hong, Y. & Wang, X. (2015). Teaching the area of a circle from the perspective of HPM. In E. Barbin, U. T. Jankvist, and T. H. Kjeldsen (Eds.): *History and Epistemology in Mathematics Education – Proceedings of the Seventh European Summer University* (pp. 403-415). Copenhagen: The Danish School of Education, Aarhus University.
Isoda, M. (2000a). Inquiring mathematics with history and software. In J. Fauvel and J. van Maanen (Eds.), *History in mathematics education, The ICMI Study* (pp. 351–358). Dordrecht: Kluwer Academic Publishers.
Isoda, M. (2000b). The use of technology in teaching mathematics with history – teaching with modern technology inspired by the history of mathematics. In W.-S. Horng W.-S. and F.-L. Lin (Eds), *Proceedings of the HPM 2000 Conference: History in Mathematics Education: Challenges for a New Millennium*, vol 1, (pp. 27-34). Taipei: Department of Mathematics, National Taiwan Normal University.
Isoda, M. (2004). Why we use historical tools and computer software in mathematics education. In F. Furinghetti, S. Kaijser and C. Tzanakis (Eds.), Proceedings HPM2004 & ESU4, revised edition (pp. 229-236). Iraklion: University of Crete.
Jahnke, H. N. (2019). Hermeneutics and the question of "how is science possible?". In E. Barbin, U. T. Jankvist, T. H. Kjeldsen, B. Smestad, and C. Tzanakis (eds.), *Proceedings of the Eight European Summer University on the History and Epistemology in Mathematics Education* (ch. 1.1). Oslo: Oslo Metropolitan University.
Jahnke, H. N. et al. (2000). The use of original sources in the mathematics classroom. In: J. Fauvel & J. van Maanen (Eds.): *History in Mathematics Education, The ICMI Study* (pp. 291-238). Dordrecht: Kluwer Academic Publishers.
Jankvist, U. T. (2013). History, Application and Philosophy in mathematics education: HAPh – a use of primary sources. *Science & Education*. *22*(3), 635-656.
Jankvist, U. T. (2014). On the use of primary sources in the teaching and learning of mathematics. In M. R. Matthews (Ed.): *International Handbook on Research in History, Philosophy and Science Teaching, Vol. 2*, pp. 873-908. Dordrecht: Springer Publishers.
Jankvist, U. T., Clark, K. & Mosvold, R. (in review). Developing mathematical knowledge for teaching teachers: Potentials of history of mathematics in teacher educator training.
Jankvist, U. T., Misfeldt, M., & Aguilar, M. S. (2019). Tschirnhaus' transformation: mathematical proof, history and CAS. In E. Barbin, U. T. Jankvist, T. H. Kjeldsen, B. Smestad, and C. Tzanakis (eds.), *Proceedings of the Eight European Summer University on the History and Epistemology in Mathematics*

Education (ch. 2.15). Oslo: Oslo Metropolitan University.

Kidron, I. (2004). Polynomial approximation of functions: Historical perspective and new tools. *International Journal of Computers for Mathematical Learning, 3*(8), 299-331.

Lagrange, J.-B (2005). Curriculum, classroom practices, and tool design in the learning of functions through technology-aided experimental approaches. *International Journal of Computers for Mathematical Learning, 10,* 143-189.

Olsen, I. M., & Thomsen, M. (2017). *History of mathematics and ICT in mathematics education in primary education*. Master's Thesis. Copenhagen: Danish School of Education, Aarhus University.

Papadopoulos, I. (2014). How Archimedes helped students to unravel the mystery of the magical number pi. *Science & Education*, *23*(1), 61-77.

Sfard, A. (2008). *Thinking as Communicating*. Cambridge: Cambridge University Press.

Siu, M. K. (2011). 1607, a year of (some) significance: Translation of the first European text in mathematics – Elements – into Chinese. In E. Barbin, M. Kronfellner, and C. Tzanakis (Eds.): *History and Epistemology in Mathematics Education Proceedings of the 6th European Summer University* (pp. 373–389). Vienna: Holzhausen Publishing Ltd.

Thomsen, M. & Olsen, I. M. (2019). Original sources, ICT, and mathemacy. Poster presented in TWG-12, CERME-11, Utrecht.

Trouche, L. (2004). Managing the Complexity of Human Machine Interactions in Computerized Learning Environments: Guiding Students' Command process Through Instrumental Orchestrations. *International Journal of Computers for Mathematical Learning, 9*, 281–307.

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In: D. Guin, K. Ruthven & L. Trouche (Eds.), *The Didactical Challenge of Symbolic Calculators: Turning a Computational Device into a Mathematical Instrument* (pp. 197–230). New York, NY: Springer.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, *52*(2), 83–94.

Zengin, Y. (2018). Incorporating the dynamic mathematics software GeoGebra into a history of mathematics course. *International Journal of Mathematical Education in Science and Technology*, *49*(7), 1083-1098.

ON MATHEMATICAL REASONING

Arto MUTANEN
Finnish Naval Academy & Finnish National Defence University, Suomenlinna, Helsinki, Finland
arto.mutanen@gmail.com

ABSTRACT

Mathematics and logic are subjects of a special kind, which stems from their formal character. Basically, mathematics and logic are often viewed as empty formal manipulation of symbols. However, this opinion hides the constructive character of the topics. The constructive steps in mathematical and logical reasoning bring new information into the reasoning process which allows us to see mathematical and logical reasoning as dialog. Connecting this dialogic character to Platonic approach we can understand the dialog as foundation of education of mathematics. The approach of the paper is basically philosophical. However, we connect philosophical discussion to present-day pedagogical discussion in which dialog is taken seriously. We will show that there are interesting interconnection between (formal) mathematical reasoning and proper dialog.

1 Introduction

According to Popper (1979, p. 133), we may be interested in mathematics either by being interested in theorems or being interested in proofs. The first viewpoint emphasizes the truth or falsity of mathematical statement, and the latter emphasize the existence of proofs. Mathematical statements, as with all statements, are either true or false, and this in principle, can be listed.[1] The listing of mathematical truths, or given truths, makes mathematics static and, as such, does not teach mathematical reasoning. Reasoning is a factual process in time and space. Proofs express the reasoning process, and the search for proofs exemplifies mathematical reasoning. Hence, there might be some truth in a common opinion that says that we learn mathematical reasoning while learning mathematics.

In logic, mathematical proofs are defined as sequences of statements in which each statement is an axiom (or a premise) or is achieved from earlier statements by the application of an inference rule.[2] This kind of “statement view” of mathematical proofs, therefore, entails an opinion that mathematics is about manipulation of symbols. This opinion is supported by textbooks of mathematics in high school and in elementary school. At the same, this opinion hides that mathematical argumentation can be not only linguistic, but also visual (or pictorial). Therefore, there is a need to study mathematical reasoning more closely.

Mathematical reasoning is a more general kind of human reasoning, which is a kind of human mental process that takes place in someone’s mind or brain. Unfortunately (or fortunately), we do not have access to what happens inside someone’s mind or in someone’s brain.[3] Haack (1978, pp. 240–241) specifies the focus of logic as follows:

---

[1] In principle, a computer can list all the mathematical truths. However, it would take an infinite amount of time to do so. Mathematical truth has a complexity of Δ^1_1, which is the logical complexity of the mathematical truth that cannot be avoided. For details, see, for example, Mutanen 2004.
[2] See any textbook of logic to confirm this.
[3] Of course, present-day study of brain processes has access, but we will not consider this question here.

Thoughts that are in someone's mind are something subjective, and we have no access to them. The second possibility is that logic would be concerned with propositions that are objective but, as the philosophy of logic shows, they are not accessible and hence cannot be the topic of logic. Therefore, the only possibility in logic is to focus on sentences, i.e., syntactical expressions that are both objective and accessible. This separates logic from psychologism, in which logic was understood as a theory of how human reasoning works.

The discussion of how logic is related to human reasoning is still reasonable, even if psychologism is not a serious possibility. In the philosophy of logic, the relationship between logic and reasoning (or thinking) is basically understood in three different ways: (i) logic describes mental processes (strong psychologism); (ii) logic prescribes mental processes (weak psychologism); or (iii) logic has nothing to do with mental processes (anti-psychologism) (Haack 1978, p.238). According to Haack, Kant represents opinion (i), Peirce represents (ii), and Frege represents (iii). In fact, in the late nineteenth century, especially in Germany, there was strong debate concerning whether logic is descriptive, i.e., whether strong psychologism was true. Echoes of the debate can still be recognized in our understanding of mathematics and logic. Anti-psychologism can be understood as a reaction against strong psychologism. The idea was to develop logic as a formal science. For example, Frege understood logic as *lingua characteristica* and, hence, not formal science in a present-day sense. However, this Fregean understanding is a version of anti-psychologism (Haaparanta 1985). The distinction between strong psychologism and anti-psychologism is not very fruitful, since it seems to be obvious that strong psychologism is not true. However, all the same, it seems evident that logic has something to do with human mental processes (reasoning), and hence anti-psychologism also seems to be wrong (Haack 1978). The discussion of the nature of logic and mathematics has not been restricted to discussion of psychologism and its alternatives; it also includes the relationship between logic (and mathematics) and the sciences. For example, Russell (1903) said that mathematics is a science like other (experimental) sciences, such as zoology; of course, mathematics is more abstract than the other sciences.

Mathematics is very special kind of science, and hence, mathematical reasoning is not easy to characterize. To start, let us take a look at, for example, axioms of Peano arithmetic, which contains four sentences and one sentence schema. According to the basic idea of axiomatization, they say everything about the topic: Let Ω be a set of axioms of Peano arithmetic. Then, the Peano arithmetic (PA) is the set of theorems of the set Ω of axioms that can be expressed $PA = \{\varphi: \Omega \vdash \varphi\}$. However, this does not tell us anything substantial about mathematical reasoning. Mathematical reasoning is coded into the formula "$\Omega \vdash \varphi$" which means that for a given φ, there is a finite sequence $\varphi_1, \ldots, \varphi_n$ such that $\varphi_\mathrm{n} = \varphi$ and for all $i < n$, $\varphi_i \,\varepsilon\, \Omega$ or is achieved from $\varphi_1, \ldots, \varphi_{i\text{-}1}$ by application of an inference rule. This shows the formal character of logical and mathematical inference.

When Hermann Weyl (1956, p. 1832) says that "mathematicians are no Ku Klux Klan with a secret ritual of thinking," he is intending to denote the fact that mathematical reasoning is something public and "objectively" recognizable that can be achieved if attention is focused on linguistic expressions. According to Haack (1978, p. 239) "logic is primarily concerned with *arguments*." Arguments are linguistically expressed formal structures whose strength is of logical interest. Arguments have a dual structure: a set of premises and conclusions inferred, according to inference rules, from the premises as expressed above. In logic, argumentation is analyzed and evaluated; the strongest argument is one in which the relationship between premises and conclusion is deductive.

Arguments in mathematics and in logic are formal and well-structured, which makes them explicit and transparent; from the argument, anyone can see all the information used in the argument. However, as formal structures or deductions do not constitute reasoning, the explicitness and transparency of argument does not make mathematical reasoning similarly explicit and transparent. As Herman Weyl (1956, p. 1832) says, nobody should expect him "to describe the mathematical way of thinking much more clearly than one can describe, say, the democratic way of life." We may not mystify mathematical reasoning – or the mathematical way of thinking.

Mathematics and logic must not be confused with empirical research into human reasoning, even if there is some (external) connection between the two. Rather, in mathematics and logic, the question "What is mathematical reasoning?" seeks a normative answer. In the study of mathematics education, the focus is on learning mathematics: Questions like "How can one learn mathematics?", "What kind of learning strategies do students have?", and "How can we teach mathematics effectively?" are important. Hence, central problems in mathematics education consider the relationship, for example, between mathematical concepts and psychology (Ben-Hur, 2006), or between reasoning and communication (Berinderjeet & Toh, 2012). We are here, basically interested in mathematical reasoning as part of mathematics and logic itself which has interesting consequences to the education of mathematics.

2 About Logic and Mathematics

To understand mathematical reasoning better, we will consider more closely some aspects of logic. Basically, logic (and mathematics) can be seen from two different points of view: Logic and mathematics consist in some factual inferences and calculations, which are the everyday practice of mathematicians and logicians. Often, exercises in school mathematics are focused on this area. Let us call this "micrologic." On the other hand, the focus might be on the consideration of mathematical and logical reasoning from an "external" perspective. Questions like whether mathematics or logic are decidable, i.e., whether mathematics or logic have a decision method. It is well known that mathematics (any system that contains elementary arithmetic) is not decidable, and it is well known that, for example, sentence calculus and elementary geometry are decidable. Another example of such an "external" perspective is to consider the kinds of model that theories (i.e., sets of sentences) have. The well-known *Löwenheim–Skolem theorem* says that each theory that has an infinite model also has a denumerable model. So, the theory of the reals, which is known to be uncountable, has a denumerable model. These are metatheorems that characterize mathematics and logic from the "macro level."

The internal point of view of logical and mathematical reasoning shows how to do mathematics and logic, that is, how to prove mathematical and logical results, which is emphasized by Weyl (1956). In school mathematics and logic, this aspect is emphasized. For example, Usiskin (2015) shows that this kind of logic has several interesting aspects that are essential in understanding mathematics, and in teaching and learning mathematics. Unfortunately, formal theorems do not show how to find proofs or how to construct proofs. Maybe this is a reason why mathematics remains such a remote and difficult topic in schools.

Besides mastery of formulating proofs and calculations, we need some general understanding of what mathematics as a whole is, which is the subject of metamathematics and metalogic. Therefore, it is not enough that one can answer mathematical questions, but one has to understand what kinds of questions are mathematical. Unfortunately, as *Gödel's incompleteness theorem* (1931) shows[4], not all mathematical questions are answerable within mathematics.

Why it is not enough that one can just answer mathematical questions? It is obvious – based on school mathematics – that mathematics means answering given mathematical questions, and that all the questions have a correct and true answer. In the case of applications of mathematics, like physics in schools, the problem is not to understand mathematics but to understand physics; so, in applications, mathematics is just a tool that is used. There is no need to understand mathematics or mathematical reasoning.

The need for metatheoretical logic become evident when we speak about the character of mathematical reasoning. Of course, the practice of mathematical reasoning lies in proving theorems and single computations, but all this does not characterize the foundational character of mathematical reasoning. The metalogic is, by definition, a key to understanding the foundations of mathematics, as the famous metalogical results (like the theorems of Löwenheim and Skolem, of Gödel, or of Tarski) demonstrate. These metalogical results give information about mathematical reasoning, and about mathematics more generally.

Metalogical results, at the same, give important information about the methodology of science. In fact, this allows us to see the connection between logic and metalogic (Hendricks, 2007; Shapiro, 2002), which deepens our understanding of mathematics and mathematical reasoning (Usiskin, 2015). Metalogical knowledge also deepens our pedagogical understanding; it helps us to develop the teaching methods of mathematics, but also of the natural sciences (Sieg, 2002; Koponen & Kokkonen, 2014). Metalogic is an important branch of mathematical study that has great theoretical importance in understanding mathematics and methodology of science. For example, the metalogic allows us to analyze the reasonability of structuralism in the philosophy of science: in structuralism, the intention is to generate a metalogical framework without using explicit logic.

3 Mathematical Reasoning

Neither the "micrologic" nor "macrologic" characterized above give us a good understanding how to reason logically or mathematically. This can be seen if we consider more closely how to construct mathematical and logical arguments. Geometry is an excellent example in which mathematical reasoning becomes actual, as the presentation by "Capone, Del Sorbo, Ninni, Fiore & Adesso" at ESU-8 clearly demonstrated. The very idea of geometrical proof is its constructive character, which becomes evident via the pictorial nature of the proofs. In geometry, there is a long tradition of using pictures in the proofs. The pictures and the auxiliary constructions are essential parts of geometrical proofs. The proofs are demonstrative in the sense that the fact to be proved can be seen from the picture constructed by the proof. As the presentation referred to showed, there are

[4] Any consistent formal system *F* within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of *F* that can neither be proved nor disproved in *F* (Raatikainen, 2018).

several different constructions that can be created to prove even a very simple geometrical statement, like the Pythagorean theorem.

Geometrical constructions bring new information into the reasoning process. The information is formulated in pictorial form. In logic the proofs are constructed symbolically (or linguistically). However, the similar increase of information as new geometrical constructions do in geometry can be achieved by instantiation of new individuals. As Hintikka (1973, pp.188-190) shows there is precise measure for the information which can be used to characterize the depth of the given proof. This is connected to aesthetic value of the mathematics (Sinclair 2011). Mathematics uses both pictorial and symbolic argumentation (De Toffioli, 2017; Hintikka & Remes, 1974), but also even bodily argumentation as Sinclair (2011) refers. This shows the importance of understanding of the character of mathematical reasoning. To develop education of mathematics one need to understand the multiplicity of mathematical reasoning.

Unfortunately, there are some restrictions in generating the proofs. For example, there is no effective procedure to find out the best construction for a given proof. However, as Michie (1961) shows, there can also be syntactical proofs for geometrical theorems. Michie refers to the fact that a computer discovered a new proof for a simple geometrical statement.[5] Now we know that geometry can be expressed as an axiomatic syntactical theory that does not need pictorial arguments (Tarski 1968).

Frege explicitly separated axioms from rules of inferences, which is one of the first explicit formulations of the present-day understanding of logic. The formal way to explicate logic (and mathematics) is further developed, for example, by Hilbert. “Hilbert’s program” is an overall metalogical approach in which Hilbert tried to explicate the very character of mathematical and logical reasoning. The use of formal methods made it possible to generate explicit metalogic that studies logic within logic itself, as Gödel’s proof (1931) demonstrates. However. metalogic is not only a specific area in mathematics and logic; several very common results are, in fact, metalogical. (See Quine, 1981; Kneale & Kneale, 1962; Mancosu, 2010.)

Hilbert distinguishes the following three levels of mathematics: (i) ordinary mathematics; (ii) proper mathematics (mathematics in strict sense); and (iii) metamathematics. The intention was not to generate different kinds of reasoning, but these are, as Hilbert says, “the familiar modes of logical inferences” (Mancosu, 2010). This may be a reason why in school mathematics, there is only one kind of reasoning. However, metamathematical results tell us about mathematics – what can be done and what cannot be done. For example, Gödel’s incompleteness theorem tells us that there cannot be a general method to check computer programs (if the programming language is as complex as elementary arithmetic).

4 Argumentation as Computation

Arguments are basically sequences of sentences. The end point of an argument is called its conclusion, and the foundational sentences are called premises. The evaluation is to consider the logical relationship between the conclusion and premises. If the conclusion logically follows, or if the conclusion can be deduced, from the premises, then the

[5] Michie (1961) said that even if the proof was a new one for them, it had already been generated by a human mathematician.

argument is valid. Deduction is a syntactical process, and validity, instead, is a semantical notion. According to Gödel's completeness theorem (1930), a given sentence is deducible if and only if it is valid.

To say an argument is valid means that if the premises are true, then the conclusion must be true. Therefore, logic preserves truth. The sequence of sentences does not presuppose any agent who argues or infers them. Hence, logic is non-personal or objective. Therefore, it is interesting that Hodges (1977, p36) gives the following personal characterization:

> An *argument*, in the sense that concerns us here, is what a person produces when he or she makes a statement and gives reasons for believing the statement. The statement itself is called the *conclusion* of the argument (through it can perfectly well come at the beginning); the stated reasons for believing the conclusion are called *premises*. A person who presents or accepts an argument is said to *deduce* or *infer* its conclusion from its premises.

According to Hodges, argument is related to argumentation. However, in logic, the study of argument is a study of the logical relationship between premises and conclusion. Syntactically, a central problem is to make a deduction from the premises of the conclusion. If one can find a deduction, then the deductive relationship is demonstrated, but if one does not find a deduction, this does not show that there is no such deduction. In fact, Gödel's (and Turing's, among others) achievement was to formalize logical inference such that it is possible to prove that it is not possible to find a deduction (incompleteness theorem). A foundational question behind Turing's, Gödel's, and Church's work was "What is an effectively calculable function?" The history of mathematics provides excellent examples of algorithms that show how to compute certain functions. Moreover, there was a certain consensus about what computability means – the consensus gave an "intuitive" notion of computability, which was not well-specified. However, the need for a formal definition becomes evident, at least partly because of the search for an answer to the famous open problems in mathematics formulated by Hilbert.

The computational approach and logical approach have different roots. In logic, the historic roots are connected to the tradition of a universal language, which had historical advocates like Raymond Lull, Leibniz, and Frege (Kneale & Kneale, 1962). The computational tradition is connected to the algorithmic tradition, which also has its roots in the history of mathematics, Rogers (1967, p. 29) gives examples of well-specified algorithms from Ancient Greece.[6] A specific mathematical study of algorithms was started in the ninth century by the Persian mathematician al-Khowrazmi (Russell & Norvig, 1995, p. 8). Rogers (1967, p. 1) gives a good intuitive characterization of an algorithm by saying that it is "a clerical (i.e., deterministic, book-keeping) procedure".

The idea of formalizing the notion of computation was to formalize deduction such that it becomes a mechanical process that does not presuppose intellect. Turing, in particular, used very attractive language. For example, he suggested to "compare a man in the process of computing a real number to a machine." Turing's intention was to formulate a mechanical computing machine that computes essentially the same way as a human using a paper and pen. The resulting notion of computation was of epistemic character. Being mechanical, it was also independent of the formalism chosen. This was recognized by

[6] Rogers's examples are Eratosthenes's method and the Euclidean algorithm.

Gödel (1946) when he said that the importance of the explication of the notion was that it had “for the first time succeeded in giving an absolute definition of an interesting epistemological notion” (Davis 1965, p. 84).

The identification of the notion of computability with Turing machine computability is known as the *Church–Turing thesis* (Copeland, 2017). There are several formulations of the notion of computation, but they have all been proved to capture the same class of functions. Of course, there are formulations of the notion of computation that extend the computing power of Turing machines, which are both logically and philosophically interesting (Hintikka & Mutanen, 1998; Syropoulos, 2018).

The theory of computation is very abstract and extremely complex field. Still it has deep pedagogical significance. Computational approach emphasizes both agenthood of a learner and process of learning (Hendricks, 2007; Hendricks & Symmons, 2015; Mutanen, 2004). So, there are interesting connections between the computational approach and different kinds of constructive approaches.

In logical reasoning, the intention is to explicate and to make transparent the reasoning. The syntactical formulations, as Frege says, bring “to light every axiom, assumption, hypothesis or whatever else you want to call it on which a proof rests; in this way we obtain a basis for judging the epistemological nature of the theorem” (as quoted in Sieg, 2002, p. 228). The computational approach emphasizes more explicitly the methodical and epistemological aspects of mathematical and logical reasoning that were emphasized by Gödel, as the quotation above shows.

5 Mathematical and Logical Reasoning

Formal arguments are static; hence they do not give an adequate characterization of reasoning. A computational approach brings dynamics into the picture. However, the agent is still missing. Computations are nonpersonal, formal algorithmic processes, even if, as Turing’s example shows, computations allow us to consider agenthood. We have seen that geometrical reasoning is a paradigmatic example of mathematical and logical reasoning. First, its strict logical structure is clear. Second, its pictorial character makes the reasoning process informative.

The logical, or arithmetic, approach has a more formal character. It appears as formal manipulation of symbols and formulas. However, this is not the whole truth. As we have seen, geometry can be seen as a formal syntactical theory, just like any other formal logical theory. On the other hand, logical and arithmetical reasoning can be interpreted similarly to geometrical reasoning. Kant speaks of intuition in mathematics, referring to the use of individuals. According to him, in arithmetical reasoning, the intuitive step is to use singular numbers, which is usual in arithmetic (Hintikka, 1973).

This can be generalized such that the use of the existential instantiation rule is a Kantian intuitive step in reasoning. We can further analyses this as separating instantiation of the “dummy name” and real name, in which the first is a formal stem and the second is a substantial step in the reasoning process that brings ne substantial information into the reasoning process (Hintikka, 2007). A similar thing can be seen in mathematics, where we have ε-γ-argumentation, which is usually read as “for a given $\varepsilon > 0$, a $\gamma > 0$ can be found such that …” There, the constructive or substantial step is “can be found,” which shows how mathematical reasoning brings new information into the reasoning process by using Kantian intuition.

This shows that pictures and substantial intuition are part and parcel of mathematical and logical reasoning (De Toffoli, 2017; Bråting, 2012). This is an important observation. Pictures and other intuitive steps are substantial steps in mathematical reasoning that convey the information needed to make the inferences needed to complete the intended proof. This is an important observation both because of logic (Hintikka & Remes, 1974) and because of pedagogy (Hintikka, 1982; Plato). This shows how important it is to analyze logical and mathematical reasoning.

In *Meno*, Plato shows how to have a logico-pedagogical dialog. In *Meno*, the dialog is logically strict and pedagogically motivated. The dialog shows the power of logico-pedagogical dialog. Hintikka has generated this approach in such a way that it can be applied to scientific reasoning (Hintikka, 2007), to pedagogy (Hintikka, 1982), and to general human reasoning (Hintikka, Halonen, & Mutanen, 2002). The approach has a firm logical basis (Hintikka & Remes, 1974; De Toffoli, 2017).

The logico-pedagogical dialog belongs to a more general dialogical tradition in science, which can be contrasted with the formal-scientific tradition in science. These two traditions have different roots: the first is connected to the Platonic tradition, and the second is connected to the Parmenidean tradition (Mutanen, 2018). The present-day science and pedagogy are basically seen as separate approaches, which is connected to the Parmenidean tradition. In the Platonic tradition, science and pedagogy are essentially connected, which entails that scientific research, as such, is a dialogical process, which is exemplified in *Meno*. The Parmenidean tradition emphasizes that truth, as such, is the goal of scientific research. The pedagogical understanding is something external to the scientific research.

Dialog has been used more generally in pedagogical literature, which enriches pedagogical and scientific understanding. Dialog may be connected to science and pedagogy in different ways. On one hand, dialog has been connected to the methodology of science and mathematics, as in the works of Hintikka and De Toffioli referred to above. On the other hand, dialog has been understood more generally and the dialog has been connected to dialog within a classroom (Bråting, 2012) or to more general dialog (Radford, 2011). Both enrich our understanding of science and the pedagogy of science.

We have considered dialog from two different points of views. First, we looked at the language and recognized that in mathematics argumentation is based on different kinds of notions (symbolic (or linguistic), pictorial, and bodily). Dialog is based on these different kinds of notions. Proofs are formal expressions of the dialog in this sense which De Toffoli (2017) shows. Second, dialog can be understood as part of general narration as, for example, Burton (2012) and Radford (2011) show. However, to develop education of mathematics these two must be unified which can be done using different kinds of approaches. The notions of information and especially, understanding play central role. The most clear-cut example of the unification is Plato's *Meno*. However, this can be done also within context of modern (formal) logic (Hintikka, 1982).

6 Closing Words

We have seen that mathematics and logic can be, and usually have been, understood as formal sciences, which is well justified: mathematics and logic are formal sciences. Metaresults give some formal restrictions that must be recognized. However, the formal character of mathematics and logic does not entail that there is nothing to be understood in mathematics and logic. The formal character entails that the content is “thin.” However, mathematics and logic are constructive sciences in a proper sense, which entails that the proofs, as such, provide the keys to understanding. However, we must emphasize the pedagogical role of the construction of proofs. This can be done by dialogical methods, which has been increasing in the present-day study of pedagogy of mathematics and logic. This pedagogical approach opens new ways to understand mathematics and logic, and their pedagogy.

REFERENCES

Aczel, Amir D. (2006). *The Artist and the Mathematician: The Story of Nicholas Bourbaki, the Genius Mathematician Who Never Existed*, NY: Thunder Mouth Press

Ben-Hur, M. (2006). *Concept-Rich Mathematics Instruction: Building a Strong Foundation for Reasoning and Problem Solving*, Alexandria: ASCD

Berinderjeet, K & Toh, T. L. (2012). *Reasoning, Communication and Connections in Mathematics: Yearbook 2012*, Association of Mathematics Educators, Singapore: World Scientific

Blakemore, C. & Greenfield, S. (Eds.) (1987). *Mindwaves: Thoughts on Intelligence, Identity, and Consciousness*, Oxford: Basil Blackwell.

Boyer, Carl B. (1991/1968). *A History of Mathematics* (Second Edition), NY: John Wiley & Sons.

Brodie K. (2010). Classroom Practices for Teaching and Learning Mathematical Reasoning. In: Brodie K. (eds.) *Teaching Mathematical Reasoning in Secondary School Classrooms*. Boston, MA: Springer.

Bråting, K., (2012). Visualizations and Intuitive Reasoning in Mathematics, *The Mathematics Enthusiast*, vol. 9(1), http://uu.diva-portal.org/smash/get/diva2:447773/FULLTEXT01.pdf. (Read 28.8.2018.)

Burton, L. (2012). The Implications of a Narrative Approach to the Learning Mathematics, in Burton, L. (ed.), Learning Mathematics: From Hierarchies to Networks (pp. 21-35). Routledge, ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/helsinki-ebooks/detail.action?docID=16529 (accessed 27/2/2019.)

Copeland, B. J. (2012). *Turing: Pioneer of the Information Age*, Oxford: Oxford University Press

Copeland, B. J. (2017). The Church-Turing Thesis. In *The Stanford Encyclopedia of Philosophy* (Winter 2017 Edition), E. N. Zalta (Ed.), URL = <https://plato.stanford.edu/archives/win2017/entries/church-turing/>.

Cutland, N. J. (1980). *Computability*, Cambridge: Cambridge University Press

Davis, M. (1965). *The Undecidable: Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions*, New York: Raven.

De Toffoli, S. (2017). ‘Chasing’ the Diagram – the Use of Visualizations in Algebraic Reasoning, *The Review of Symbolic Logic*, *10*(1), 158-186.

Dreyfus, H. (2006). How Far is Distance Learning from Education?, in Evan Selinger and Robert P. Crease (Eds.), *The Philosophy of Expertise* (pp. 196-212). New York: Columbia University Press.

Edwards, H. M. (2010). Kronecker’s Algorithmic Mathematics, in Mircea Pitici (ed.), *The Best Writings on Mathematics*, Princeton: Princeton University Press

Geach, P. (1979). *Reason and Argument*, Oxford: Basil Blackwell.

Haack, S. (1978). *Philosophy of Logic*, Cambridge: Cambridge University Press

Haaparanta, L. (1985). *Frege’s Doctrine of Being*, Acta Philosophica Fennica. Helsinki: Ylopistopaino.

Hendricks, V. F. (2007), *Mainstream and Formal Epistemology*, New York: Cambridge University Press.

Hendricks, V. F. & Symons, J. (2015). Epistemic Logic. In *The Stanford Encyclopedia of Philosophy* (Fall 2015 Edition), E. N. Zalta (Ed.), URL = <https://plato.stanford.edu/archives/fall2015/entries/logic-epistemic/>.

Hintikka, J. (1973). *Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic*, Oxford: Clarendon Press.

Hintikka, J. (1982). Dialogical Model of Teaching, *Synthese 47*, 39-59.

Hintikka, J. (2007). *Socratic Epistemology: Explorations of Knowledge-Seeking by Questioning*, Cambridge University Press

Hintikka, J. (2011). What is the Axiomatic Method?, *Synthèse 183*, 69-85.

Hintikka, J. (2012). Method of Analysis: A Paradigm of Mathematical Reasoning?, *History and Philosophy of Logic*, *33*(1), 47-67.

Hintikka, J., Halonen, I. & Mutanen, A. (2002). Interrogative Logic as a General Theory of Reasoning, in Johnson R. H. and Woods, J., (eds.), *Handbook of Practical Reasoning*, Dordrecht: Kluwer Academic.

Hintikka, J. & Mutanen, A. (1998). An Alternative Concept of Computability. In J. Hintikka (Ed.), *Language, Truth and Logic in Mathematics* (pp. 174-188). Dordrecht: Kluwer Academic.

Hintikka, J. & Remes, U. (1974). *The Method of Analysis: Its Geometrical Origin and Its General Significance*. Dordrecht: D. Reidel Publishing Company.

Hodges, W. (2001) *Logic: An Introduction to Elementary Logic* (Second Edition), London: Penguin Books

Horsten, L. (2018). Philosophy of Mathematics. In *The Stanford Encyclopedia of Philosophy* (Spring 2018 Edition), E. N. Zalta (Ed.), URL = <https://plato.stanford.edu/archives/spr2018/entries/philosophy-mathematics/>. (accessed 26/10/.2018).

Jacquette, D. (Ed.) (2002). *Philosophy of Mathematics: An Anthology*. Oxford: Blackwell Publishers.

Jacquette, D. (2002a). Introduction: Mathematics and Philosophy of Mathematics. In Jaquette, 2002.

Kneale, W. & Kneale, M. (1962). *The Development of Logic*, Oxford: Oxford University Press

Koponen, I. & Kokkonen, T. (2014). A systemic view of the learning and differentiation of scientific concepts: The case of electric current and voltage revisited. *Frontline Learning Research*, *4*, 140-166. DOI: 10.14786/flr.v2i2.120

Mancosu, P. (2010). Hilbert and Bernays on Metamathematics. In P. Mancosu, (Ed.), *The Adventure of Reason: Interplay Between Philosophy of Mathematics and Mathematical Logic, 1900–1940*, Oxford: Oxford University Press.

Michie, D. (1961). Trial and Error. In *Science Survey*, part 2 (pp. 129-145). Harmondsworth: Penguin.

Mutanen, A. (2004). *From Computation to Truth via Learning*. Helsinki: Helsinki University Printing House

Mutanen, A. (2018). Constructive Methods in Economics, *Synthesis Philosophica* (forthcoming)

Nagel, E. & Newman, J. R. (1989). *Gödel's Proof*. London: Routledge

Noss, R. & Hoyles, C. (2006). Exploring Mathematics through Construction and Collaboration, in Sawyer 2006.

Nouwen, R., Brasoveanu, A., van Eijck, J. & Visser, A. (2016). Dynamic Semantics. In *The Stanford Encyclopedia of Philosophy* (Winter 2016 Edition), E. N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/win2016/entries/dynamic-semantics/>.

Plato. *Meno*, http://classics.mit.edu/Plato/meno.html (accessed 4/11/2018)

Poincaré, H. (1989). On the Nature of Mathematical Reasoning. In P. Benacerraf & H. Putnam (Eds.), *Philosophy of Mathematics: Selected Readings*. New York: Cambridge University Press.

Popper, Karl, 1979, *Objective Knowledge: An Evolutionary Approach*, revised edition, Oxford: The Clarendon Press.

Quine, W. V. (1981). *Mathematical Logic* (Revised Edition), London: Harvard University Press [Originally published 1940.]

Quine, W. V. (1986). *Philosophy of Logic*, Cambridge: Harvard University Press.

Raatikainen, P. (2018). "Gödel's Incompleteness Theorems". In T*he Stanford Encyclopedia of Philosophy* (Fall 2018 Edition), E. N. Zalta (Ed.), URL = <https://plato.stanford.edu/archives/fall2018/entries/goedel-

incompleteness/>.
Radford, L., (2011). Dialogism in Absentia or the language of Mathematics. In Silvia Sbaragli (Ed.), *La Mathematica e la sua Didattica Quarant'anni di Impegno; Mathematics and its Didactics: Forty Years of Commitment. In Occasion of the 65 Years of Bruno D'Amore*. Bologna: Pitagora Editirice.
Ray, Y. (2007). A Critique of a Formalist-Mechanist Version of the Justification of Arguments in Mathematicians' Proof Practices, *Philosophia Mathematica 15*(3), 291-320.
Robinson, Abraham, 1969, *Metaphysics of the Calculus*. In Hintikka, J. (Ed.), *The Philosophy of Mathematics* (pp. 153-163). Oxford: Oxford University Press.
Rogers, H. (1967). *Theory of Recursive Functions and Effective Computability*, Cambridge: MIT Press.
Russell, S. & Norvig, P. (1995). *Artificial Intelligence: A Modern Approach* (Second Edition), New Jersey: Pearson Education International.
Sawyer, K. R. (Ed.) (2006). *The Cambridge Handbook of the Learning Sciences*, Cambridge: Cambridge University Press.
Scardamalia, M. and Bereiter, C. (2006). Knowledge Building: Theory, Pedagogy, and Technology. In Sawyer 2006.
Shafer, G. (2001). Nature's possibilities and Expectations. In V. F. Hendricks, et. al. (Eds.), *Probability Theory: Philosophy, Recent History and Relations to Science*, Dordrecht: Kluwer Academic Publishers.
Shapiro, S. (2002). *Philosophy of Mathematics: Structure and Ontology*, New York: Oxford University Press.
Shapiro, S. & Kouri Kissel, T. (2018). Classical Logic. In *The Stanford Encyclopedia of Philosophy* (Spring 2018 Edition), E. N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/spr2018/entries/logic-classical/> (accessed 4/7/2018).
Sieg, W. (2002). Mechanical Procedures and Mathematical Experience. In Jaquette 2002.
Sinclair, N. (2011). Aesthetics as a Liberating Force in Mathematics Education, in Pitici, M (Ed.). The Best Writings on Mathematics 2010 (pp. 206-235). Princeton: Princeton University Press.
Suppes, P. (1969). The Psychological Foundations of Mathematics, in Patrick Suppes (Ed.), *Studies in the Methodology and Foundations of Science: Selected Papers from 1951 to 1969*, Dordrecht: D. Reidel. [Originally in *Les Modeles et la Formalisation du Comportement*, 1967.]
Syropoulos, A. (2018). On TAE Machines and Their Computational Power, *Logica Universalis*, https://doi.org/10.1007/s11787-018-0196-5 (accessed 4/11/2018).
Tarski, A. 1969, *What Is Elementary Geometry?*. In Hintikka, J. (Ed.), *The Philosophy of Mathematics*. Oxford: Oxford University Press, pp. 164-175.
Terrace, H. (1987). Thought Without Words. In Blakemore and Greenfield 1987
Usiskin Z. (2015). What Does It Mean to Understand Some Mathematics?, In S. Cho (Ed.) *Selected Regular Lectures from the 12th International Congress on Mathematical Education*. Cham: Springer
Vaina, L. & Hintikka, J. (eds.) (1984) *Cognitive Constraints on Communication: Representations and Processes*, Dordrecht: D. Reidel.
van Heijenoort, J. (1967). *From Frege to Godel: A Source Book in Mathematical Logic, 1879-1931*. London: Harvard University Press.
Weyl, H. (1956/1951) The Mathematical Way of Thinking, in Newman, J. R. (ed.). World of Mathematics (pp. 1832-1849). New York: Simon and Schuster.

PHILOSOPHICAL AND DIDACTIC PRACTICE IN THE UNIVERSE OF FRACTIONS

Trace and Icon

Petronilla BONISSONI[1], Marina CAZZOLA[2], Paolo LONGONI[3], Gianstefano RIVA[4], Ernesto ROTTOLI[5], Sonia SORGATO[6]
Gruppo di Ricerca sull'insegnamento della matematica,
Università Milano Bicocca, Milano, Italy
[1]nillarock@alice.it, [2]marina.cazzola@unimib.it, [3]plong172@gmail.com,
[4]gianstefano.riva@gmail.com, [5]ernerott@tin.it, [6]sonia.sorgato@unimib.it

ABSTRACT

The reflection on the long persistence of unsatisfactory results, has led us to upset the most common idea of fraction: fraction-of-something. To this, we have: (a) corrected the "primitive intuition" of fraction by constructing an "intuitive representation", (b) set the familiarization of children with fractions as a goal, and (c) practiced the idea of fraction as megaconcept. In our enquiring we have focused on the Pythagorean statement: the comparison is a logos. This latter is like a modern act of mathematisation of the comparison and it is the starting point of our didactic practice. It has the following features: (a) it is imposed, (b) it is a leap, (c) it is elementary, (d) it is an axios, (e) it has the characteristic of presence/absence along the path towards the megaconcept.

KEY WORDS: Familiarization – Megaconcept – Trace – Icon – Plurality of truth.

1 Introduction

This reflection concerns the didactics of fractions in primary school. Our attention to this topic has been attracted by the observation of the long persistence of unsatisfactory results in the teaching and learning fractions; persistence that is widely recalled in the scientific literature and which continues, notwithstanding the efforts over decades both in research and in practice.[1]

"One reaction to the prolonged history of poor results in rational number instruction is that … instruction in rational-numbers should be postponed ..." [Kieren, 1980]. In addition to the question of postponing, the long persistence of unsatisfactory results also generates social considerations: although Western knowledge on the didactics of fractions is wide and deep, teaching is ineffective; a discrepancy between knowledge and effectiveness that raises ethical and democratic issues.

These considerations constituted the foundation on which our activity related to the fractions arose. Reflecting on them, we have obtained the following indications: (a) It is important to start teaching fractions already in primary school; this requires adjustments both in some didactic principles and in structuring of the content. (b) We have abandoned the common teaching and learning practice, pursuing a higher level of mathematisation without introducing a higher level of difficulty and opening a wider horizon of potentiality toward the formation of critical citizenship.

[1] "The concept of fraction has manifested itself in education as a refractory one." [Streefland, 1978]. "As investigated in a variety of research and demonstrated with standardized tests, learning fractions is difficult across countries" [Tunç-Pekkan, 2015].

2.1 Two distinct groups of practice

Our teaching practice has had the particular characteristic of proceeding through the interaction of two distinct groups of practice: a group of teaching practice and a group of "reflective philosophical practice". The first group has carried out an enquiring activity, practicing directly in the classes, from the third to the fifth of the primary school. The second group has carried out both a process of exploration and a process of reflection on the progress of the dialogical interaction[2] that involves teacher, children and the proposed idea of fraction.

The interaction between the two groups of practice involves a double aspect of the report on our activity: one concerns the way of designing the curriculum and practicing it in the classroom[3]; the other concerns the historical and philosophical reflections and discussions that have accompanied its planning and practice. In this conference we give priority to historical and philosophical considerations.

1.2 Fraction as fraction-of-something

Our practice and considerations upset the most common idea of fraction that pervades the ordinary curricula for primary school: the fraction as fraction-of-something. This idea is excellently summarized in Bobos and Sierpinska. "Fraction of a quantity is a mathematical theorization of the visual and intuitive idea of fraction of something. … The idea of fraction-of-something stays in its <u>primitive, intuitive state</u> and functions as an <u>obstacle</u> to the construction of a <u>systemically connected knowledge</u> about fractions." The fraction-of-something remains the primitive and intuitive representation on which the knowledge of fractions of nearly all people is built.

The idea of fraction as fraction-of-something constrains the way of thinking fractions and practicing with them, favoring situations of division associated with the part-whole subconstruct [Kieren, 1980].[4] In this way, the subconstruct part-whole becomes the only scheme of action that innervates the entire teaching process. Scientific literature has now largely confirmed that these situations have limited teaching effectiveness.

Our alternative proposal to the idea of fraction-of-something is based on two basic believes that we here present as reaction to the statement of Bobos and Sierpinska : (a) we believe that the "<u>primitive, intuitive state</u>" of fraction can be corrected by constructing an "<u>intuitive representation</u>" that <u>does not function as an obstacle</u>; (b) the construction of a <u>systemically connected knowledge</u> about fractions is not a primary school goal; the goal is rather to come up with a process of <u>familiarization</u> aimed to the cohesion of the different subconstructs.

1.3 Intuitive representations versus primitive intuition

We have mentioned the possibility, according to some authors, of postponing the instruction concerning the fractions to when the students have reached the stage of formal operations. We have instead tried to find alternatives to the hypothesis of

[2] We have derived the term "dialogical interaction" from the presentation by D. Guillemette at ESU-8 (ch. 1-8 in this volume).

[3] The curriculum has been presented and discussed at ICMI Study 24.

[4] Kieren identifies five "subconstructs of the rational number construct": part-whole, quotients, measure, ratios, and operators; but there are other possible subconstructs: proportionality, point on the number line, decimal number and so on.

postponement, involving the idea itself of intuition. If primitive intuition works as an obstacle, it is necessary to proceed already in primary school to the construction of an intuitive representation that elaborates a larger and more effective idea of fraction. The distinction between primitive intuition, understood as “commonsense representation”, and intuitive representation, has been proposed by Fishbein, who contrasts to the intuitions firmly correlated with the primitive feeling, with the idea that “intuitions are deeply rooted in our previous, practical and mental, experience”. Proper practice with fractions must start in primary school, in order to construct intuitive representations, and to avoid the formation of those obstacles that accompany the idea of fraction-of-something.

1.4 Familiarization

Our teaching proposal is characterized by the choice to avoid, in primary school, to set the aim of constructing a systemically connected knowledge about fractions; we have instead set the familiarization of children with fractions as a goal. The term familiarization was derived from Davydov, and requires an explicit specification when referring to the teaching fractions in primary school.

Practices. To clarify the main features of the practice of familiarization with fractions in primary school, we refer to the following classification of practices: conceptual, algorithmic or executive, strategic or resolutive, semiotic, communicative [D’Amore & Radford, 2017]. The process of familiarization with fractions differs from the usual process of teaching and learning with regard to some of these.

(a) *Conceptual practices*. Familiarization is not directly aimed at the cognitive construction of mathematical concepts; it is rather aimed at shaping the learning environment; cognitive construction will start in the following years.

(b) A*lgorithmic or executive practices*. Speaking of familiarization means “freeing children from reliance on schooled algorithms” [Erlwangers, 1973], and carrying on the practice “without using pre-established formal rules” [Pitkethly & Hunting, 1996].

(c) *Semiotic practices*. The practice of familiarization with fractions is aimed at the construction of the “forms”, that is, of those mental structures that precede the concept and the formula.

(d) The other practices, *strategic or resolutive*, and *communicative*, develop in the dialogical interaction among teacher, children and the proposed idea of fraction.

Teaching principles. The difference between the familiarization with fractions in primary school and the usual process of teaching and learning also manifests itself in innovations with respect to the indications contained in the usual teaching principles.

Here are some examples. (a) The process of familiarization proposed by us, introduces a fracture in the historical process of development of the concept of fraction, rediscovering, thanks to a historical/philosophical reflection, an initial act of mathematisation that had been put aside by science; this fact suggests to critically take into account the *principle of scientificity* [Blažková, 2013], according to which a school subject is based on scientific math. (b) Moreover, the initial act of mathematisation constitutes a leap into the didactic process, interrupting systematicity and gradualness and introducing “a form of reflection qualitatively different from the previous one” [Davydov, 1990]. This does not entirely agree with the *principles of systematicity* (the Math curriculum is organized systematically in a logical succession which is necessary to respect), and *gradualness* ("step by step", Skinner; "natura non facit saltum, gradatim procedit", Comenio). (c) The initial act of mathematisation is not obtained by generalization and abstraction but it is imposed. This does not tune with the *principle of the concrete operativeness*, according to which all mathematical concepts arise from problems, so ... the rules, the formulas ..., are not imposed by the teacher but naturally conquered by the students [Faggiano, 2008]. (d) The idea of a trace, which, with its presence/absence, is central to our teaching proposal, partially puts into question the *principle of purposefulness*. According to this principle "the aims of each lesson should be formulated in the way to be concrete, achievable, and checkable; in our teaching proposal, the focus is instead on the centrality of dialogical interaction. (e) Finally, the very fact that the aim of our teaching proposal is the familiarization, redefines the content of the *principles of adequacy and verification*, enhancing, in particular, the role of lightness and flexibility. The dialogical interaction, that involves the teacher, the children and the contents, is finalized to guide the children in practicing with "leggerezza" (lightness); that is with serenity, quiet and confidence, achieving adequate results despite the complexity of the theme. Flexibility in practicing has to be favored: in choosing the most appropriate manipulative; in varying the conditions under which the practice is developed; in making stronger the link with real demands; in outlining how some concepts find different realizations in different contexts.

1.5 A different "base of belief": Fraction as megaconcept

In constructing an intuitive representation that allows avoiding the a priori formation of obstacles, we have explored and practiced the idea of fraction as megaconcept. This idea is suggested by Wagner (1976): "... for the person rational numbers should be a megaconcept involving many interwoven strands" [in Kieren, 1980]. It contrasts with the idea of fraction-of-something exposed by Bobos and Sierpinska and corresponds to a different "base of belief" [Bell in Fischbein, 1982].

Megaconcept demands that all strands/subconstructs contribute to the determination of its nature, and constitute its structural elements. In the megaconcept, the subconstructs find cohesion: they tune in to each other, so that the practice can naturally switch among them.

Developing the path towards the cohesion of the megaconcept, we have started from a fundamental strand and then we proceeded at an appropriate "interweaving rhythm", assuming the times and the ways of involving the different strands: (a) The subconstruct ratio is the first to be practiced, taking advantage of a meaning of the fraction hidden in Pythagorean mathematics; this meaning is brought to light in the form of an initial act of mathematisation. (b) The appropriate choice of the manipulative allows developing the subconstruct ratio in the subconstruct measure. (c) The practice with appropriate manipulative also allows to support children in discovering the link with the division by themselves; the teacher reinforces this link and proposes activities that lead pupils to practice the Euclidean division. The interweaving of the subconstructs ratio, measure and division is so achieved; these subconstructs attune in the Euclidean division. (d) In our practice, the part-whole scheme is no longer a subconstruct but rather it is an important instance; moreover, the choice of familiarization minimizes the role of the subconstruct operator. (e) The Euclidean division becomes the core of the subsequent interweaving process that involves other subconstructs: point on the number line, decimal number and so on. These practices have taken into account the specificity of the enacting process and the peculiarities of the context.

1 Historical and philosophical considerations

The central part of our presentation concerns the historical and philosophical reflections and the discussions that have accompanied the planning and the practice of our proposal. In the planning, two have been the main authors of reference: Davydov and Toth.

2.1 Davydov

Our exploration of fractions is started by practicing in classroom the situations proposed by Davydov; many of the activities enacted by us, resume the activities indicated by him. Two fundamental ideas have been taken from Davydov and have become the object of our reflection: the ideas of familiarization and of essence of the concept of fraction.

The research of the essence of a concept, central to Davydov's philosophy, and in particular, his research for the "objective content of the concept of fraction" deserve separate reflection. This research suggests two considerations: a) To speak of "essence" obliges to deal also with the "violence of the ontology", treated by Levinas, and to ask ourselves in what form and to what extent the concept of essence has a role in contemporary culture. In the continuation of the presentation we will discuss some considerations on this point. B) Davydov's direction of research raises the question "What to teach?" before the question "How to teach?". This has strongly influenced our investigation and has directed us towards the search for the "originary"[5] content of the concept of fraction.

2.2 Toth

In the course of our investigation we met Imre Toth and we were fascinated by his suggestion to listen to hidden meanings that could still be kept in the Pythagorean mathematics. This indication harmonizes with his basic conviction of the plurality of truth; an idea strongly linked to his discovery of non-Euclidean affirmations in the text of Aristotle. In the pre-Euclidean debate, the Euclidean hypothesis has been successful, certainly due to its effectiveness, but possibly due to philosophical reasons, too: the fourth postulate of Euclid, "all the right angles are equal", is justified, according to some authors, if completed in the form "all the right angles are equal, while the acute and obtuse angles are multiple"; it's the One of Parmenides that has imposed its primacy. Koiré's statement that the birth of science took place with an act of purely philosophical-metaphysical foundation, finds in Euclidean truth a place of implementation. The birth and development of Euclidean geometry have kept hidden non-Euclidean hypotheses for thousands of years. Only during the nineteenth century these hypotheses were unveiled and became "truths", alongside the Euclidean truth; here is the plurality of truth. It is therefore possible to hypothesize similar occurrences for other truths: there are still hidden truths that could be brought to light alongside

[5] The word "originary" is not an English word. Nevertheless, some authors [Roth & Radford, 2011] are beginning to use it. In this way the wealth of meaning possessed by the corresponding term "originario/originaire" that is used in continental philosophy, is recovered.

the known truth. It is this hypothesis that underlies Toth's suggestion to listen to hidden meanings.

The meeting between the searching for the essence of the concept (Davydov) and the listening to hidden meanings (Toth) has become the source of our proposal concerning fractions. History has become “the site” where our project has found its structure; in this site we have looked for the “originary” meaning of the concept of fraction and we have found some foundational aspects that allow rethinking its didactics. At the same time, the plurality of truth, generated by the disclosure of hidden meanings, responds to the need, according to Lévinas, to dissolve the violence associated with the search for the essence of a concept.

2.3 Anthyphairesis

After hearing Toth at the conference held in Bergamo,[6] while he was exhorting to listen to hidden meanings, we discovered, thanks to the reading of the *Menone* proposed by him, the anthyphairesis, the Pythagorean procedure of comparison. The practice with anthyphairesis provided us the “ladder” (Wittgenstein) to climb to a higher level and from here to look at fractions. However, we do not present here the anthyphairesis, due to the fact that there is no need for teachers and children to know and practice it.[7]

Walking step by step the concrete actions of the anthyphairetic process, we have met some indications stored in it and still potentially significant: a) indications of historical type, as this walking allows to listen again to not secondary aspects of Greek philosophy; b) indications of mathematical type, as it highlights a “physical” language for rational numbers and it enables an unusual outlook on the intrinsic reciprocity of a primitive concept of measure; c) indications of pedagogical type, which have moved our project.

2.4 The mathematisation of the comparison

The reflection of historical type and the practice of anthyphairesis have led us to focus on the Pythagorean statement: “The comparison is a logos (ratio)”. This statement, in its flatter formulation: “The comparison is a pair of natural numbers”, turns out to be extraordinarily modern. If we compare it with Eddigton's statement that a relativistic event is a quadruplet of numbers,[8] it becomes an act of mathematisation that operates an universal synthesis concerning the concept of comparison.

The method of anthyphairetic comparison did not last long. Already at the time of Euclid it had been forgotten; its crisis came with the discovery of incommensurable quantities and its difficulties were contrasted by the effectiveness of the Euclidean algorithm. This latter overshadowed the anthyphairetic comparison, which was consequently forgotten. But with the anthyphairesis, even the act of mathematisation of the comparison has been forgotten: scientific practice and teaching have put it aside.

Our enquiring led us to conclude that the ‘originary’ content of the concept of fraction is ascribable to the comparison of quantities. So we have recovered the forgotten act of mathematisation: “the comparison between two quantities is a pair of numbers”. This act

[6] “Matematica, Storia e Filosofia: quale dialogo nella cultura e nella didattica?” Bergamo, 1999, May 19.

[7] We briefly presented the procedure of antyphairetic comparison at HPM 2016, in Montpellier.

[8] “An event in its customary meaning would be the physical happening which occurs at and identifies a particular place and time.” (Eddington, p. 45)

has therefore become the starting point of our didactic practice. Two considerations: (a) Our choice corresponds to placing the subconstruct “ratio” as starting point of our classroom activity. But, while usually “Ratio is a complex concept, which demands a long-lasting learning process” [Streefland, 1984], historical reflection allowed us, instead, to make it elementary; elementary because turned to “the originary elements”, but also by reason of the "lightness" with which the children have lived the proposed acts: their answer has been quiet, serene, with adequate results. (b) Furthermore, while the analysis and the study of comparison are central in psychology and in teaching, its mathematisation is not part of the common way of considering it. So the mathematisation of the comparison is a “new” way of looking at the world.

2.5 The new universe of fractions

Our approach to fractions is diversified from that of Davydov: Davydov considers the measure as a juxtaposition and “The relationship between one quantity and any other that is taken as a measure is recorded in the form of a number”; the historical - philosophical practice instead leads us to consider measure as comparison; then measure is an ordered pair of numbers.

This implies a substantial difference in the didactics of numbers in primary school. While in Davydov the teaching of numbers proceeds through “extension, intensification, and expansion” starting from natural numbers, in our approach there is a split: the initial acts concerning the concept of number are two: a) counting is a number, b) comparing is a pair of numbers. This splitting breaks the categorical framing in which acts concerning the number teaching are usually blocked.

In our approach, the universe of fractions is not obtained as an extension of the universe of natural numbers; it is new universe, with its own rules and properties.

2.6 Reflections about the act of mathematisation

In this presentation we do not describe the classroom practice to introduce the act of mathematisation of the comparison. We show instead some features of this act.

a) The mathematisation of the comparison is imposed. This act of mathematisation is not present in Western culture, and in some workshops we have kept (in Milan Bicocca and at CIEAEM 66, Lyon), no one spontaneously has hinted at it. Despite its simplicity it is not a spontaneous consequence of the common observing and acting. Then it cannot be discovered by the children.
b) The mathematisation of the comparison is a leap. Another form of recording of activities in the exercise book, the representation by segments, is a form of mathematisation obtained through generalization and abstraction from concrete experiences. The representation of the comparison by a pair of numbers is instead a leap: the special symbolic representation A;B = 9;5 (“the comparison between A and B is the pair of numbers 9;5) is non-spontaneous; it consists “in an “interruption in the gradualness,” in a “leap,” in the appearance of a new form of reflection that is qualitatively different from the preceding stage in knowing. …” (Davydov).
c) The mathematisation of the comparison is elementary. There are no difficulties for children. Children have lived these activities with "lightness – leggerezza" (Calvino).
d) The mathematisation of the comparison is an “axios”, “dignum”, worthy, because it keeps and opens the trace of the didactic process.

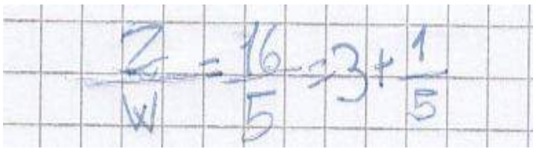
e) **Trace**: the mathematisation of the comparison is “pregnant with significance”. To affirm that the act of mathematisation of the comparison keeps and opens the trace of the didactic process, means two things: first, the process of interweaving that articulates the different strands of the megaconcept into a cohesive and effective structure, begins from this act; second, this act enacts a presence that directs the subsequent didactic path; this presence is due to inescapable indications that this act keeps; indications that orientates in assuming the times and the ways of involving the different strands. However, the fact that the indications find different realities in specific actualization, causes an absence to be breathed, while seeking the most effective enacting in the particular context. Teacher is forced to “suspend” [Bobos & Sierpinska, 2017] his knowledge on fractions and to carry on a process of dialogical interaction with children, in order to unveil the path that the initial act indicates. This presence/absence has prompted us to make use of the name “trace”, echoing the philosopher Lévinas.

2.7 Inescapable indications

In order to highlight inescapable indications imposed by the initial act, let us briefly retrace the whole path of mathematisation we have practiced. (a) The act of mathematisation of the comparison leads the children to practice the concept of common unit. (b) Practicing with common unit, children arrive to measure as ordered pair of numbers, and to fraction as measure. (c) The idea of common unit guides towards the rethinking of the usual language related to the concept of measure. This compels a reconceptualisation that involves the names “whole”, “unit”, and “quantity” in a linguistic mathematisation. Their corresponding formulas are a safe reference for didactic activities but are not directly used; children do not know them but they grasp the corresponding “forms”. (d) Practice with appropriate manipulative brings children to discover the link with the division by themselves. (e) The teacher reinforces this link and proposes activities that lead children to practice the Euclidean division.

2.8 Euclidean division: an icon

The mathematisation process comes realized in the Euclidean division.



This latter is the historical milestone that has created confidence in our approach.

In our didactics, the Euclidean division is not lived by children as a formula to memorize; it is rather the "icon" of their active process of learning. The evocative value we ascribe to the word "icon", refers to art history, to which we have recourse to highlight and enhance deeper meanings that this word has in the context of our cultural training.

So the word "icon" houses many "indications":

a) icon as “memorative (mnemonic) synthesis” of own history of active learning;
b) icon as target towards which the enacting steps attune;
c) icon as opening meanings and then making the “trace” to the future activity;
d) icon as medium between teacher and pupil, as it allows the understanding between the two.

3 Concluding observations and perspectives

We conclude our presentation with two types of considerations: about the role of history in our enquiring on fractions, and about some key words.

3.1 Role of history

In this conference we have given priority to historical and philosophical considerations. In particular, the reference to history is characterized by three features:

a) *Break in the historical process of development*. The most widespread idea of fraction is that of “fraction-of-something”. This idea is the result of the "historical process of development" of the concept of fraction. Reflecting on history has allowed us to create a break in the historical process of development, rediscovering an act of mathematisation put aside.
b) *Wittgenstein’s ladder*. “... He must, so to speak, throw away the ladder after he has climbed up it.” Practice with anthyphairesis provided us the ladder to rise to a higher level and from here to design our teaching proposal.
c) *Clues of history*. Different actualizations of the teaching path regain unity in the Euclidean division. So the Euclidean division is the clue left behind by history, and the milestone that has created confidence in our approach.

3.2 Key words

We now recall some key words, trying to briefly indicate any prospects.

Long persistence of unsatisfactory results. Our enquiring try to react to the long persistence of unsatisfactory results in teaching and learning fractions: although Western knowledge on the didactics of fractions is wide and deep, teaching is ineffective. The discrepancy between knowledge and effectiveness requires reflections on the responsibility of knowledge in today's society.

Familiarization. In our teaching proposal about fractions in primary school, the familiarization is opposed to the usual teaching/learning process: familiarization as shaping the learning environment precedes the proper cognitive construction. In our classroom practice we have tried to clarify how the familiarization with the fractions in primary school works. This opens topics of investigation and discussion: (a) how do the ordinary teaching principles change in the presence of a process of familiarization? (b) Does the familiarization work with other subjects in primary school?

Megaconcept – Cohesion. The idea of fraction as a megaconcept that we have practiced in our classroom activities, requires the search for cohesion between the different subconstructs. Cohesion acquires a fundamental role because it gives unity to the interweaving of the strands that integrate into the megaconcept. It depends on the dialogical interaction between teacher, pupil and content, and the nature and quality of the teaching proposal, rely on it. Therefore cohesion must be subject to a constant process of investigation.

Trace. To bring the construction of the new didactic universe of fractions to the initial act of mathematisation of the comparison reveals an attitude to think of a principle pervading that universe. Fragmentation[9] and singularity[10] prevail today in enacting

[9] “Modeling is the application of a fragment of mathematics to a fragment of reality” [Israel, 2002]
[10] “The singular as knowledge actualized in activity” [Radford & Sabena, 2015]

mathematics. However, there is also something else that deserves to be investigated. To talk about the trace, about of the presence/absence that it brings within itself, means believing in the presence of a principle that indicates, and comes back continuously throughout the path of enacting.

Icon. We resort to the word “icon” to denote the Euclidean division. This choice finds justification in the reference we make to art history and to deeper meanings that this word has in the context of our cultural training. The word icon highlights the roles of evocation (memorative synthesis), indication (trace), and mediation (medium between teacher and children) that the Euclidean division possesses. These roles make Euclidean division the core of the didactic process of fractions. This interpretation of the word icon raises the question of the possibility that icons are also identified for didactic processes related to other topics.

Plurality of truth. Following the indications of Toth, we have discovered a hidden meaning in Pythagorean mathematics, in the form of a modern principle of mathematisation: the mathematisation of the comparison. This allowed us to highlight an "other" concept of measure, more primitive, with reciprocity, different from the usual one. The bringing to light another definition of measure, recalls the plurality of truth, so dear to Toth, and echoes the freedom (Cantor) of mathematics: mathematics can progress not only by “extension, intensification, and expansion”, but also by working up an unveiled, originary meaning. This make sure that universality replaces totality, plurality of truth replaces uniqueness, the violence of ontology fades away, while the power of truth is preserved. This makes mathematics the paradigm on which to rebuild critical citizenship.

REFERENCES

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 296-333). New York: Macmillan Publishing.

Blažková, R. (2013) *Didactics of mathematics*. www.ped.muni.cz/wmath/interma/blazkova_en.doc

Bobos, G., & Sierpinska, A. (2017). Measurement Approach to Teaching Fractions: A Design Experiment in a Pre-service Course for Elementary Teachers. *International Journal for Mathematics Teaching and Learning*, *18*(2), 203 – 239.

D’Amore, B., & Radford L. (2017) *Enseñanza y aprendizaje de las matemáticas: problemas semióticos, epistemológicos y prácticos*. Bogotá: Universidad Distrital Francisco José de Caldas.

Davydov, V. V. (1990) *Types of Generalization in Instruction: Logical and Psychological Problems in the Structuring of School Curricula*. Soviet Studies in Mathematics Education, vol. 2. Reston Virginia: National Council of Teachers of Mathematics,.

Davydov, V. & Tsvetkovich, Z. H. (1991). The object sources of the concept of fractions. *Focus on Learning Problems in Mathematics*, *13*(1), 13-64.

Eddington, A. (1920). *Space, Time and Gravitation. An outline of the General Relativity Theory*. Cambridge: Cambridge University Press.

Erlwangers, S. H. (1973). *Benny's Conception of Rules and Answers in IPI Mathematics*. http://www.wou.edu/~girodm/library/benny.pdf

Faggiano, E. (2008). *Elementi di Didattica della Matematica*. https://www.dm.uniba.it/Members/faggiano/didattica/introduzione.pdf

Fishbein, E. (2005) *Intuition-in-Science-and-Mathematics*. https://www.scribd.com/doc/163196879/ Intuition-in-Science-and-Mathematics-E-Fischbein-2005-WW

Fowler, D. H. (1979). Ratio in Early Greek Mathematics. *Bulletin (New Series) of the American Mathematical Society*, *1*(6), 807-846.

Gabriel, F. C., Szucs D., & Content, A. (2013). The Development of the Mental Representations of the Magnitude of Fractions. *PLOS ONE.* Volume 8 (Issue 11) e80016.

Kerslake, D. (1986). *Fractions: Children's Strategies and Errors. A Report of the Strategies and Errors in Secondary Mathematics Project* (pp.1-139). Chelsea College, University of London: The NFER-NELSON Publishing Company Ltd.

Kieren, T. E. (1980). *Recent Research on Number Learning*. Edited by Kieren T. E. (pp. 1-175) University of Alberta, ERIC Clearing house for Science, Mathematics, and Environmental Education, Columbus, Ohio. Washington, DC: National Institute of Education (ED).

Longoni, P., Riva, G., & Rottoli, E. (2016). Anthyphairesis, the "Originary" Core of the Concept of Fraction. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting – HPM 2016* (pp. 477-487). Montpellier: IREM de Montpellier.

Pitkethly, A., & Hunting R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics 30*, 5-38.

Radford L., & Sabena C. (2015). *The Question of Method in a Vygotskian Semiotic Approach*. http://luisradford.ca/publications/

Streefland, L. (1978). Some observational results concerning the mental constitution of the concept of fraction. *Educational Studies in Mathematics*, *9*(1), 51-73.

Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long term learning process (towards ... a theory). *Educational Studies in Mathematics*, *15*(4), 327-348.

Tenuta U. (1996). *I principi metodologico-didattici dell'educazione matematica*. www.edscuola.it/archivio/didattica/emprin.pdf

Toth, I. (1998). *Lo schiavo di Menone*. Editore: Vita e Pensiero. Milano.

Tunç-Pekkan, Z. (2015) An analysis of elementary school children's fractional knowledge depicted with circle, rectangle, and number line representations. *Educational Studies in Mathematics*, *8* (3), 419-441.

Wagner, H. (1976) *An analysis of the fraction concept.* An unpublished report, Chicago.

DIFFERENT FACETS OF PRE-SERVICE TEACHERS' BELIEFS ON THE HISTORY OF MATHEMATICS

Sebastian SCHORCHT[1,3], **Nils BUCHHOLTZ**[2,4]
[1]Justus-Liebig-University of Giessen, Karl-Gloeckner-Str. 21c, 35394 Giessen, Germany
[2]University of Oslo, Postboks 1099 (Blindern), 0317 Oslo, Norway
[3]sebastian.schorcht@math.uni-giessen.de, [4]n.f.buchholtz@ils.uio.no

ABSTRACT

The study ÜberLeGMa examined the beliefs of 141 mathematics pre-service teachers on the structure of mathematics, the history of mathematics, and the teaching and learning of the history of mathematics. The paper presents the key findings of the study, including statements on the structure and distribution of these beliefs and their relationships. The aim of the paper is to provide empirical foundations for studying higher education learning opportunities in this area and to formulate recommendations for incorporating appropriate learning arrangements into teacher education.

1 Introduction

At many universities, mathematics pre-service teachers study the history of mathematics in addition to subject-related, didactic and pedagogical learning content during their study. In their overview, Clark, Kjeldsen, Schorcht, & Tzanakis (2018) highlight the benefits of the so-called HPM perspective not only for the learning of mathematics, but also for teacher education. There is widespread consent among mathematicians and mathematics educators that the inclusion of the historical, philosophical, and developmental context of mathematics in the university curriculum provides prospective teachers with learning opportunities that can enrich the teachers' future pedagogical practice (Mosvold, Jakobsen, & Jankvist, 2014; Burns, 2010; Smestad, 2011). For example, the AMTE recommend in its standards for teacher education in mathematics (2017, p. 38): "Well-prepared beginning teachers of mathematics realize that the social, historical, and institutional contexts of mathematics affect teaching and learning and know about and are committed to their critical roles as advocates for each and every student."

Universities offer different ways of dealing with mathematical-historical content during teacher education. Study content can be anchored in seminars or lectures in the mathematics teacher education program for primary and secondary level as well as for upper secondary level.

From the perspective of evidence-based teacher education, however, the question of impact of these learning opportunities remains open. It has not yet been sufficiently clarified how pre-service teachers perceive these study contents and to what extent the university learning opportunities contribute to professionalization processes of the teachers. Specifically, the question arises as to whether pre-service teachers later refer to mathematics historical references in their pedagogical practice and are prepared to address them explicitly in the classroom, or whether the study content as a whole has only little sustainability.

Previous research into the integration of historical references in mathematics education and in mathematics teacher education is mainly concerned normatively with the questions of why historical references are meaningful and how these references can be addressed in teaching (Clark, Kjeldsen, Schorcht & Tzanakis, 2018; Clark, Kjeldsen, Schorcht, Tzanakis & Wang, 2016; Fauvel & van Maanen, 2002). With regard to empirical work, empirical task analyzes by Schulte (2016) or Schorcht (2018) at the level of textbooks used in practice have been able to identify various types of tasks that allow for the integration of historical references into mathematics education. A wide range of international empirical studies also address teachers' perceptions of mathematics history in the classroom (Alpaslan, Işıksal, & Çiğdem, 2014; Bütüner, 2018; Burns, 2010; Charalambous, Panaoura, & Philippou 2009; Furinghetti, 2007; Goodwin, 2007; Ho, 2008; Jankvist, 2010; Philippou & Christou 1998; Smestad, 2011). Although the empirical research on the history of mathematics in the classroom suggests that the embedding of historical references has advantages (Furinghetti, 2007; Glaubitz, 2011), the meaning and significance of the content of history of mathematics depends on how mathematics teacher's education are developed (Nickel, 2013). Due to the fact that only little is known about how and if respective measures in teacher education can influence pre-service teachers' attitude towards the history of mathematics, there are diverse implementations of history of mathematics in teacher's education.

Against the backdrop of the issue of impact, the aim of our article is to create empirical foundations for a discussion of study content on the history of mathematics and to create opportunities for researchers to empirically analyzing the impact of university learning arrangements in this area. As part of our study ÜberLeGMa ("Überzeugungen von Lehramtsstudierenden zur Geschichte der Mathematik" – "Beliefs of pre-service teachers on the history of mathematics", Schorcht & Buchholtz, 2015), we have developed an instrument to analyze pre-service teachers' beliefs on the history of mathematics and the teaching and learning of history of mathematics. With the help of this instrument, we have collected the beliefs of 141 pre-service teachers and examined their connection to further beliefs.

We hope that colleagues working in teacher education in the history of mathematics will use this instrument to examine the impact of seminars or learning activities in history of mathematics. Ideally, we hope that whatever university learning opportunity in the history of mathematics will result in pre-service teachers becoming open to this study content and willing to address it later in the classroom. In the ongoing didactic discussion, normative ideas and demands can thus be systematically substantiated with empirical results – at least on the level of beliefs of pre-service teachers.

2 Theoretical Framework

2.1 Strategies for addressing mathematics history in the classroom

In terms of dealing with history of mathematics, teachers use two perspectives on mathematics: on the one hand, developments of mathematics can be focused as a product and, on the other hand, as a process of changes.

Furinghetti (2007) states in her study that prospective teachers (15) used two different modes during a course of 42 hours in a teacher program to integrate historical sources into mathematics teaching. In the first mode, prospective teachers pursued the goal of clarifying mathematical concepts through their genesis over time. Hereby, the product of the concepts at the end of this process is used as a starting point to create a path of development. Furinghetti (2007, p. 137) calls this mode “evolutionary”. In the second mode, the prospective teachers used selected original sources of history of mathematics from a writer or author to situate the foundations of a concept in its particular historical situation. Thus, the cognitive origins are traced back to historical roots and thus prevent adherence to the product of a completed genesis of a concept. This mode is called “situated” (Furinghetti, 2007, p. 137). Prospective teachers therefore use the history of mathematics to either present a current process in its historical genesis or to introduce students to mathematical thinking based on a mathematical-historical example. These two perspectives can also be related to the distinction of "history" and "heritage" proposed by Grattan-Guiness (2004a, b), with evolutionary mode rather emphasizing an emphasis on "heritage" in the classroom, and situated mode rather emphasizing the "history" aspect. Similar conclusions states Lakoma (2002, p. 28ff) for Polish textbooks, but calls these two modes “discursive style” or “dogmatic style”. She argues that the choice of mode dependents on authors’ beliefs about teaching and learning mathematics. Depending on which goals are pursued in lessons contains the history of mathematics, tasks can contribute to the mediation of a mathematical discourse culture or emerge as a “set of curious details” for the purpose of motivating the learners. Summarizing this might lead us to the assumption that pre-service teachers therefore might use history of mathematics to either present a current process in its historical genesis or to introduce students to mathematical thinking based on a mathematical-historical example. From this and other empirical studies on the beliefs of teachers and pre-service teachers of mathematics, it is known that mathematics teachers either understand mathematics as a static, rather un-changeable product or understand mathematics as a dynamic, continuous process of change initiated by human mathematical activities (Blömeke, Kaiser, & Lehmann, 2010; Voss, Kleickmann, Kunter, & Hachfeld, 2013). The exciting question that emerges from this finding is how different views on mathematics are related to different attitudes towards the use of history of mathematics in education.

Previous studies on the history of mathematics in teaching do not fully address this notion and focus, e.g. rather on the relations between teachers’ worldviews and different methodical ways to use mathematics history or the interaction in the classroom (Alpaslan et al., 2014; Furinghetti 2007; Goodwin, 2007). Studies by Buehl, Alexander, and Murphy (2002) or Hofer and Pintrich (1997) for example show a dependency between epistemological beliefs and teacher-student interaction in the classroom.

2.2 Reasons for and against mathematics history in the classroom

There are various reasons for the use of history of mathematics in the education. Bütüner (2018, p. 9) presents a list of these justifications as a synopsis of his literature research, which is briefly described below. This is to demonstrate the possible intentions of teachers to use history of mathematics in the classroom (see Fried, 2001; Liu, 2003; Tzanakis et al 2000). Mathematics history should accordingly

(1) make students clear that mathematics is a human activity and a human product,
(2) increase motivation and positive attitude towards mathematics,
(3) open up perspectives on the nature of mathematics to students and broaden the subject-didactic repertoire of teachers,
(4) provide a deeper understanding of mathematical concepts, problems and solutions.

This list includes *cognitive*, *affective*, and *evolutionary* justifications. The students should be cognitively challenged with regard to mathematical or mathematical-historical content in order to expand their mathematical discourse skills. Likewise, they should be motivated by the history of mathematics to deal with a specific subject area, and thus to understand the fundamental change processes of mathematics influenced by protagonists, culture, or social environment.

On the other side, studies by Ho (2008) or Panasuk and Horton (2012) revealed barriers of teachers to using history of mathematics. The teachers interviewed in both studies most often cited insufficient training in dealing with the history of mathematics as an obstacle to the integration of mathematical-historical topics. Likewise, teachers do not use mathematics history because they try to use the available time in class for other content. Teachers also report that mathematics history can confuse students. In the end, mathematics history can be considered history and therefore is not content of mathematics education. In addition to these critical arguments, some teachers also mentioned inadequate opportunities for assessment or a lack of teaching materials.

In their articles, Tzanakis et al (2000) and Siu (2006) put together lists of obstacles that teachers encounter in the classroom if they want to integrate the history of mathematics. Tzanakis et al distinguish *philosophical* and *practical* objections. They add to philosophical objections the ontological distinction between mathematics and history, which leads to a prioritization of mathematical learning content by teachers, insufficient historical prior knowledge of students and their lack of motivation, as well as the danger of having history can help cultivate cultural chauvinism and narrow-minded nationalism. As a practical objection, Tzanakis et al add the inadequate expertise of teachers, which is related to a lesser self-concept with regard to mathematical-historical content (Tzanakis et al 2000, p. 203).

Siu (2006) describes a similar list, but differentiates it further. He additionally refers to teachers' doubts about the value of mathematics history for mathematics education. Overall, the discussion on the use of mathematics history involves affirmative and negative beliefs. Affirmative beliefs are based on affective, cognitive, or evolutionary benefits, while negative beliefs include practical or philosophical objections. Following the findings of Buehl et al. (2002), Furinghetti (2007), Goodwin (2007), Hofer and Pintrich (1997) or Lakoma (2002), these reasons show this two-fold nature, depending on the epistemological beliefs teachers have in their field.

2.3 Research on Teachers Beliefs

The prospect that study content on the history of mathematics is used in the classroom ties in with the hope that university learning opportunities can change the beliefs of pre-service teachers in such a way that they thematize mathematics history in teaching. This hope is supported by the existing assumption for the field of research on teacher actions that the application of professional knowledge in context situations is only successful if there are corresponding subjective beliefs among the teachers. Beliefs are given an orienting and action-guiding function for the application of learned content (Ernest, 1989; Schmotz, Felbrich, & Kaiser, 2010; Schoenfeld, 1998, 2010; Thompson, 1992).

Despite intensive research into beliefs of teachers, especially in the context of pedagogical-psychological research, there is yet no clear and precise definition of the concept of beliefs (see, for example, Pajares, 1992). Richardson (1996) therefore proposes an area-unspecific definition of beliefs, based on a broader understanding. He understands beliefs as "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p. 103). Following Richardson, we also understand beliefs as subjective opinions and attitudes of a person to an object, which also include affective stances and the willingness to act (see Grigutsch, Raatz, & Törner, 1998). With regard to the long-term development of beliefs, it can be assumed that they are relatively stable to restructuring and, to a certain extent, can act as psychological "filters" and / or "barriers" (Reusser, Pauli, & Elmer, 2011). On the other hand, however, justifications for beliefs may change in the professional development of teachers (Eichler & Erens, 2015). For mathematics teachers, however, despite the blurring of the term, there is a broad consensus on the differentiation of professional beliefs (Ernest, 1989). It is assumed that beliefs can be domain-specific (Eichler & Erens, 2015; Törner, 2002) or even situation-specific (Kuntze, 2011; Schoenfeld, 2010). In addition to epistemological beliefs on the structure of mathematics (see, Grigutsch et al., 1998), beliefs on the acquisition of mathematical knowledge or on the teaching and learning of mathematics (Buchholtz & Kaiser, 2017; Handal, 2003; Kuntze, 2011; Staub & Stern, 2002) are other important dimensions of epistemological beliefs.

For the study of beliefs on the history of mathematics, it made sense to expand these existing dimensions by further specific beliefs. To this end, in the present study, based on the above-described theoretical framework on the history of mathematics and its use in the classroom, tools have been developed that specifically capture beliefs about the history of mathematics and the teaching and learning of history of mathematics. Similar to the beliefs on the structure of mathematics, the beliefs on the history of mathematics include both static and dynamic perspectives (Buehl et al., 2002; Furinghetti, 2007; Goodwin, 2007; Hofer & Pintrich 1997; Lakoma, 2002).

Static perspectives include, for example, the assumption that mathematical findings are axiomatic and therefore have ideal or eternal existence. Mathematics is understood as a perfect logical and consistent system. This view can also suggest an anecdotal understanding of the history of mathematics, limited to the narrative of the work of eminent personalities or their biographies. However, the perspective frequently neglects that mathematical findings are often the subject of disputes, and without questioning mathematical theorems, it would hardly be possible to uncover contradictions and initiate further developments.

Dynamic perspectives on the history of mathematics emphasize this aspect in particular. They have a critical attitude to mathematical findings and do not exclude that today's mathematics can be questioned and further developed. On the other hand, they regard mathematics as an intellectual creation of humans in their respective historical and cultural context and see the origins of mathematical thinking in a strong reference of the discipline to every day's life.

The beliefs on the teaching and learning of history of mathematics pick up the different justifications described above (see Bütüner, 2018; Siu, 2006; Tzanakis et al, 2000).

Affirmative beliefs are fed by affective, cognitive or evolutionary reasons, emphasizing the motivational nature of mathematics history in the classroom, the cognitive added value of using the history of mathematics in class, or overall a processual image of mathematics. Teachers who share these benefits use mathematics history to engage students in a mathematical subject area and get to know the genesis of and relations between mathematical content. Here, especially historical references, which can motivate the students, play a role. Mathematics should be understood as a human product and provide a specific perspective on the nature of mathematics. Overall, these references should favor a positive attitude towards mathematics.

Negative beliefs refer to philosophical or practical objections. Teachers who share these reasons tend to reject mathematics history in the classroom. They see the high complexity, the low motivation and the lack of learning prerequisites of the students as well as the high time pressure as obstacles to thematize relevant content in the classroom. It can be assumed that dynamic beliefs about mathematics and constructivist teaching-learning beliefs are related to a more strongly emphasis of a process-based, iterative operation with mathematics in lesson design (Reusser et al., 2011). For this reason, we suspect that open-mindedness to incorporating historical aspects in mathematics education is most likely to be found among teachers with appropriate dynamic beliefs. Conversely, in the field of mathematics history, however, one could argue, that teachers with a more static belief of mathematics might also be open to historical references in mathematics teaching, because these put the spotlight on the universal and eternal validity of mathematical theorems.

Ultimately, this raises the empirical question of how teachers' different facets of beliefs are interrelated and whether convergent or more differentiated structures can be identified between beliefs on the structure of mathematics, the history of mathematics, and the teaching and learning of history of mathematics. In the present study, these structures could be examined empirically, at least at the level of pre-service teachers.

2.4 Research questions

Existing research on the beliefs of teachers refers to a differentiated image of beliefs in the field of mathematics and mathematics as a school subject. For instance, research describes beliefs on the origin of mathematical knowledge and the nature of mathematical problems (Grigutsch et al.; 1998, Törner, 2002). Empirical studies highlight the importance of such beliefs in teaching (Buehl et al., 2002; Hofer & Pintrich, 1997; Staub & Stern, 2002). Despite this research background, the research discussion lacks of empirical findings in the area of teachers' beliefs on history of mathematics. Most of studies in this area are either normative studies on the epistemological foundations of teachers' beliefs on history of mathematics (e.g, Siu, 2006; Tzanakis et al, 2000). Other, more qualitative case studies engage with students' learning processes in classroom contexts about history of mathematics (e.g, Chorlay, 2016; Glaubitz, 2011; Jankvist, 2009). A third kind of studies analyses historical documents, textbooks and teacher materials for didactical implications (e.g. Biegel, Reich & Sonar, 2008; Clark, et al., 2018; Clark et al., 2016;, Fauvel & van Maanen, 2000). The aim of our study, and thus of this article is therefore, to deepen the research findings on teachers' beliefs on history of mathematics in education and to support the discussion with empirical results. Our research questions are:

1) What kind of beliefs on mathematics, on history of mathematics and on teaching and learning of history of mathematics have pre-service teachers?
2) How are their beliefs on the history of mathematics related to their epistemological beliefs about mathematics, and what relations exist to their beliefs on teaching and learning of history of mathematics?

3 Methodology

The following section outlines the method of data collection. The instrument is further provided as a whole in the appendix.

3.1 Sample

By means of an online survey in the summer term 2015 and in the winter term 2015/2016, the study examined the beliefs of 159 German mathematics pre-service teachers studying both for primary and secondary level. For the administration of the survey, pre-service teachers at various universities received a link via email. We asked colleagues at the respective universities working in the field of history of mathematics and mathematics education to administer the link to their respective groups of pre-service teachers, aiming for a convenient sample. As is it the case with studies in tertiary education, it is not uncommon to gather different sized subsamples at individual universities due to limited access to the field (for example, we had no insight in how many pre-service students were informed). Participation in the study was voluntary and the data was collected anonymously, so that we have to assume a positively selected sample. Eighteen pre-service teachers had to be excluded from our analysis due to missing values or unfinished surveys. Overall, the study is based on a sample of 141 pre-service teachers from nine universities, the numbers in brackets indicate the sample size at the respective university: University of Hamburg (6), Justus Liebig University Giessen (12), University of Siegen (11), University of Wuppertal (37), Technical University Dresden (28), University of Kassel (6), University of Vechta (1), University of Bielefeld (1), Technical University of Dortmund (39). No comparisons between universities were foreseen, however, the different sample sizes at the universities may affect the results of our study. On average, the pre-service teachers were about 24 years old with a standard deviation of slightly more than 4 years, studying mostly in the 6th semester (with a relatively large span from the 1st to the 33rd semester) and predominantly female (111 pre-service teachers, 79%). The evaluation of the study degree of the pre-service teachers showed a differentiated picture of the sample. The vast majority of pre-service teachers (108, 77%) were studying for primary or lower secondary level. 23 pre-service teachers (16%) studied for upper secondary or vocational level and 10 students (7%) studied for special needs education.

3.2 Instrument

Pre-service teachers' beliefs about mathematics, the history of mathematics, and the teaching and learning of history of mathematics were collected in a survey using three scales. With reference to empirical research on attitudes and beliefs (Grigutsch et al., 1998), and empirical and theoretical work on beliefs on the history of mathematic (Alpaslan et al., 2014; Siu, 2006; Tzanakis et al, 2000), we developed scales on the history of mathematics (26 items) and the teaching and learning of the history of mathematics (21 items) as part of the piloting of the study (Schorcht & Buchholtz, 2015). Appendix A and B shows the developed scales. An already existing 12-item component of the instrument included questions on beliefs about mathematics, which included both static (formalism aspect, schema orientation) and dynamic beliefs (process orientation, application aspect) (Grigutsch et al., 1998). For all scales, pre-service teachers should indicate their agreement on a five-point Likert scale (1 = "strongly disagree" to 5 = "strongly agree"). The instrument was implemented in the form of an online survey for the main study in the software Questback Table 4.1 gives an overview of the descriptive statistics and reliabilities of the developed scales and illustrates them with example items.

3.3 Data analysis

With the help of confirmatory factor analyses (CFA), the assumed factor structure from the piloting of the study was empirically tested individually for each facet of the beliefs. The models were specified in the form of structural equation models for each belief facet in mplus (Mplus, Muthén & Muthén, 1998-2012). For model evaluation, the statistical significance was tested using the likelihood ratio χ^2 fit test, and other global fit indices were used to describe the quality of the model (RMSEA, CFI, SRMR). RMSEA values less than .05, CFI values greater than .90, and SRMR values less than .08 indicate a good model fit (Hu & Bentler, 1999). For all inference statistical tests $p <.05$ was set as the significance level. For the scale of beliefs on the history of mathematics, 6 items had to be excluded due to bad item-fit. The results of the CFAs provide a measure of the relationship between the individual dimensions of the different scales at latent level (latent correlations) and the descriptive statistics that provide us with information about the distribution of beliefs within the sample. For the subsequent correlation analyzes between different belief facets, factor scores were exported from mplus for all 141 pre-service teachers, which are estimated there and output as standard. These numerical variables are weighted scores that reflect the individual's latent standing on a factor based on their agreement to the items. The further correlation analyzes were based on the consideration of the manifest correlations of these factor scores.

4 Results

4.1 Beliefs on the structure of mathematics

In essence, we were able to replicate the four-factor solution of beliefs about the structure of mathematics based on the work of Grigutsch et al. (1998). We identified the factors *formalism*, *application*, *process*, and *scheme orientation* (see Table 1), where formalism and scheme orientation represent static perspectives and application and process dynamic perspectives. The model had an acceptable to good fit (χ^2/df = 1.85, RMSEA = 0.07, CFI = 0.91, SRMR = 0.06). In addition, we identified significant correlations (Figure 4.1) between the factors already revealed by Grigutsch et al. (1998). Since these are correlations at the latent level, the correlation coefficients were correspondingly a bit higher. Figure 4.1 shows how static beliefs, such as the formalism aspect and scheme orientation, are positively related and distinct from other related dynamic beliefs, such as the application and process aspects. Despite the fact that there were clear correlations between the factors, not all correlations between the individual factors were significant.

From Table 4.1 it can be seen that the pre-service teachers on average agree slightly more with the dynamic beliefs than with the static beliefs, with the highest, average value (4.31) in the process-view, but the agreement with static beliefs is relatively high (> 3.38).

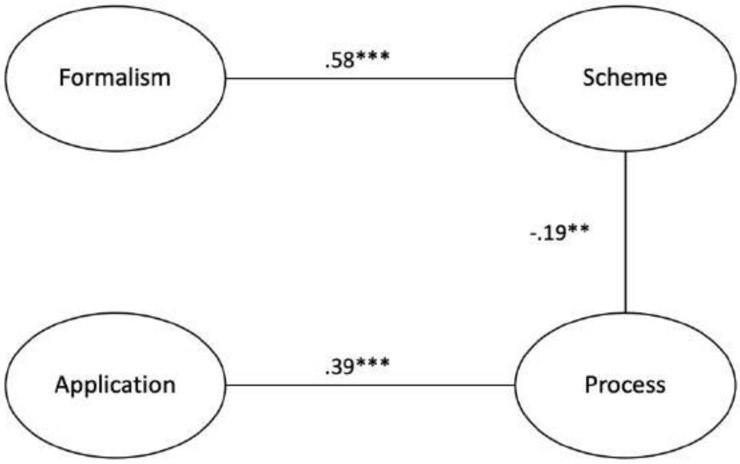


Figure 4.1: Model for the beliefs on the nature of mathematics

4.2 Beliefs on the history of mathematics

For the beliefs on the history of mathematics, a five-factor solution of different perspectives on the history of mathematics based on the assumptions could empirically be confirmed (see Figure 4.2).

The *protagonist view* includes items that focus on the work of mathematicians or show how people have used mathematics in the past.

With the *perfectionist view*, pre-service teachers agreed with the statements that formulas have always played a significant role in mathematics, and that mathematics history describes a move toward perfect mathematics.

On the other hand, the *real-life view* focuses on the high everyday value of mathematics for humans. This includes the cultural significance of mathematics and application problems that arise within mathematical development.

With a *process-oriented view*, pre-service teachers see mathematics undergoing constant change. They would accept the refutation of todays' valid mathematical knowledge, if it is proven wrong. Mathematics history shows accordingly that mathematical knowledge must constantly be questioned.

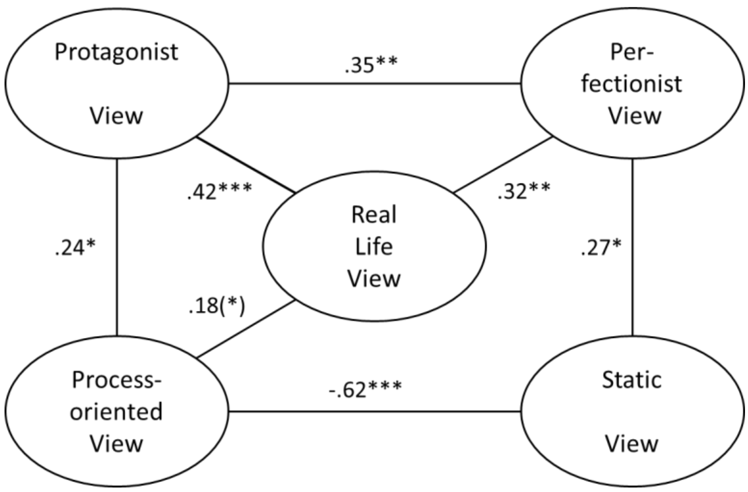


Figure 4.2: Model for the beliefs on the history of mathematics

Within the *static view*, items are summed up that no longer give mathematics any meaningful insights in the future. The pre-service teachers agreed with the statements that there is only one "right" mathematics that has not changed over time. This view understands mathematics history essentially also as a collection of biographies.

The model in Figure 4.2 fits well (χ^2/df = 1.33, RMSEA = 0.05, CFI = 0.91, SRMR = 0.07). While the static view correlates negatively with the process-oriented view, this negative correlation does not apply to all dynamic beliefs – such as the real-life view.

There were still no significant relations between the protagonist view and the static view. Rather, the protagonist viewpoint is associated with both the beliefs of the perfectionist view and the real-life and process-oriented view of the history of mathematics. Interpreting these correlations, this could mean that the work of important personalities in mathematics is more strongly associated with the dynamic development of the discipline than with the creation of eternally valid theorems. The perfectionist view, on the other hand, correlates both with the protagonist view and the real-life view, as well as with the static view. All in all, it was thus possible to identify a much more differentiated structure of the beliefs on the history of mathematics, which cannot simply be attributed to the distinction between static and dynamic beliefs.

Scale	Items	M	SD	Cronbach's α	Example Items
Beliefs on the structure of mathematics (Grigutsch et al. 1998)					
Formalism	4	3,67	,62	.77	Fundamental to mathematics is its logical rigor and precision.
Application	2	3,88	,61	.63	Many aspects of mathematics have practical relevance.
Process	4	4,31	,49	.70	Mathematical problems can be solved correctly in many ways.
Scheme	2	3,38	,80	.68	When solving mathematical tasks, you need to know the correct procedure, else you would be lost.
Beliefs on the history of mathematics (Schorcht & Buchholtz, 2015)					
Process-oriented View	5	3,63	,62	.78	The history of mathematics shows us that mathematical knowledge must constantly be scrutinized.
Real-Life View	4	3,96	,53	.67	The history of mathematics shows how people solved everyday problems with mathematics
Protagonist View	4	3,64	,48	.54	The history of mathematics shows us the work of outstanding personalities.
Static View	5	2,02	,58	.71	The history of mathematics shows that there is nothing new to explore in mathematics.
Perfectionist View	2	2,99	,76	.54	The history of mathematics describes the path of mathematics towards a consistent system without contradictions.
Beliefs on the teaching and learning of history of mathematics					
Application reasons	4	3,65	,68	.76	Historical references in mathematics lessons teach students the practical applicability of mathematical concepts.
Deepening reasons	4	3,50	,73	.81	Historical references in mathematics lessons help students recognize interconnections between mathematical

					concepts.
Motivation reasons	3	3,59	,77	.75	Students should learn historical references in mathematics education, because it also allows mathematically less interested students to learn mathematics.
Critical stance reasons	2	3,59	,66	.37	In dealing with the erroneous paths of mathematics, students can develop a critical attitude towards mathematical insights.
Time reasons	3	2,74	,80	.78	Historical references depend on many interpretations and require too much time in mathematics lessons.
Relevance reasons	3	2,50	,76	.76	Students do not need to learn historical references in mathematics lessons because they usually are not subject to exams.
Complexity reasons	2	2,44	,80	.70	Historical references to earlier errors and fallacies of mathematics only confuse students.

Table 4.1 : Scales and descriptive statistics

However, the relationships are only very cautiously interpretable, since some scales showed poor reliability (see Table 4.1). We attribute this to the difficulty of developing scales of very heterogeneous and nuanced content. Table 4.1 shows that the pre-service teachers agreed least on the history of mathematics in the static (2.02) and the perfectionist (2.99) view-points, while the agreement on the real-life view (3.96) is much higher.

4.3 Beliefs on the teaching and learning of history of mathematics

For the beliefs on the teaching and learning of history of mathematics, we here present a seven-factor solution that showed satisfying model fit (χ^2/df = 1.53, RMSEA = 0.06, CFI = 0.93, SRMR = 0.05). Although the model had a good fit, very high correlations could be identified between the factors (multicollinearities), which makes it difficult to distinguish between individual belief facets. There was a clear correlation pattern between the three objections (time, relevance and complexity) and the four affirmative belief facets (application, deepening, motivation and critical stance). This suggests an underlying structure of higher order (Byrne, 2012), which we elaborated on elsewhere (Buchholtz & Schorcht, submitted). Note that the factor critical stance also showed unsatisfactorily reliability (see Table 4.1). However, we decided to take up this factor here and to present the whole seven-factor model in order to present our findings to a higher degree of detail and to better map the theoretical anchoring of the instrument (see Figure 4.3).

Affirmative beliefs therefore contain four justification patterns: deepening reasons, motivation reasons, application reasons and critical stance reasons.

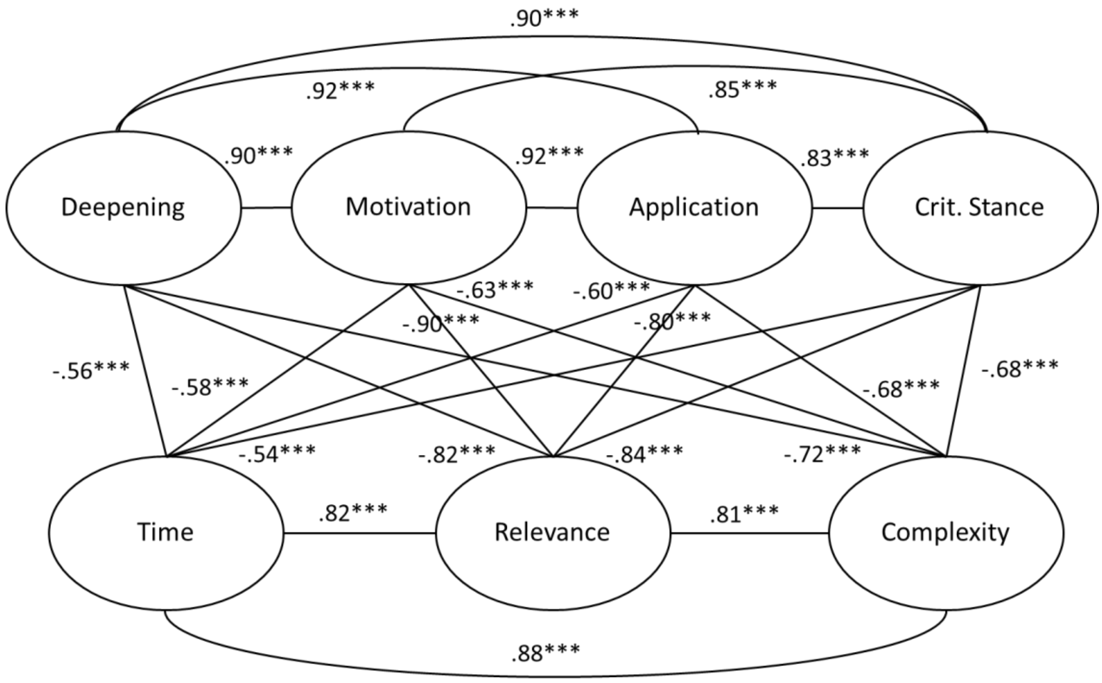


Figure 4.3: Model for the beliefs on the teaching and learning of history of mathematics

Deepening reasons are a collection of beliefs that aim to help students understand the genesis and inter-connectedness of mathematical terminology in the classroom.

Reasons for justification, which are more likely to be attributed to *motivation*, point to the functional role of mathematics history in mathematics teaching like when used in problem-posing for a lesson or the motivation of less interested students.

The *application* reasons focus on the training of problem-solving skills. Pre-service teachers who agreed with these statements would claim to use mathematics history to clarify the applicability of mathematical concepts. The students would thereby better recognize a sense of their own learning.

Other reasons point to the evolution of a *critical stance* towards mathematics and the development of an inquiry-oriented mind-set.

Beliefs about objections also have three justification patterns: time reasons, relevance reasons, and complexity reasons.

Under *time* reasons, we find beliefs, which attach too much time to the treatment of mathematical historical references, which is therefore not available when teaching. At best, mathematics history is seen as a digression in the lesson and is considered time-consuming due to its complexity.

The *relevance* factor includes statements that assume that the history of mathematics is boring students and that the content of mathematical-historical references is not relevant to tests and exams. The knowledge of the historical development of a mathematical concept is therefore of little relevance as long as one knows the definition of the concept.

Under the *complexity* factor, pre-service teachers also regard mathematics history as too complex to handle in class. Especially, they agree on the statement that mathematical errors and fallacies, often mentioned in mathematics historical content, could rather confuse the learners than help to build up mathematical understanding.

In the seven-factor model, the very high significant correlations between all affirmative factors and between all factors of objections and the very high negative correlations between the four affirmative and the three factors of objections are striking. This belief pattern indicates that affirmative reasoning as well as objections patterns are closely related, but pre-service teachers tend to emphasize one or the other aspect of their beliefs in a complementary manner. Interestingly, the correlations between the time factor and the affirmative factors in the range of -.54 to -.60 are somewhat smaller than between the other two negative factors and the affirmative factors (-.68 to -.90). This may suggest that even though pre-service teachers see a benefit in the thematization of mathematics-historical content in the classroom, concerns about lack of time cannot be completely dispelled. However, particularly in the case of objections, to interpret this as a dichotomy between the different reasons is supported by the descriptive statistics (see Table 4.1). For all facets without the exception of the critical stance factor we achieved good reliability, which could be because the factor only consists of two items. Interestingly, the factors of objections consistently averaged less agreement and a higher standard deviation, suggesting a larger divergence of pre-service teachers' answers. The lowest agreement could be identified in the complexity (2.44) and relevance aspect (2.50), and the application aspect (3.65) received the strongest agreement.

4.4 Relations between different dimensions of beliefs

In analyzing the relations of beliefs on the history of mathematics and beliefs on the structure of mathematics, we find significant correlations between the different facets, but with the exception of a mid-high correlation between real life view and application orientation (.49) all other significant correlations are relatively low (see Table 4.2).

	Process-oriented View	Real-life View	Protagonist View	Static View	Perfectionist View
Formalism	n.s.	n.s.	.23**	n.s.	.32**
Application	.18*	.49**	.23**	-.22**	n.s.
Process	.22**	.41**	.27**	-.25**	.18*
Scheme	n.s.	n.s	.23**	n.s.	n.s.

Table 4.2: Relations between beliefs about the history of mathematics and the structure of mathematics

Between the dynamic beliefs on mathematics - application and process - and the process-oriented view and the real-life view on the history of mathematics we find relationships as expected. This also applies to the negative relations between the static view on the history of mathematics and the dynamic beliefs on mathematics. Interestingly, the protagonist view correlates significantly with all beliefs on the structure of mathematics, which can be taken as an indication that (anecdotal) beliefs about outstanding personalities and their work in the development of mathematics may be overarching beliefs that are independent from whether mathematics is perceived as a more logical-deductively ordered structure or as applied science. However, these relationships are only carefully interpretable here due to the poor reliability of the scale. Another interpretable result is that the belief that the history of mathematics witnesses the evolution of mathematics toward a perfect system is related to structural beliefs about formalism. Epistemological similarities such as the orientation towards (the development of) universal formulas and logical statements, which contain both facets of beliefs, are likely to be decisive here. Interestingly, however, formalistic beliefs and

	Process-oriented view	Real-life view	Protagonist view	Static View	Perfectionist View
Application	.23**	.45**	.29**	-.24**	.18*
Deepening	.27**	.41**	.29**	-.26**	n.s.
Motivation	.21*	.41**	.29**	-.22**	.17*
Critical Stance	.25**	.35**	.26**	-.25**	n.s.
Time	n.s	-.24**	n.s.	.26**	n.s.
Relevance	n.s.	-.29**	-.16(*)	.26**	n.s.
Complexity	n.s.	-.28**	-.15(*)	.29**	n.s.

Table 4.3: Relations between the beliefs on history of mathematics and the teaching and learning of history of mathematics

static beliefs on the history of mathematics (including the belief that mathematics does not change over time) are not related.

The analysis of the relations between beliefs on the history of mathematics and the beliefs on the teaching and learning of history of mathematics initially revealed a clear pattern (see Table 4.3).

The process-oriented view, the real-life view and the protagonist view (and at 10% significance level also the perfectionist view for some instances) are weakly to medium highly positively correlated to affirmative beliefs on the teaching and learning of history of mathematics (see Table 4.3).

Partly - as in the real-life view and in the protagonist view - the beliefs on the history of mathematics are also negatively correlated with the negative beliefs about teaching and learning. On the other hand, the static perspective is positively related to the rather negative beliefs, although the size of the relation-ship does not differentiate between the aspects of justification. Overall, however, the correlations are only low to medium high here.

5 Discussion

Using confirmatory factor analyzes, our study has elucidated pre-service teachers' various beliefs about the structure of mathematics, as well as various views and justifications in the beliefs of history of mathematics and in the teaching and learning of the history of mathematics.

The results on the beliefs on the structure of mathematics (see section 4.1) have already replicated well-known structural results with regard to static and dynamic beliefs of (pre-)service teachers on mathematics (Blömeke et al., 2010; Voss et al., 2013). However, static or dynamic beliefs on the structure mathematics do not translate clearly into beliefs on the history of mathematics. With regard to the beliefs on the history of mathematics, a more differentiated picture of different static and dynamic points of view was found, which are however interrelated. A process-oriented view of the history of mathematics, a real-life view, a protagonist view, a perfectionist view and a static view could be distinguished (see section 4.2).

Our first research question asked for what beliefs could be identified for the pre-service teachers in our sample. For the beliefs on mathematics we could identify static as well as dynamic beliefs among our pre-service teachers. However, for the beliefs on the history of mathematics we might see the impact of the positively selected sample. The majority of the pre-service teachers in the sample clearly agreed with the process-oriented and the real-life view on the history of mathematics, the static view of the history of mathematics received the lowest approval. Among the reasons in favour or against the use of mathematics history in the classroom were affirmative as well as negative beliefs among the pre-service teachers. The affirmative beliefs for the teaching and learning of history of mathematics (see section 4.3) included various justification patterns, such as application, motivation, deepening and critical stance reasons. The beliefs of objections were captured in three typical justification patterns: timing, relevance and complexity reasons.

Our second research question asked for the relations between different facets of beliefs. Correlation analyzes of the different beliefs provided interesting insights into the structural relationships of the various facets. Overall, the majority of pre-service teachers in our sample supported the use of the history of mathematics in the classroom. Our convergent findings show that the affirmative justifications for this advocacy are also related to dynamic views on the history of mathematics, which in turn are linked to dynamic views of the structure of mathematics. For the inclusion of the history of mathematics in the classroom, this means that in this case, pre-service teachers will use the history of mathematics in the classroom to emphasize the dynamic aspects of mathematics and its history, and to provide a deeper understanding of mathematics. On the other hand, if pre-service teachers increasingly take a static view on the history of mathematics, this relates to their greater rejection of the use of mathematics history in teaching. However, this relationship seems to be independent of static beliefs about the structure of mathematics, such as the formalism aspect or schema orientation. Pre-service teachers with a static view reject mathematics history in the classroom because they think it is too complex for the students. In addition, they see the available time in the classroom as too tight, as that mathematics history could be integrated in addition. Also, the lack of relevance for exams is a justification for not using mathematics history.

Interestingly, we were able to identify differentiated relationships between static beliefs about the structure of mathematics and dynamic beliefs on the history of mathematics. Thus, beliefs on the formalism aspect and scheme orientation are related, albeit only weakly, to the protagonist view and the perfectionist view of the history of mathematics, but not to the static view of the history of mathematics. All in all, we infer from our results that prospective teachers specifically locate the reflection about people and outstanding figures in the history of mathematics within two different views on the history of mathematics, which are related differently to beliefs on the teaching and learning of history of mathematics and the structure of mathematics. On the one hand, people and their influence on mathematics are reflected within the protagonist view, with the focus then being on the mathematical work of the persons in their time. Appropriate beliefs are agreed on by pre-service teachers with all sorts of structural beliefs and are positively associated with affirmative beliefs about teaching and learning about the history of mathematics. On the other hand, pre-service teachers with a static view perceive mathematics history also as a pure collection of biographies (which may have anecdotal value for teaching at best). These pre-service teachers may not regard the knowledge of human achievements in the development of mathematics as very relevant, and accordingly, they might assess the importance of this knowledge as less important for teaching and learning of mathematics.

Overall, however, the results of the study ÜberLeGMa can only be interpreted with caution. Some scales showed poor reliability, so it seems appropriate to replicate the findings in further studies. Moreover, the small sample of pre-service teachers does not allow a generalization of the results. Since we did not forge any direct comparisons between universities, we refrained from displaying a site-specific presentation of the individual sub-samples, but it cannot be ruled out that results are distorted by the influence of locations with high numbers of pre-service teachers. In further follow-up studies our quantitative findings could be extended by additional qualitative studies. For example, new research questions arise about the relationships of beliefs or the impact of courses in history of mathematics on respective beliefs.

REFERENCES

Alpaslan, M., Işıksal, M., & Çiğdem, H. (2014). Pre-service mathematics teachers' knowledge of history of mathematics and their attitudes and beliefs towards using history of mathematics in mathematics education. *Science & Education, 23*, 159–183. doi: 10.1007/s11191-013-9650-1

Association of Mathematics Teacher Educators (AMTE) (Ed.) (2017). *Standards for Preparing Teachers of Mathematics*. Available online at amte.net/standards (03/13/2019).

Biegel, G., Reich, K., & Sonar, T. (2008). *Historische Aspekte im Mathematikunterricht an Schule und Universität*. Stuttgart & Göttingen: Termessos.

Blömeke, S., Kaiser, G., & Lehmann, R. (Hrsg.) (2010). *TEDS-M 2008: Professionelle Kompetenz und Lerngelegenheiten angehender Mathematiklehrkräfte für die Sekundarstufe I im internationalen Vergleich*. Münster: Waxmann.

Buchholtz, N., & Schorcht, S. (submitted). *Welche Überzeugungen haben Lehramtsstudierende zur Geschichte der Mathematik? Ergebnisse der Studie ÜberLeGMa*. To appear in *mathematica didactica*.

Buchholtz, N., & Kaiser, G. (2017). Ein Mixed-Methods-Evaluations-Ansatz zur Untersuchung von Makro-Mikro-Interaktionen: Die Entwicklung von lehr- und lernorientierten Überzeugungen von Mathematiklehramtsstudierenden in der Studieneingangsphase. *Kölner Zeitschrift für Soziologie und Sozialpsychologie, 69*(Suppl. 2), 435–458. doi: 10.1007/s11577-017-0465-y

Buehl, M., Alexander, A., & Murphy, P. (2002). Beliefs about schooled knowledge: Domain general or domain specific? *Contemporary Educational Psychology, 27*, 415–449. doi: 10.1006/ceps.2001.1103

Burns, B. (2010). Pre-service teachers' exposure to using the history of mathematics to enhance their teaching of high school mathematics. *Issues in the Undergraduate Mathematics Preparation of School Teachers. The Journal, 4*, 1–9.

Bütüner, S. Ö. (2018). Secondary school mathematics teachers' knowledge levels and use of history of mathematics. *Journal of Education and Training Studies, 6*(1), 9–20. doi: 10.11114/jets.v6i1.2722

Byrne, B. M. (2012). *Structural equation modeling with Mplus: Basic concepts, applications, and programming*. New York, USA: Routledge.

Charalambous, C. Y., Panaoura, A., & Philippou, G. (2009). Using the history of mathematics to induce changes in preservice teachers' beliefs and attitudes: Insights from evaluating a teacher education program. *Educational Studies in Mathematics, 71*(2), 161–180. doi: 10.1007/s10649-008-9170-0

Chorlay, R. (2016). Historical sources in the classroom and their educational effects. In Radford, L., Furinghetti, F., & Hausberger, T. (Eds.). *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (pp. 5–23). Montpellier, France: IREM de Montpellier.

Clark, K., Kjeldsen, T. H., Schorcht, S., & Tzanakis, C. (Eds.) (2018). *Mathematics, Education and History: Towards a harmonious partnership*. Cham: Springer. doi: 10.1007/978-3-319-73924-3

Clark, K., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang, X. (2016). History of Mathematics in Mathematics Education: Recent developments. In Radford, L., Furinghetti, F., & Hausberger, T. (Eds.). *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (135–179). Montpellier, France: IREM de Montpellier.

Eichler, A. & Erens, R. (2015). Domain-specific belief systems of secondary mathematics teachers. In Pepin, B. & Rösken-Winter, B. (Eds.). *From beliefs to dynamic affect systems in mathematics education. Exploring a mosaic of relationships and interactions* (179–200). Cham: Springer. doi: 10.1007/978-3-319-06808-4_9

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In Ernest, P. (Ed.). *Mathematics teaching. The state of the art (249–254)*. New York: Falmer Press.

Fauvel, J. & van Maanen, J. (2000). *History in mathematics education: the ICMI study*. Dordrecht: Kluwer. doi: 10.1007/0-306-47220-1

Fried, N. M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education, 10*(4), 391-408. doi: 10.1023/A:1011205014608

Furinghetti, F. (2007). "Teacher education through the history of mathematics". *Educational Studies in Mathematics, 66*(2), 131–143. doi: 10.1007/s10649-006-9070-0

Glaubitz, M. R. (2011). *Mathematikgeschichte lesen und verstehen: Eine theoretische und empirische Vergleichsstudie*. (Vol. 1./ Dissertation). Universität Duisburg-Essen, Germany. Available online https://duepublico2.uni-due.de/servlets/MCRFileNodeServlet/duepublico_derivate_00027471/DissertationGlaubitzBd1.pdf (03/13/2019)

Goodwin, D. M. (2007). *Exploring the relationship between high school teachers' mathematics history knowledge and their images of mathematics* (Dissertation). University of Massachussets, USA: Lowell MA.

Grattan-Guinness, I. (2004a). The mathematics of the past. Distinguishing its history from our heritage. *Historia Mathematica, 31*, 163–185. doi: 10.1016/S0315-0860(03)00032-6

Grattan-Guinness, I. (2004b). History or heritage? An important distinction in mathematics for mathematics education. *The American Mathematical Monthly,111*(1), 1–12. doi: 10.2307/4145010

Grigutsch, S., Raatz, U., & Törner, G. (1998). Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematik-Didaktik, 19*(1), 3–45. doi: 10.1007/BF03338859

Handal, B. (2003). Teachers' mathematical beliefs. A review. *The Mathematics Educator, 13*, 47–57.

Ho, W. K. (2008). *Using history of mathematics in the teaching and learning of mathematics in Singapore*. Paper presented at the 1st RICE, Singapore: Raffles Junior College.

Hofer, B. K., & Pintrich, P. R. (1997). The development of epistemological theories. Beliefs about knowledge and knowing and their relation to learning. *Review of Educational Research, 67*, 88–140. doi: 10.2307/1170620

Hu, L. & Bentler, P.M., (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling: A Multidisciplinary Journal, 6*(1), 1–55. doi: 10.1080/10705519909540118

Jankvist, U. T. (2010). Students' beliefs about the evolution and development of mathematics. In Durand-

Guerrier, V., Soury-Lavergne, S., & Arzarello, F. (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (2732–2741). Lyon: INRP. doi: 10.5948/UPO9781614443001.013

Jankvist, U. T. (2009). *Using history as a ‚goal‘ in mathematics education. Dissertation. Roskilde: Roskilde University*. Available online http://milne.ruc.dk/imfufatekster/pdf/464.pdf (03/13/2019)

Kuntze, S. (2011). Pedagogical content beliefs: Global, content domain-related and situation-specific components. *Educational Studies in Mathematics, 79*(2), 273–292. doi: 10.1007/s10649-011-9347-9

Lakoma, E. (2002). History of mathematics in curricula and schoolbooks: a case study of Poland. In Fauvel, J. & van Maanen, J. (Eds.). *History in mathematics education: the ICMI study* (19–29). Dordrecht, Netherlands: Kluwer.

Liu, P.-H. (2003). Do teachers' need to incorporate the history of mathematics in their teaching? *Mathematics Teacher, 96*(6), 416–421.

Mosvold, R., Jakobsen, A., & Jankvist, U. T. (2014). How mathematical knowledge for teaching may profit from the study of history of mathematics. *Science & Education, 23*(1), 47–60. doi: 10.1007/s1119 1-013-9612-7

Muthén, L. K., & Muthén, B. O. (1998–2012). *Mplus user's guide* (7th ed.). Los Angeles: Muthén & Muthén.

Nickel, G. (2013). Vom Nutzen und Nachteil der Mathematikgeschichte für das Lehramtsstudium. In Allmendinger, H., Lengnink, K., Vohns, A., & Wickel, G. (Eds.), *Mathematik verständlich unterrichten: Perspektiven für Unterricht und Lehrerbildung* (253–266). Wiesbaden: Springer. doi: 10.1007/978-3-658-00992-2_16

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*, 307–332. doi: 10.3102/00346543062003307

Panasuk, R. M., & Horton, L. B. (2012). Integrating history of mathematics into curriculum: what are the chances and constraints? *International Electronic Journal of Mathematics Education, 7*(1), 3–20.

Philippou, G. N. & Christou, C. (1998). The Effects of a preparatory mathematics program in changing prospective teachers' attitudes towards mathematics. *Educational Studies in Mathematics, 35*, 189–206. doi: 10.1023/A:1003030211453

Reusser, K., Pauli, C, & Elmer, A. (2011). Berufsbezogene Überzeugungen von Lehrerinnen und Lehrern. In Terhart, E., Bennewitz, H. & Rothland, M. (Eds.). *Handbuch der Forschung zum Lehrerberuf* (pp.478–495). Münster: Waxmann.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In Sikula, J., Buttery, T., & Guyton, E. (Eds.). *Handbook of research on teacher education* (2nd ed., pp. 102–106). New York: Macmillan.

Schmotz, C., Felbrich, A., & Kaiser, G. (2010). Überzeugungen angehender Mathematiklehrkräfte für die Sekundarstufe I im internationalen Vergleich. In Blömeke, S., Kaiser, G., & Lehmann, R. (Eds.), *TEDS-M 2008: Professionelle Kompetenz und Lerngelegenheiten angehender Mathematiklehrkräfte für die Sekundarstufe I im internationalen Vergleich* (pp. 279–305). Münster: Waxmann.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education, 4*(1), 1–94. doi: 10.1016/S1080-9724(99)80076-7

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge. doi: 10.4324/9780203843000

Schorcht, S. (2018). *Typisierung mathematikhistorischer Beispiele in deutschen Mathematikschulbüchern der Klassenstufen 1 bis 7*. Münster: WTM.

Schorcht, S. & Buchholtz, N. (2015). Ergebnisse einer Pilotstudie zu Überzeugungen von Lehramtsstudierenden zur Geschichte der Mathematik. In Caluori, F., Linneweber-Lammerskitten, H., & Streit, C. (Eds.). *Beiträge zum Mathematikunterricht* (pp. 1150–1151). Münster: WTM-Verlag.

Schulte, L. (2016). Eine Querschnittsanalyse zur Rolle der Mathematikgeschichte in aktuellen Schulbüchern.

SieB: Siegener Beiträge zur Geschichte und Philosophie der Mathematik, 7, 99–129.

Siu, M.-K. (2006). No, I don't use history of mathematics in my classroom: Why? In Furinghetti, F., Kaisjer, S., & Tzanakis, C. (Eds.). *Proceedings of HPM 2004 & ESU 4* (pp. 268–277). Iraklion, Greece: University of Crete.

Smestad, B. (2011). Teachers' conceptions of history of mathematics. In Katz, V. J. & Tzanakis, C. (Eds.). *Recent Developments on Introducing a Historical Dimension in Mathematics Education* (pp. 231–240). Washington, USA: Mathematical Association of America. doi: 10.5948/UPO9781614443001.022

Staub, F. C. & Stern, E. (2002). The nature of teachers' pedagogical content beliefs matters for students' achievement gains: Quasi-experimental evidence from elementary mathematics. *Journal of Educational Psychology, 94*, 344-355. doi: 10.1037/0022-0663.94.2.344

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In Grows, D. A. (Ed.). *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: Macmillan.

Törner, G. (2002). Mathematical beliefs – A search for a common ground: Some theoretical considerations on structuring beliefs, some research questions, and some phenomenological observations. In Leder, G., Pehkonen, E., & Törner G. (Eds.). *Beliefs: A hidden variable in mathematics education?* (pp. 73–94). Dordrecht: Kluwer. doi: 10.1007/0-306-47958-3_5

Tzanakis, C., Arcavi, A., de Sá, C. C., Isoda, M., Lit, C.-K., Niss, M., de Carvalho, J. P., Rodriguez, M., & Siu, M. K . (2000). Integrating history of mathematics in the classroom: an analytic survey. In Fauvel, J. & van Maanen, J. (Eds.). *History in mathematics education: the ICMI study* (201–240). Dordrechts: Kluwer. doi: 10.1007/0-306-47220-1_7

Voss, T., Kleickmann, T., Kunter, M., & Hachfeld, A. (2013). Überzeugungen von Mathematiklehrkräften. In Kunter, M., Baumert, J., Blum, W., Klusmann, U., Krauss, S., & Neubrand, M. (Eds.), *Professionelle Kompetenz von Lehrkräften. Ergebnisse des Forschungsprogramms COACTIV* (pp. 235–257). Münster, Germany: Waxmann.

APPENDIX A

26 Items: "Beliefs on the history of mathematics"
The history of mathematics shows that the basic mathematical findings are centuries old and have survived the time.
The history of mathematics shows us the work of outstanding personalities.
The history of mathematics for me is a collection of interesting anecdotes.
The history of mathematics testifies that formulas have always played a significant role in mathematics.
The history of mathematics shows us that mathematical knowledge must constantly be scrutinized.
The history of mathematics shows that mathematics has its origins in application problems.
How people used mathematics at their own time is shown by the history of mathematics.
Mathematics has also gone astray in the course of its development.
Mathematics history shows that mathematics is undergoing constant change.
The history of mathematics testifies to the development of mathematical ideas towards a perfect mathematics.
In the future, no fundamentally new mathematical findings will be discovered.
The history of mathematics shows us that you have to deal critically with mathematical findings.
History of mathematics is essentially a collection of biographies.
The history of mathematics shows the constant elimination of mathematical inconsistencies.
The history of mathematics shows that there is nothing new to explore in mathematics.
The history of mathematics shows that mathematics does not change over time.
The history of mathematics testifies that there is only one "correct" mathematics.
The history of mathematics shows how people solved everyday problems with mathematics.
Mathematics history shows that mathematical discoveries are eternally valid and unchangeable.
The history of mathematics illustrates the high everyday benefits that mathematics has for people.
The history of mathematics shows the high cultural significance of mathematics.
In the future, today's accepted mathematical findings could be discarded again.
History of Mathematics describes how people practiced math in their time.
The history of mathematics describes the path of mathematics towards a consistent system without contradictions.
The history of mathematics documents the constant progress in mathematics.
The history of mathematics describes mathematics as the spiritual creation of humanity.

APPENDIX B

21 Items: "Beliefs on the teaching and learning of history of mathematics"
Historical references in mathematics education contribute to an application-oriented image of mathematics.
In mathematics it is not only important to know a concept, but also its historical development.
At best, historical references can be used for a digression in the classroom.
Historical references in mathematics lessons teach students the practical applicability of mathematical concepts.
Historical references in mathematics classes can take away students' fear of the "scientific" mathematics.
Students should learn historical references in mathematics education, because it also allows mathematically less interested students to learn mathematics.
Dealing with the struggle for solutions to mathematical problems make students understand the meaning of their own learning.
Historical references are ideal as an introduction to a substantive mathematical topic.
Dealing with the history of mathematics trains problem solving abilities of students.
Dealing with historical references in mathematics teaching motivates students.
Historical references in mathematics lessons help students recognize interconnections between mathematical concepts.
Students will gain a deeper understanding of mathematical procedures as they see how they have changed over time.
By dealing with the historical genesis of mathematical concepts, they can be better memorized and understood.
The historical development of mathematics is too complex to handle in class.
Historical references depend on many interpretations and require too much time in mathematics lessons.
The knowledge of the historical development of a mathematical concept is of little relevance as long as one knows the definition of the concept.
In dealing with the erroneous paths of mathematics, students can develop a critical attitude towards mathematical insights.
Historical references in mathematics education are too time consuming.
Historical references in mathematics lessons bore students.
Students do not need to learn historical references in mathematics lessons because they usually are not subject to exams.
Historical references to earlier errors and fallcies of mathematics only confuse students.

HPM LESSON STUDY IN THE CONTEXT OF AN HPM LEARNING COMMUNITY

A Case Study in Chinese Senior High School

Zhongyu SHEN
East China Normal University, 3663 N. Zhongshan Rd, Shanghai, China
344080566@qq.com

ABSTRACT

Considerable work has been done over the last 15 years on how to integrate History of Mathematics (HM) into Mathematic Education (ME). However, integrating HM into ME from the perspective of teaching research system has been less researched. In Mainland China, a teaching research system has been practiced nationally since the 1950s. China has a long history of collaboration among teachers. This paper deals with the teaching research system that integrates HM into ME at Chinese education system. The conceptual framework in this research including HPM Learning Community (HPMLC) and HPM Lesson Study (HPMLS). We used the research method of case study. Based on the conceptual framework above, we selected one case of HPM lesson that integrates the history of conic section into teaching practice. Through case analysis, the conceptual framework is modified to obtain the result of this study that how the HPMLS work under the context of HPMLC.

1 Introduction

Considerable work has been done over the last 15 years on how to integrate History of Mathematics (HM) into Mathematic Education (ME) (Clark et al., 2016), and there are various classifications of approach (Tzanakis et al., 2000; Jankvist, 2009). However, integrating HM into ME from the perspective of teaching research system has been less researched.

In Mainland China, a teaching research system has been practiced nationally since the 1950s, which refers to various activities of professional development institutionalized by four hierarchical organizations: province/city, district/county, school, and lesson plan group. The system focuses on "guiding teaching research, overseeing teaching administration in schools on behalf of educational bureaus, providing consultation for educational authorities, mentoring the implementation and revision of new curricula, building the bridge between modern educational theories and teaching experiences, and promoting high-quality classroom instruction" (Huang, Ye & Prince, 2017).

The core components of Chinese teaching research activities is studying lessons which include Keli (Exemplary Lesson Development) (pronounced: Ker-Lee) and public lessons, Both Keli and public lessons are known as Chinese Lesson Study (LS) (Huang & Shimizu, 2016). Chinese LS is a form of school-based professional development that aims to update ideas of teaching and learning, to design new learning situations, and to improve classroom practice through Keli, which is a community-mediated process of developing an exemplary lesson, including the planning, delivery, debriefing, revision and re-teaching of the lesson, and this form develops a model known as action education (Gu & Gu, 2016).

Unlike the West, China has a long history of collaboration among teachers (Wong, 2010). Recently, concepts like "Profession Learning Communities" (PLC) are flourishing (Cheng & Wu, 2016). Hord define PLC as defined it as ‘‘teachers in a school and its administrators continuously seek and share learning and then act on what they learn. The goal of their actions is to enhance their effectiveness as professionals so that students benefit (Hord, 1997). So, it is meaningful to research LS in the context of PCL in China.

This paper deals with the teaching research system that integrates HM into ME at Chinese education system. This paper discusses the following questions:

1. What is the structure and characteristics of PLC that integrates HM into ME?
2. How does Chinese Lesson Study integrate HM into ME work in the context of PCL?

2 The conceptual framework

2.1 HPM Learning Community

The whole process was done by two teams (see fig. 1), HPM Research Team and School Teacher Team, and each team has its own expertise. The research team includes HPM expert, Ph.D. and M.A. students in the direction of HPM, whose team was responsible for theoretical guidance and providing historical materials, meanwhile, the school teacher team was mainly responsible for designing discussions and teaching practice. They related each other by HPM seminar.

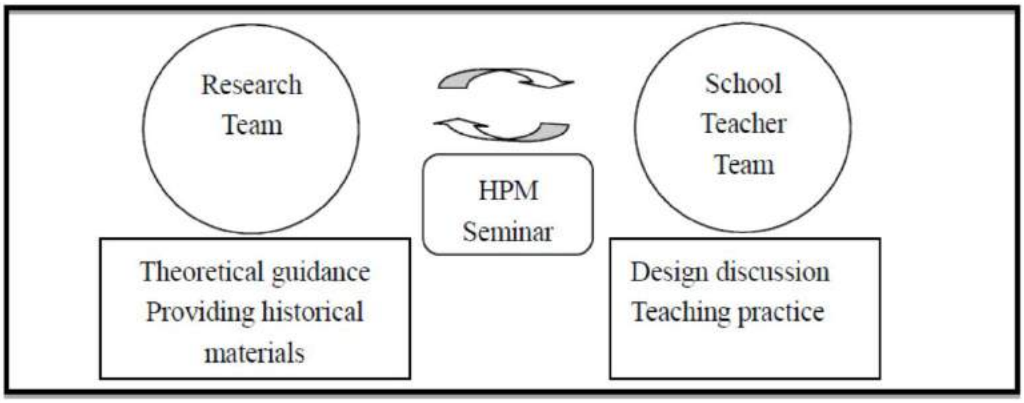


Figure 2.1: HPM-LC

These two teams form a community, and as shown in table 2.1, it has the characteristics of PLC (Huffman, Hipp & Hord, 2003), so we call it HPM learning community (HPMLC).

Table 2.1: Characteristics of PLC

Characteristics	connotation
Shared and Supportive Leadership	●Nurturing leadership among staff ●Shared power, authority and responsibility ●Board-based decision-making that reflects commitment and accountability
Shared Values and Vision	●Espoused values and norms ●Focus on student learning ●High expectations ●Shared vision guides teaching and learning
Collective Learning and Application	●Sharing information ●Seeking new knowledge, skills and strategies ●Working collaboratively to plan, solve problems and improve learning opportunities
Shared personal practice	●Peer observation to offer knowledge, skill and encouragement ●Feedback to improve instructional practices ●Sharing outcomes of instructional practices ●Coaching and mentoring
Supportive Condition	●Relationships ●Structure

2.2 HPM Lesson Study

By using the concept of Chinese LS and PLC, we using the concept structure of the teaching research system that integrates HM into ME in this research. We call the lessons that integrate HM into teaching as HPM lesson, and we call The Chinese LS combining HPM as HPM Lesson Study (HPMLS). The development process of HPMLS includes four stages (see Fig 2) (Wang, 2017).

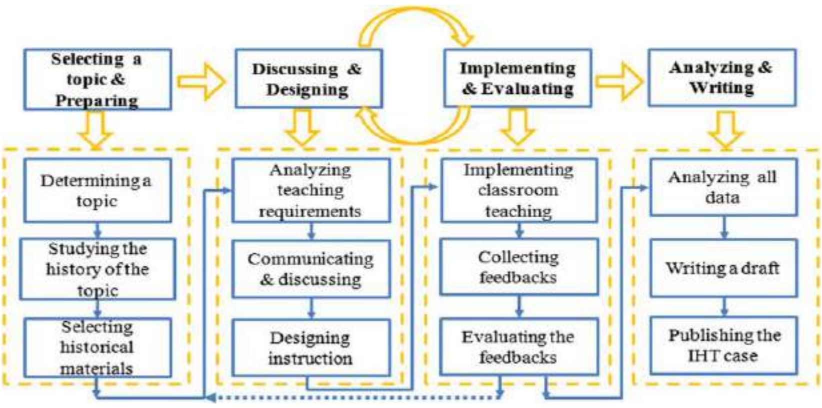


Figure 2.2: HPM-LS

All concepts mentioned above constitute the concept structure of this research.

3 Methodology

In this research, we used the research method of case study, and the research process is shown in the following figure 3.1.

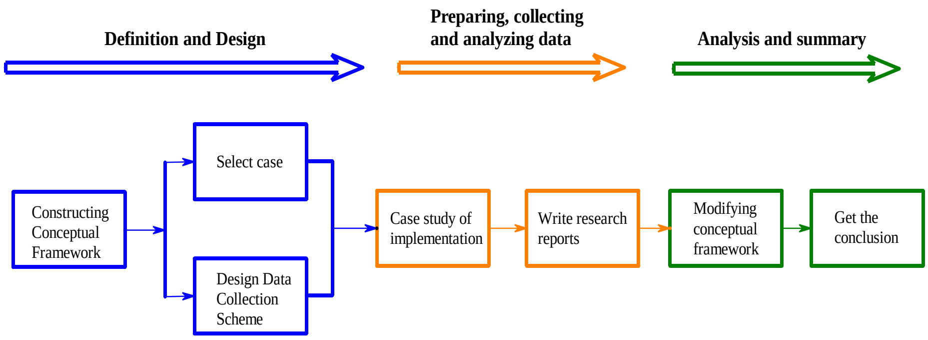


Figure 3.1: The research process

Based on the conceptual framework above, we selected one case of HPM lesson that integrates the history of conic section into teaching practice and was conducted by an HPM studio in a Chinese senior high school. Through case analysis, the conceptual framework is modified to obtain the result of this study.

3.1 The case school and participants

The research sample of this study is a HPM studio in a senior high school in China. The school was founded in 2005, and its students here are of the medium level. The participants include the research team and school teacher team, and are Research Team consists of University teachers and graduate students, while School Teacher Team includes three teachers and a teaching expert in senior high school. The basic information of the members in school teacher team is shown in the following table 3.1.

Table 3.1: Basic information of the members in school teacher team

members	Gender	Teaching age	title	Educational background	Position
A	Male	25	senior	master	Leader
B	Male	10	Intermediate	undergraduate	Teacher*
C	Male	11	senior	undergraduate	Teacher*
D	Male	13	senior	undergraduate	Teacher

* Teachers that responsible for implement the HPM lesson

3.2 Data collection and analysis

We use the modules approach in this case (Jankvist, 2009), and integrate the history into the unit of conic section, which lasts two-class periods. Through the planning, discussing, implementing and analyzing, two HPM lessons become two exemplary lessons. The whole process has undergone three discussions and two implementations. The schedule is as following table 3.2.

Table 3.2: the schedule of HPMLS

time	activity	Data collection
2017.9.28	Discussion(before lesson)	instruction design and reflection list
2017.11.17	Implementation(1 round)	instruction design、tapes and videos
2017.11.17	discussion(after lesson)	tapes and videos
2017.11.20	Implementation(2 round)	instruction design、tapes and videos
2017.11.20	discussion(after lesson)	tapes and videos

We videotaped and recorded the whole process of the HPMLS, and collected the reflection list and instruction design of the teachers.

By analyzing the data of tapes, videos, instruction design and reflection list in the development process of this HPMLS, we can better explain what the structure and characteristics of the HPMLC are and how the HPMLS works in the context of HPMLC. Therefore, it can provide enlightenment to how to conduct HPM lessons from the perspective of teaching research system.

4 Implementation of framework

4.1 Selecting a topic and preparing

Firstly, the leader A confirms the topic of conic section, which in Chinese textbooks is the content of a chapter, including curves and equations, ellipses, hyperbolas and parabolas. The first and second class periods which are about curves, equations and ellipses in this chapter were selected

The reason of choosing the curves and equations is that some of the curves in the textbook have been studied by mathematicians in history.

"In the section of curve and equation, whether it is an example or a exercise, it is found that these are some of the curves that mathematicians have studied in history. So, why don't you directly tell the students that mathematicians have studied these curves, now it is just to let us restudy it, which can increase the historical sense of mathematics and cultural charm, and let the students understand the inheritance and development of mathematical content."(in reflection list from teacher A)

The reason of choosing the ellipses is that the introduction of ellipse in textbooks is not very appropriate (see fig. 4.1), and using trajectory to define ellipse, students will not know why ellipses are called conic curves.

Based on the reasons above, the research team provided the school teacher team with published papers about "Generation of plane analytic geometry", "Origin and Development of Conic Curves" and "Travel of Elliptic Equation". Those papers were written by the HPM expert in the research team through reading the original historical sources and secondary literature.

探究

取一条定长的细绳，把它的两端都固定在图板的同一点处，套上铅笔，拉紧绳子，移动笔尖，这时笔尖（动点）画出的轨迹是一个圆．如果把细绳的两端拉开一段距离，分别固定在图板的两点处（图 2.2-1），套上铅笔，拉紧绳子，移动笔尖，画出的轨迹是什么曲线？

在这一过程中，你能说出移动的笔尖（动点）满足的几何条件吗？

Figure 4.1: The introduction of the ellipse in the textbook

After studying the history of the conic section, School Teacher Team has selected the following historical materials.

Three classic geometry problems in ancient Greek, the Greeks very early found themselves confronted by three problems which they could not solve, at least by the use of the unmarked ruler and the compasses alone. The first was the trisection of any angle, the second problem was the quadrature of the circle and the third problem was the duplication of a cube. (Smith, 1925, pp. 297-298)

Hippocrates showed that the problem of duplicating the cube resolves itself into the finding of two mean proportionals between two given lines. If $a:x=x:y=y:b$, then $x^2=ay$, $y^2=bx$, and hence $x^4=a^2y^2=a^2bx$, or $x^3=a^2b$. If $b=2a$, then $x^3=2a^3$, that is, the cube of edge x will then have double the volume of a given cube with edge a. since we have the three equations $x^2=ay$ (parabola), $y^2=bx$ (parabola), and $ab=xy$ (hyperbola), we can evidently solve the problem by finding the intersection of two parabolas or of a parabola and hyperbola. These methods are credited to Menaechmus (Smith, 1925, p. 313).

Menaechmus may have used that property of the parabola expressed by the equation $y^2=px$, and also that property of the rectangular hyperbola expressed by the equation $xy=c^2$. Archimedes used the same relation for the parabola, Apollonius carried the method much farther, the names “ellipse”, “parabola” and “hyperbola” are probably due to Apollonnius (Smith, 1925, p. 319).

We have seen that Menaechmus solved the problem of the two mean proportionals by means of conic sections. Menaechmus came to think of obtaining curves by cutting a cone. Aristaeus used the title ‘solid loci’ instead of ‘conics’ to indicate that the main devote to conics regarded as loci. He must have discussed the locus of three-line and four-line problem. Apollonius also studied this problem in Book III of *The Conics*. Pappus studied this problem and distinguished three loci: ‘plane loci’, meaning of straight line and circle; solid loci’, meaning conic sections; and ‘linear loci’, meaning curves with a more complicated and indeed a forced or unnatural origin such as spirals, quadratrices, conchoids and cissoids (Heath, 1921).

Apollonius proved that ‘in an ellipse the sum of the focal distance of any point is equal to the long axis’ (Apollonius, 1896). The French mathematician and astronomer Lahire (1640-1719) gave the definition of the focal radius of the ellipse in his work (Lahire, 1679).

The definition of the focal radius of the ellipse is widely used. The French mathematician L'Hospital in 18th century (L'Hospital, 1720) and the British mathematician Wright in the 19th century (Wright, 1836) gave the method of deriving the the equation of an ellipse.

Dandelin's spheres were discovered in 1822. They are named in honor of the Belgian mathematician Germinal Pierre Dandelin. That the locus of points such that the sum of the distances to two fixed points (the foci) is constant is an ellipse was known to ancient Greek mathematicians (like Apollonius of Perga), but Dandelin's spheres facilitate the proof (Boag, 2010).

4.2 Discussing and Designing（First round）

After the historical materials being selected, the leader A designed three continuous lessons. They are "curves and equations" (responsible for teacher B); "solving the equation of a curve" (responsible for teacher C); "the ellipse and its standard equation" (responsible for teacher D). Each teacher carried out a preliminary teaching design. After communicating within the school teacher team, they present their teaching designs to Research team in the HPM Seminar. The designs of the three teachers are as follows.

Curves and equations: (1) To introduce the three classic geometry problem in ancient Greek to students, analyze the method of duplication of a cube by Hippocrates and tell students who the ancient Greek discovered the conic curve while solving the problem. (2) Use a plane to cut the cone and get the oval. (3) Some trajectory problems of ancient Greek research are given. (4) Teach the "purity" and "completeness" of curves and equations around the trajectory problem. (5) Solve the trajectory problem, and then summarize several steps to track equation, and then do some exercises.

Solving the equation of a curve: (1) Take the contents of the lesson above and review the steps of solving the trajectory equation. (2) Continue by finding the equation of the trajectory problem given in the first lesson. (3) Give an example: the known curvilinear equation is a circle, a moving point is on the circle, and the vertical axis of the *x*-axis is passes through the moving point. To find the locus of the midpoint of the perpendicular line (the locus is ellipse). (4) Continue to study the three line problem (if the ratio is not 1, the trajectory is an ellipse).

Ellipse and its standard equation: (1) Along with the content of the first and second lessons, the students knew that the ellipse can be cut by the cone, and then the geometric method of the second lesson is used to lead to the ellipse, and it is explored by the Dandelin spheres. (2) The standard equation of an ellipse is derived from the analytic method, and it is prepared to be deduced in three or four ways. (3) More examples and exercises about ellipses are given.

After their presentation, the research team discussed these three teaching designs with the teachers' team, and the teachers' team modified their designs afterwards.

4.3 Implementing and Evaluating (First round)

Because teacher D have another task, so he is no longer in charge of the lesson of Ellipse and its standard equation, as the result, teacher B was still in charge of the lesson of curve and equation, teacher C become in charge of the lesson of Ellipse and its standard equation. So the three lesson planed in the first place became two lessons when implementing. Those two lessons that implemented are as following. Research team and teachers' team observe those two lessons together.

The first lesson started by introducing three classical geometric problems in ancient Greek, and used a plane to cut the cone and get the oval. Then began to study the six trajectory problems in ancient Greek:

(1) The distance to a fixed-point is equal to a constant;

(2) The sum of the distances to the two fixed-point is equal to a constant;

(3) The absolute value of the difference between two fixed-points is equal to a constant;

(4) The ratio of the distances to two fixed-point is equal to a constant;

(5) The ratio of the distances to two straight lines (parallel or intersecting) is equal to a constant (two-line problem)

(6) Given three straight lines, the ratio of the product of the distances of a moving point to the two lines, to the square of the distance to the third straight line is equal to a constant (three-line problem).

The teacher said that Apollonius, the Greek mathematician, used geometry to solve the problem of "three lines". However, the process was very complicated, and the "four line" trajectory problem cannot be completely solved by geometric methods. In the early 17th century, Descartes invented the coordinate system, so he, together with Fermat, began to study the trajectory problem using algebra on the basis of the coordinate system – "analytic method".

Let the students find the conic section through the problem by three-line and four-line problem. Then the teacher explained the relationship between the curve and the equation, and then let the students get the trajectory equation of the problem 1, 5 and 6. The ellipse is obtained from question 6. Then the teacher posed the four-line problem and introduced the conic sections.

The second lesson introduced the 'ellipse' from our life. Let the student recall that by using a plane to cut the cone and get the oval in the first lesson, and use of Dandelin's spheres to get the property that 'the locus of points such that the sum of the distances to two fixed points (the foci) is constant in the ellipse', could also solve problem 2 of the first lesson. After that define the ellipse by this property.

Then the teacher let the students explore the derivation methods of equation of an ellipse. Students are given a variety of methods and teacher complement the method in the history. After that, students do some classroom exercises, and then the teacher explains some extension to the ellipse. Finally, let's students take a look at what problem of trajectory had been solved after the two lessons. Problem 3 and 4 would not be solved until the following lesson.

After those two lessons, the teachers collect feedback from students and do some preliminary analysis. Then the research team and teachers' team conducted the evaluation activity.

4.4 Discussing and Implementing (Second round)

The evaluation activity found that (1) It is difficult for students to understand the three classical geometric problems and it takes too long to introduce them. (2) The order of the trajectory problem does not conform to the cognitive order of the students. (3) Too many derivation methods are unacceptable to students. (4) Students need more explanation to the extension. So the teachers' team modified their teaching design as following.

Curve and equation: (1) Shorten the introduction of three classic geometry problems in ancient Greek. (2) Adjust the order of six trajectory problems in ancient Greek. Ellipse and its standard equation: (1) Cut down some derivation methods of equation of ellipse, and increase the time for students to explore. (2) Reduced the number of classroom exercises and add explanation to the extension.

Then those two HPM lesson become two exemplary lessons and implemented. Research team, teachers' team and teachers in this region observe these two lessons together. The following content will be focused on presenting the differences between the two rounds of implementation.

The first lesson is introduced from three classic geometry problems in ancient Greek and began to study the six trajectory problems in ancient Greek:

(1) The ratio of distance to two straight lines (parallel or intersecting) is equal to constant(two-line problem)

(2) The distance to a fixed-point is equal to a constant

(3) The ratio of the distance to the two fixed-point is equal to the constant

(4) Given three straight lines, the ratio of distance product from the moving point to the two lines the square of the distance to third straight lines equal to the constant(three-line problem)

(5) The sum of the distance to the two fixed-point is equal to the constant

(6) The absolute value of the difference between the two fixed-point is equal to the constant

The teacher said the "four line" trajectory problem cannot be completely solved by geometry method and introduced the "analytic method".

Teacher explained the relationship between the curve and the equation. And then let the students get the trajectory equation of the problem1, 2 and 3. The ellipse is obtained from question 4. Then the teacher posed the four-line problem and introduced the conic sections.

The second lesson introduce the 'ellipse' from our life and use Dandelin spheres to get the property, it is also solve the problem 5 in the first lesson, then define the ellipse by this property.

Then the teacher let the students explore the derivation methods of the equation of an ellipse. The students were given a variety of methods and the teacher focused on two methods in history. The students did some classroom exercises, and then the teacher explained some extension. Finally, it is found that problem 6 remained unsolved and that it would be solved in the following lesson.

After those two lessons, teachers collected feedback of students and do some preliminary analysis. Then the research team, the teachers' team and teachers in this region conducted the activity of the lesson's evaluation.

4.5 Analyzing and Writing

The last step of HPMLS is to analyze students' feedback and write the paper of this HPM lesson. Following present some preliminary result of student’s feedback.

About the question of ‘What kind of teaching method do you prefer? Teaching methods integrating the history of mathematics, or traditional ways? Why?’. The result is that there were 33 students (40 in total) who liked to integrate the history of mathematics, accounting for 82, 5%, and 7 students who liked traditional ways of teaching, accounting for 17.5%. There are several reasons in favor of integrating history of mathematics: (1) Classroom is more interesting, not boring (25%). (2) It is easier to understand the mathematical content (15%). (3) Understand the development of the content (12.5%). (4) Acquired much extracurricular knowledge and enhanced interest in research questions (12.5%). (5) The classroom is vivid and impressive (12.5%). (6) A study spirit that can learn from the ancient people (5%).

About the question ‘Please explain why we need to establish a coordinate system and use algebraic methods to find out the equation’. In the pretest stage, the results are (1) Because of convenience (70.3%); (2) easy to find the law (10.8%); (3) do not know (13.5%). In the post test stage the results are: (1) Because of convenience (62.5%); (2) it is difficult to solve some complicated problems by using a purely geometric method (20%); (3) do not know (10%).

About the question of ‘Please tell me why the curve like a circle and an ellipse is called a conic section’. In the pretest, the result are (1) do not know (83.8%); (2) it is because the mathematicians took the name in the first place (10.8%). In the post test the result are: (1) Because it is a curve obtained by the plane when it cuts a cone (82.5%); (2) do not know (10%).

After students' feedback, the teachers reflect on teaching and write articles.

5 Findings and discussion

After the analysis of those cases, we can modify the concept framework and get the question of what the structure and characteristics of PLC and how Chinese Lesson Study that integrates HM into ME work in the context of PCL.

5.1 The structure and character of HPMLC

From the above case, we can see that the whole process was done by two teams, HPM research team and School teachers’ team, and each team has its own expertise. There was a leader in each team; the leader in school teachers’ team is responsible for selecting a topic and planning the lessons as a whole. The research team is mainly responsible for theoretical guidance and providing historical materials, while the school teacher team is mainly responsible for teaching design and teaching practice. They became related to each other via an HPM seminar. This seminar included the discussion in the stage of teaching design and the lesson’s evaluation after the implementation of the lessons.

From the above case, we can see HPMLC have the common characteristics of the PLC, they are Shared and Supportive Leadership, Shared Values and Vision, Collective Learning and Application, Shared personal practice and Supportive Condition, but there are also some special points. The revised model is shown as follows (see Table 5.1). From the perspective of Shared and Supportive Leadership, we can see that it nurtured leadership in the school teacher team and all teachers in the team shared responsibility. On the point of Shared Values and Vision, we can see that teachers have common values and norms, focus on student learning and shared vision of HPM guides teacher's teaching and learning. About the collective learning and application, school teachers work collaboratively to plan the HPM lesson and apply the new knowledge in the teaching practice. In relation to the shared personal practice, we can say that the teacher will modify his or her teaching based on the opinions which are shared by others during the lesson's evaluation stage. In relation to the Supportive Condition, we can say that the relation between teachers is very close and the school provides a lot of support.

Table 5.1: Characteristics of HPMLC

Characteristics	connotation
Shared and Supportive Leadership	● Nurturing leadership in the school team ● Shared power, authority and responsibility
Shared Values and Vision	● Common values and norms ● Focus on student learning ● Shared vision of HPM guides teaching and learning
Collective Learning and Application	● Working collaboratively to plan the HPM lesson ● Application the new knowledge in the teaching practice
Shared personal practice	● Peer observation in HPM lesson ● Sharing opinions in the lesson evaluating ● Feedback to modify lesson plan
Supportive Condition	● Relationships ● Structure

Compared with the previous characteristics, these characteristics have more HPM features.

5.2 The current model of HPMLS

From the above case, the current model of HPMLS in the context of HPMCL needs some modifications. The revised model is shown in figure 5.1.

The first stage is "Selecting a topic & Preparing". In this stage teachers and researchers need to determine a topic and select historical materials about this topic, then teachers need to complement the Preliminary teaching design by use the historical materials.

The second stage is "Discussing & Designing". In this stage, teachers will display teaching design to researchers and discuss their design with each other. Based on the discussion, the teachers will modify their preliminary teaching design.

The third stage is "Implementing & Evaluating". In this stage teachers will implement classroom teaching. After the implementation, the teachers will get students' feedback. Then the teachers' team and the researcher team will evaluate the lesson together. Based on the lesson evaluation, the teachers modify their teaching design further. This stage is repeated; usually two to three times, until the teachers feel satisfied.

The final stage is "Analyzing & Writing". In this stage, the teachers analyze all data collected after the implementation. Based on these data, the teacher will reflect by themselves and publish a paper on the HPM lessons.

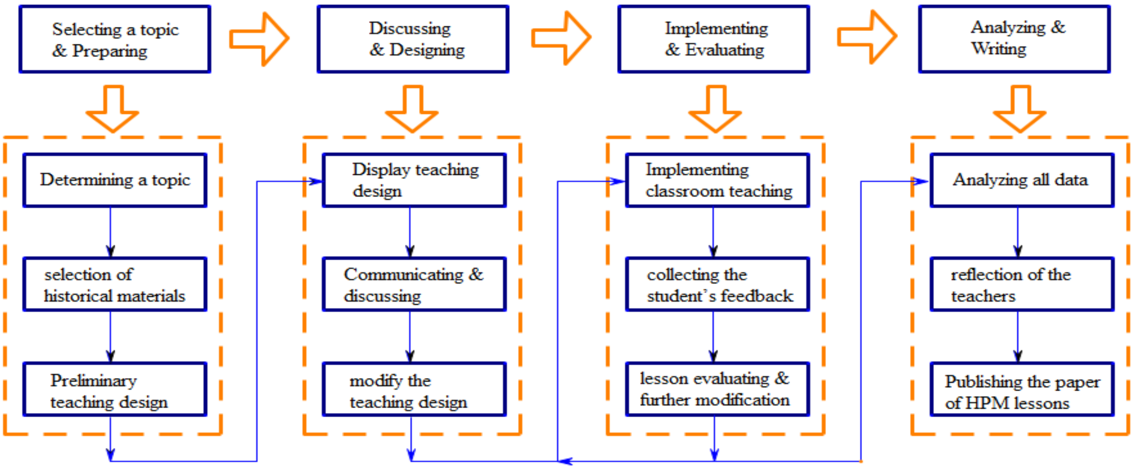


Figure 5.1: The running model of HPMLS

Compared with the previous model, the teaching design has advanced to the first stage. In the second stage, the display teaching design has been added, and in the third stage the cycle has been carried out, while it emphasizes the reflection of teachers in the fourth stage.

6 Conclusion

Integrating HM into ME from the perspective of teaching research system has been less researched. This research proposes a framework for integrating HM into ME in a teaching system. In view of the PLC, we present the HPMLC, based on the Chinese LS, and we propose the HPMLS. We portrayed how the HPMLS work in the context of HPMLC. We cannot say that this is the only framework to describe the teaching research system and it still requires further empirical studies to confirm its educational value.

The role of history of mathematics in general mathematics education research is still limited today. A possible reason for this may be that mathematics education researchers in the past decades had been striving towards a more theoretically founded discipline (Mosvold, Jakobsen and Jankvist, 2014). In this theoretical article, we aimed to theorizing the teaching research system of integrating HM into ME by the theory of PLC and LS. On the one hand, it enriches the related theories of HPM, like triangular pyramid IHT model and design-based IHT procedure (Wang et al., 2018). On the other hand, it also enriches the theories of community in mathematics education, and I hope more mathematics education researchers will pay attention to the study of HPM.

REFERENCES

Apollonius of Perga. (1896). *treatise on conic sections with introductions including an essay on earlier* *history on the subject*. New York : Cambridge University Press.

Boag, E. (2010). Dandelin spheres. *Mathematics in School*, *39*(3), 34-36.

Clark, K., Kjeldsen, T.H., Schorcht, S., Tzanakis, C.,& Wang, X. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, &T. Hausberger (Eds.), *Proceedingsofthe 2016 ICMESatelliteMeeting – HPM 2016* (pp. 135–179).Montpellier: IREM de Montpellier.

Cheng, X., & Wu, L. Y. (2016). The affordances of teacher professional learning communities: a case study of a Chinese secondary school. *Teaching & Teacher Education*, *58*, 54-67.

Gu, F., & Gu, L. (2016). Characterizing mathematics teaching research specialists' mentoring in the context of Chinese lesson study. *ZDM*, *48*(4), 441-454.

Heath, T. L. (1921). *A History of Greek Mathematics*. London: Oxford University Press. 218.

Hord, S. M. (1997). Professional learning communities: what are they and why are they important? *Issues about change*, *6*(1), 1-8.

Huang, R., Ye, L., & Prince, K. (2017). Professional Development of Secondary Mathematics Teachers in Mainland China. In B. Kaur, O. Kwon, Y. Leong (Eds.) *Professional Development of Mathematics Teachers. Mathematics Education – An Asian Perspective*. Singapore: Springer.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: an international perspective. *ZDM*, *48*(4), 393-409.

Huffman, J. B., Hipp, K. K., & Hord, S. M. (2003). *Reculturing schools as professional learning communities*. R & L Education.

Lahire, P. de. (1679). *Nouvaux Elemens des Sectiones Coniques*. Paris: Andre Pralard.

L Hospital, M. de. (1720). *Traité Analytique des Sections Coniques* (pp. 22-25). Paris: Montalant.

Smith, D. E. (1925). *History of Mathematics(Volume II)*. Boston: Ginn & Company.

Tzanakis, C., Arcavi, A., de Sá, C. C., Isoda, M., Lit, C.-K., Niss, M., de Carvalho, J. P., Rodriguez, M.,&Siu, M.K. (2000). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel& J. van Maanen (Eds.), *History in mathematics education: The ICMI study*, New ICMI Study Series, vol. 6 (pp. 201–240). Dordrecht: Kluwer.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics*, *71*(3), 235-261.

Wang, X. Q., Qi, C. Y., & Wang, K. (2017). A categorization model for educational values of the history of mathematics. *Science & Education*, *26*(7-9), 1029-1052.

Wang, K., Wang, X. Q., Li, Y., & Rugh, M. S. (2018). A framework for integrating the history of mathematics into teaching in shanghai. *Educational Studies in Mathematics*, *98*(2), 135-155.

Wong, J. L. N. (2010). What makes a professional learning community possible? a case study of a mathematics department in a junior secondary school of china. *Asia Pacific Education Review*, *11*(2), 131-139.

Wright, J. M. F. (1836). *An Algebraic System of Conic sections & Other curves* (pp.94-95). London: Black& Amstrong.

A CATEGORIZATION MODEL OF THE "HOWS" OF USING HISTORY IN MATHEMATICS EDUCATION

An Empirical Study

Dandan SUN
East China Normal University, Shanghai, China
1427300047@qq.com

ABSTRACT

The research on how to integrate History of Mathematics (HM) in Mathematics Education (ME) can guide the teaching practice directly, promoting the better use of HM in ME. Many scholars have done some researches on how to use HM in ME (e.g. Tzanakis et al, 2000; Jankvist, 2009). The previous discussions about how to use HM in ME can be divided into the following four dimensions; basically: The type of history in use (anecdotes, historical issues, etc.); the ways that HM can be used in ME (the illumination approaches, the history-based approaches, etc.); different sizes and scopes that HM can be used in ME (1 class period,10–20 class periods, etc.); the ways students work with history (worksheets, student-projects, etc.). The reason why the HM should be integrated in ME in China is that the HM has 6 values – the harmony of knowledge, the beauty of ideas or methods, the pleasure of inquiries, the improvement of capabilities, the charm of culture and the availability of moral education, which can help to achieve the goal of teaching. Because there are a lot of characteristics in Chinese mathematics classroom, such as: the teaching rhythm is fast, the teaching structure is clear, etc., so the ways that the HM can be used in mathematics classroom are somewhat different from other countries, and it is very difficult to analyze the ways of using HM in Chinese mathematics classroom with the existing classification framework. Based on this, this study aims to construct a classification framework for the ways that HM can be used in Chinese mathematics classroom.

Combining the existing researches about how to use HM in ME and the overall situation of the application of the HM in Chinese mathematics classroom, this research puts forward a new categorization model about the ways the HM can be used in Chinese mathematics classroom. Focusing on one material in HM, there are three ways of applying HM: complementation, replication and accommodation. Focusing on the role of HM in the teaching process, there are two ways of applying HM: creation and reconstruction. Afterwards, 20 well-chosen HPM lessons (the lessons featuring HM) are coded with the new categorization model. By analyzing the coding results, some subcategories of each method in the classification framework are obtained. In order to illustrate every subcategory, an example is given for every subcategory, which comes from the 20 well-chosen HPM lessons; the value of each subcategory is given too. Finally, we elaborate of the connotation of the 5 approaches in the new categorization model.

REFERENCES

Jankvist, U. T. (2009b). A categorization of the "whys" and "hows" of using history in mathematics education.*Educational Studies in Mathematics, 71*(3), 235–261.

Tzanakis, C., Arcavi, A., de Sá, C. C., Isoda, M., Lit, C.-K., Niss, M., de Carvalho, J. P., Rodriguez, M., & Siu, M.K. (2000). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study*, New ICMI Study Series, vol. 6 (pp. 201–240). Dordrecht: Kluwer.

History and epistemology in students and teachers mathematics education: Curricula, courses, textbooks, and didactical material of all kinds - their design, implementation and evaluation

EPISTEMOLOGICAL BELIEFS ABOUT MATHEMATICS

Challenges and chances for mathematical learning: Back to the future

Ingo WITZKE
University of Siegen, Herrengarten 3, Siegen, Germany
witzke@mathematik.uni-siegen.de

ABSTRACT

The transition from school to university is connected to a variety of some problems for many students. This can be attributed to different beliefs about mathematics in school and university. While mathematics teaching at school allows knowledge to be developed on the basis of real objects and empirical working methods, mathematics at universities is characterized by a rigorous axiomatic structure. The successive detachment of the connection to real objects has also occurred in the history of mathematics. From this situation, conclusions can be derived for the teaching of mathematics at school and university. The transition from school to university seems to be facilitated by the use of digital media in processes of concept development and the systematic thematisation of different beliefs about mathematics.

1 A challenge in mathematics education

When mathematics teachers as students move from school to university and then again when moving from university training back to school to teach mathematics they are often confronted with various problems. Felix Klein describes this situation as “double discontinuity”:

> *“The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching.”* (Klein, 1908)

Witzke, Struve, Clark & Stoffels (2016) describe a seminar at university level that focuses on the first discontinuity, the transition from school to university. In an empirical study, they put the following question to the participants: *What is the biggest difference or similarity between school and university mathematics?*

One male participant answered: “The fundamental difference develops as mathematics in school is taught empirical-perceptual (ger.: anschaulich), whereas at university there is a rigid modern-axiomatic structure characterizing mathematics. In general, there are more differences than similarities, caused by differing aims”.

Many similar statements from other participants could be found. Thus, the problem of the transition from retrospective student viewpoint is closely connected with the “differentness” of mathematics. These differences concern the aspects clearness, level of abstraction, evidence, formal rigor and axiomatic structure. The result is a clear distinction between school and higher education mathematics regarding its character.

2 Beliefs about mathematics – A theoretical framework

Looking at the challenges presented in the previous section, one question is particularly obvious:

How do we develop mathematical knowledge (further)?

The answer to this question is crucially related to our conceptions of mathematics. The notion about the beliefs of mathematics provides a good basis for this description.
According to Schoenfeld, the way someone works on a mathematical problem depends strongly on his beliefs about mathematics:

> *"One's beliefs about mathematics [...] determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. The belief system establishes the context within which we operate […]"* (Schoenfeld, 1985, 2011)

Steiner emphasizes the influence of the conception of mathematics on concepts for teaching and learning:

> *"Concepts for learning and teaching of mathematics [...] often implicitly are based on certain aspects of a philosophy of mathematics"* (Steiner, 1987)

Green describes teaching as the modification of the belief system of learners:

> "*The activity of teaching, at least in the sense of instructing, might therefore be defined as the effort to reconstitute the structure of our belief systems so that the number of core beliefs and belief clusters are minimized, the number of evidential beliefs are maximized, and the quasi-logical order of our beliefs is made to correspond as closely as possible to their objective logical order*."(Green, 1971)

How to build up mathematical knowledge, how to handle it and whether one is successful seems to depend essentially on the individual conceptions of mathematics (mathematical world view, attitudes, beliefs). The term beliefs of mathematics is frequently used in literature:

> "*Psychologically held understandings, premises, or propositions about the world that are thought to be true.*" (Philipp, 2007)

> "*Belief System: One's 'mathematical world view', the set of (not necessarily conscious) determinants of an individual's behavior about self, about the environment, about the topic, about mathematics.*" (Schoenfeld, 1985)

> "*individual's beliefs [...] as subjective, experienced-based often implicit knowledge and emotions on some matter or state of the art*" (Pehkonen & Pietilä, 2003)

The different explanations show the diversity of the term belief (cf. Rezat, 2009). Mathematics education is characterized by various beliefs about mathematics (cf. Grigutsch, Raatz & Törner, 1998, Schoenfeld, 2011, Witzke & Spies, 2016):

- Scheme-Aspect: Mathematics is a system consisting of rules, formulas and algorithms.
- Formalism-Aspect: Mathematics is characterized by logic, formal rigidity and precise technical terminology. It is the formal-abstract science.
- Process-Aspect: Mathematics is seen as a creative and constructive process.

- Application-Aspect: Mathematics is a tool for applications in the natural sciences and everyday life.
- Empirism-aspect: Mathematics describes a universe of discourse in reality. It is a natural science.

A formal-abstract view on mathematics is beside others represented at the university level. According to the frequently used textbook for calculus courses at university Heuser (2009), the central properties of mathematics are the brightness and sharpness of the concept formation, the pedantic care in dealing with definitions, the rigor of proofs and the abstract nature of mathematical objects that you cannot see, hear, taste or feel.

At least since Hilbert it is possible to see mathematics as an archetype of formal science with an axiomatic structure that is detached from reality. He developed mathematics as a science of uninterpreted abstract systems (focus on structures) with an absolute notion of certainty (internal consitency) (e.g. Hilbert, 1899). Thus, "the umbilical cord between reality and geometry has been cut" (Freudenthal, 1961). Geometry has become pure mathematics and the question of whether and how it can be applied to reality is answered just as in any other branch of mathematics.

The axioms are no longer self-evident truths; in fact, it does not even make sense to ask for their truth. This does not mean that there are no real applications or interpretations of the theories.

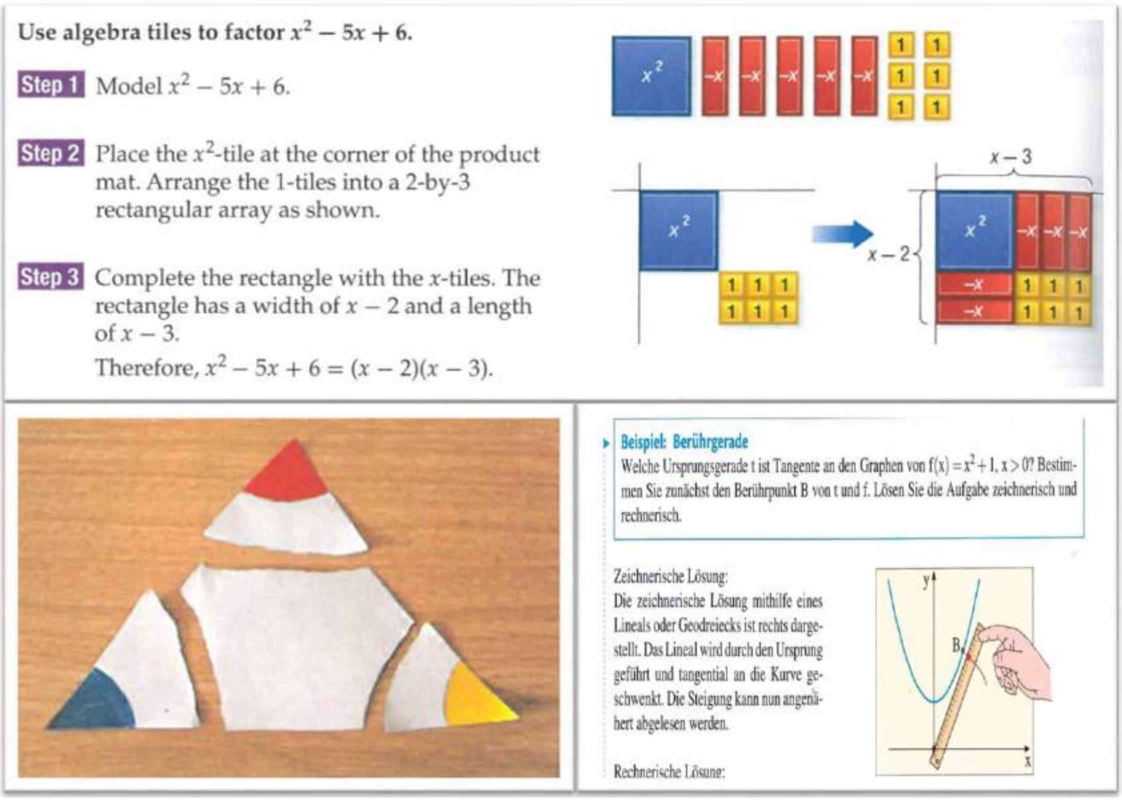


Figure 2.1: Empirical approaches to mathematical concepts and theorems in schoolbooks

0.3 Basic set theory

Note: 1–3 lectures (some material can be skipped, covered lightly, or left as reading)

Before we start talking about analysis, we need to fix some language. Modern* analysis uses the language of sets, and therefore that is where we start. We talk about sets in a rather informal way, using the so-called "naïve set theory." Do not worry, that is what majority of mathematicians use, and it is hard to get into trouble. The reader has hopefully seen the very basics of set theory and proof writing before, and this section should be a quick refresher.

0.3.1 Sets

Definition 0.3.1. A *set* is a collection of objects called *elements* or *members*. A set with no objects is called the *empty set* and is denoted by $\emptyset$ (or sometimes by $\{\}$).

Think of a set as a club with a certain membership. For example, the students who play chess are members of the chess club. However, do not take the analogy too far. A set is only defined by the members that form the set; two sets that have the same members are the same set.

Most of the time we will consider sets of numbers. For example, the set

$$S := \{0, 1, 2\}$$

is the set containing the three elements 0, 1, and 2. By ":=", we mean we are defining what S is, rather than just showing equality. We write

$$1 \in S$$

Definition 2.1.2. A sequence $\{x_n\}$ is said to *converge* to a number $x \in \mathbb{R}$, if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq M$. The number x is said to be the *limit* of $\{x_n\}$. We write

$$\lim_{n\to\infty} x_n := x.$$

A sequence that converges is said to be *convergent*. Otherwise, we say the sequence *diverges* or that it is *divergent*.

The Completeness Axiom

It is one thing to define an object and another to show that there really is an object that satisfies the definition. (For example, does it make sense to define the smallest positive real number?) This observation is particularly appropriate in connection with the definition of the supremum of a set. For example, the empty set is bounded above by every real number, so it has no supremum. (Think about this.) More importantly, we will see in Example 1.1.2 that properties **(A)**–**(H)** do not guarantee that every nonempty set that is bounded above has a supremum. Since this property is indispensable to the rigorous development of calculus, we take it as an axiom for the real numbers.

(I) If a nonempty set of real numbers is bounded above, then it has a supremum.

Property **(I)** is called *completeness*, and we say that the real number system is a *complete ordered field*. It can be shown that the real number system is essentially the only complete ordered field; that is, if an alien from another planet were to construct a mathematical system with properties **(A)**–**(I)**, the alien's system would differ from the real number system only in that the alien might use different symbols for the real numbers and $+$, $\cdot$, and $<$.

Figure 2.2: Formal-abstract representations in lecture notes for analysis at the university level

In contrast, this clear distinction between reality and mathematics does not take place for school mathematics. Hefendehl-Hebeker (2016) states in this context:

> "*The concepts and contents of school mathematics have their phenomenological sources predominantly in our surrounding reality. […] All in all the ontological bounding to reality is in place because of the educational and psychological purposes and aims of school. School mathematics barely surpasses the conceptual niveau and state of knowledge of the 19th century […]. Mathematics as a scientific discipline has today become a network of highly specialized abstract sub-areas.*"

This considerations lead to the following research thesis:

Research thesis I: *„Mathematical knowledge of pupils is generated in a constructive process - through interaction and the work with the offered learning material."* (cf. Bauersfeld, 1983)

At school, mathematics appears as an empirical science of concrete objects, it is not an abstract science of uninterpreted systems of terms as in modern mathematics. The empirical character of school mathematics (argumentation, models, experiments, term, etc.) is on epistemic grounds comparable to the character of natural sciences. Argumentations are based on real objects. This results in the following thesis:

Research thesis II: *If mathematics is consequently taught with the support of visual representations and illustrative material, students acquire an empirical belief system about mathematics. It is a theory about these representations – a quasi - 'natural science'.*

This kind of mathematics describes a universe of discourse in physical reality. The notion of truth relies in empirical facts gained through observation and experiments. Nevertheless, empirical mathematics needs logical reasoning to avoid a pure empiricism and pure phenomenology. The empirical characteristic is a fundamental difference to the above described university mathematics. The question arises, if this ‘non-abstract’ point of view is a reasonable one for the developing of mathematical knowledge.

3 Epistemological beliefs about mathematics in the past

A first possible answer to the above mentioned question is provided by an insight into beliefs in the history of mathematics. Substantial pieces of historical mathematics can be reconstructed as empirical mathematics (e.g. Witzke, 2009) with the help of structuralism (cf. Balzer, Moulines & Sneed, 1987).

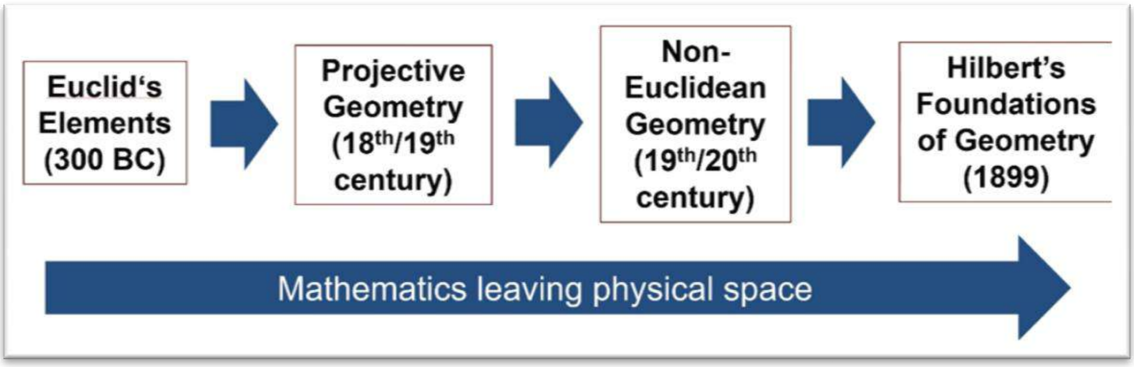
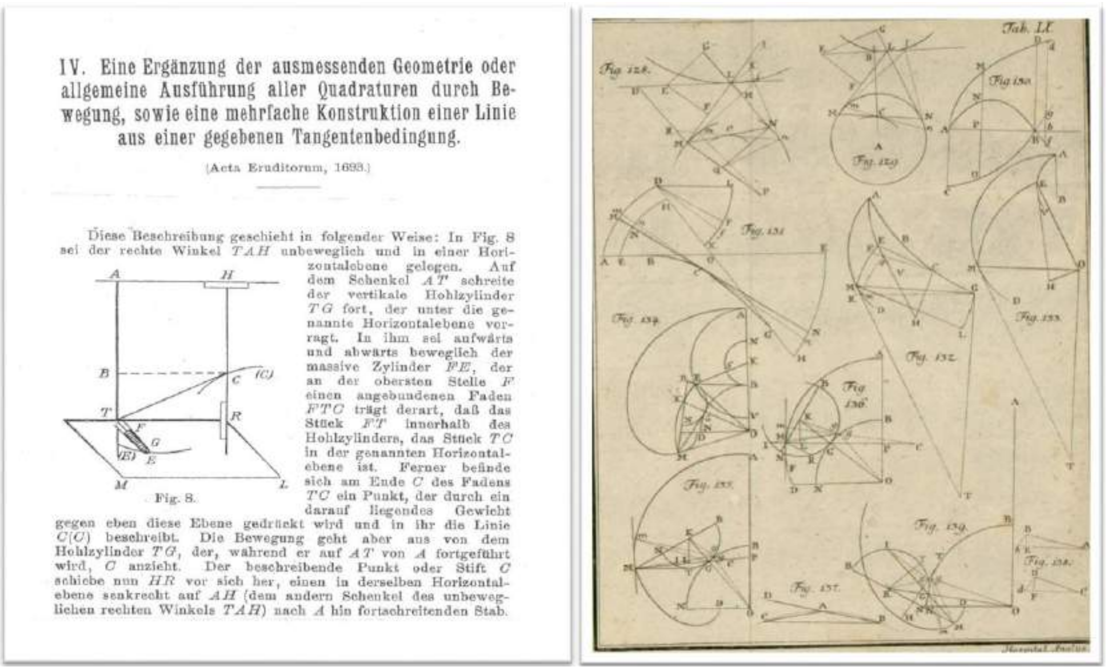


Figure 3.1: Development of geometry in the history of mathematics



IV. Eine Ergänzung der ausmessenden Geometrie oder allgemeine Ausführung aller Quadraturen durch Bewegung, sowie eine mehrfache Konstruktion einer Linie aus einer gegebenen Tangentenbedingung.

(Acta Eruditorum, 1693.)

Diese Beschreibung geschieht in folgender Weise: In Fig. 8 sei der rechte Winkel TAH unbeweglich und in einer Horizontalebene gelegen. Auf dem Schenkel AT schreite der vertikale Hohlzylinder TG fort, der unter die genannte Horizontalebene vorragt. In ihm sei aufwärts und abwärts beweglich der massive Zylinder FE, der an der obersten Stelle F einen angebundenen Faden FTC trägt derart, daß das Stück FT innerhalb des Hohlzylinders, das Stück TC in der genannten Horizontalebene ist. Ferner befinde sich am Ende C des Fadens TC ein Punkt, der durch ein darauf liegendes Gewicht gegen eben diese Ebene gedrückt wird und in ihr die Linie $C(C)$ beschreibt. Die Bewegung geht aber aus von dem Hohlzylinder TG, der, während er auf AT von A fortgeführt wird, C anzieht. Der beschreibende Punkt oder Stift C schiebe nun HR vor sich her, einen in derselben Horizontalebene senkrecht auf AH (dem andern Schenkel des unbeweglichen rechten Winkels TAH) nach A hin fortschreitenden Stab.

Fig. 8.

Figure 3.2: The development of calculus based on curves as empirical objects, constructed and drawn on paper

The development of modern views of mathematics can be illustrated particularly well using the example of geometry (cf. Witzke, Struve, Clark & Stoffels, 2016). The first axiomatic structure of geometry can be found in Euclid's elements in 300 B.C. The justification of the axioms occured by evidence (cf. Garbe, 2001). Thus they had a clear relation to the objects of the real world, e.g. a line drawn on a sheet. The further development of the Euclidean geometry took place in the 18th and 19th centuries. The famous mathematician Moritz Pasch wrote in 1882: “The geometric terms [...] serve to describe the world around us [...]. Geometry is nothing more than a part of the natural sciences”. One goal of geometry is the description of the physical world, although there is an increasing axiomatization. The relationship of geometry to the real world changed dramatically with the development of non-Euclidean geometries in the 19th and 20th centuries. These internal consistent theories are based on axioms that are initially independent from the surrounding world. However, by striving to find the geometry that describes the physical space, there is still a connection to reality. The underlying axiom system and the physical world were then disconnected consciously by the development of Hilbert’s foundations of geometry in 1899. The axioms no longer need any connection to reality. It is a pure inner-mathematical theory.

The previous explications can be described simplified in a bipolar model of belief systems. On the one hand there is the empirical-concrete mathematical belief system. It can be found in the history of mathematics as well as in school mathematics and is based on didactical (learning theory, e.g. Gopnik et al. , 2007; educational theory, e.g. Winter, 1969; empirical reasons, e.g. Schoenfeld, 2011, Struve, 1990) and epistemological reasons (parallels with natural science, e.g. Einstein, 1921; historical reconstructions, Witzke, 2009; structuralistic reconstructions, Balzer, Moulines & Sneed, 1987). On the other hand, there is a formal-abstract mathematical belief system that can be found in mathematics courses at universities and in the history of mathematics since Hilbert.

4 Epistemological beliefs: Back to the future

The questions remain what we can learn from these perspectives and how history can inform modern mathematical education. One possible answer can be provided by looking at the use of digital media in mathematics classroom. The use of digital media is usually connected with an emphasis on qualitative and empirical working methods. The potential of digital media can be illustrated using the example of calculus. Textbooks at school contain a large number of graphical representations. Often, they form the basis for argumentations; questions of existence such as continuity or differentiability become less relevant (cf. Witzke, 2014). Graphic calculators and function graphing software enable the dynamic investigation of curves. In this way, concepts can be developed qualitatively in a first step, so that the students can develop sustainable ideas (e.g. function microscope by Elschenbroich, 2015). On epistemological grounds, these objects represented in an iconic way constitute parallels to the construction of curves at the time of Leibniz.

Another example results from graphical differentiation and integration. This means the qualitative drawing of the graph of a primitive integral or the derivative by graphical determination of the integral or the derivative at single points which is somewhat problematic because of the discreteness. The 3D printing technology offers the possibility to develop a so-called integraph (cf. Witzke & Dilling, 2018). This is a device that continuously draws the graph of a primitive integral of a graphically given function in a mechanical way. It enables the students to justify the first part of the fundamental theorem of calculus visually. First concepts of an integraph reach back to Leibniz (1693).

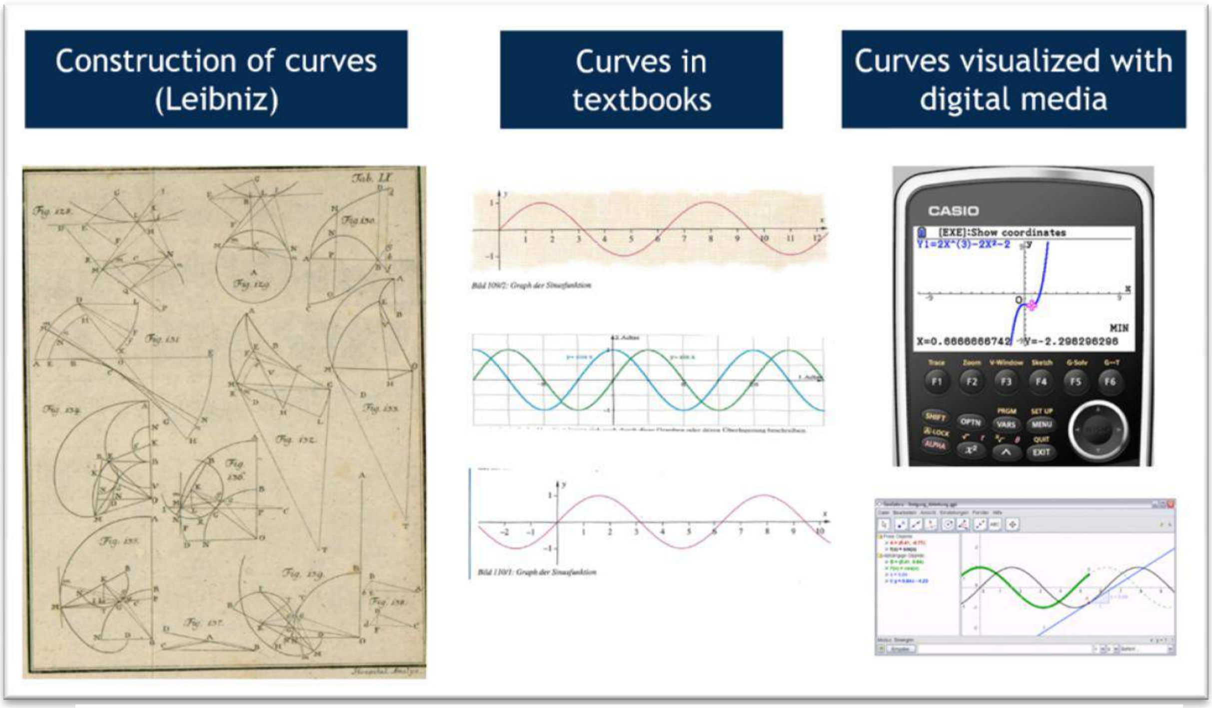


Figure 4.1: Curves in the history of mathematics, in the mathematics textbook and visualized with digital media

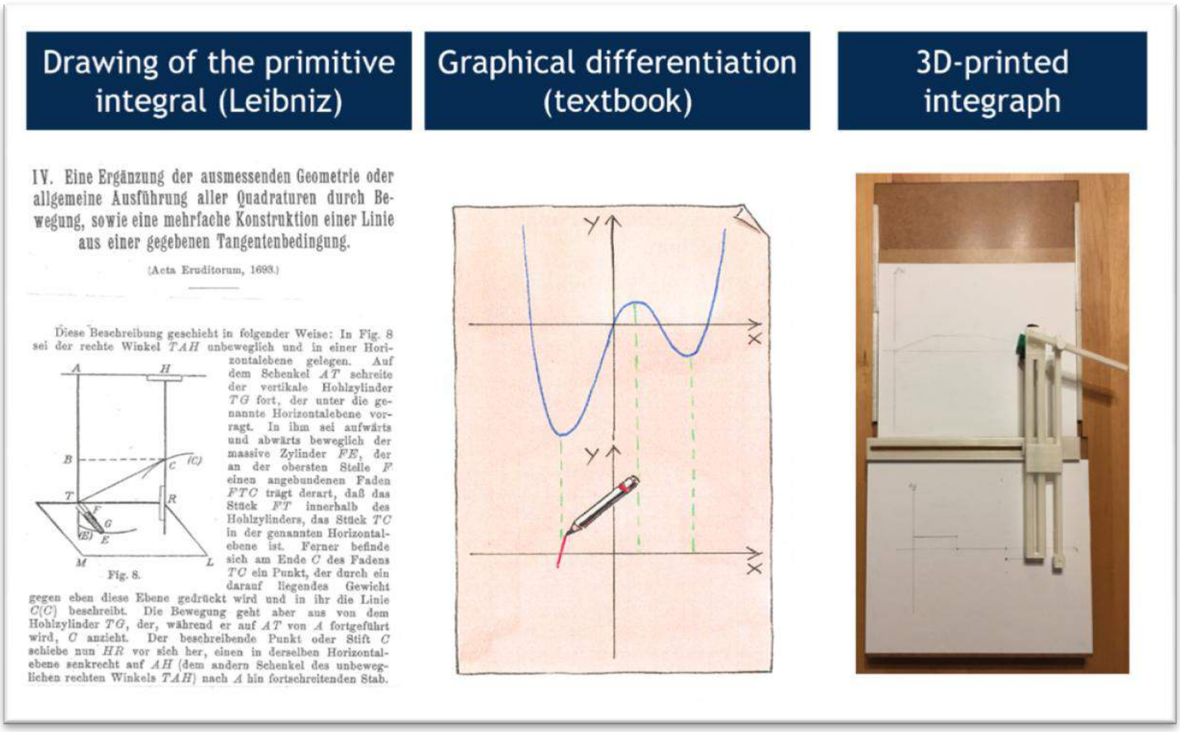


Figure 4.2: Graphical determination of the derivative or primitive integral in the history of mathematics, in the mathematics textbook and by the use of an integraph

The examples show that working with digital media promotes an empirical view of mathematics that was established in the history as well. However, these empirical approaches are not to be equated with pure empiricism, since concepts are consciously emphasized and systematically developed. The aim of the authors is not the equalization of school and university mathematics. Instead, the differences and the resulting obstacles for the transition from school to university should be specifically addressed. This is connected with the hope that in this way more students bridge the gap and develop an adequate perspective regarding the nature of mathematics in different contexts.

Acknowledgement: I am particularly grateful to Frederik Dilling for his comments on the manuscript and the support in the development of the paper.

REFERENCES

Balzer, W., Moulines, C. U. & Sneed, J. D. (1987). *An Architectonic for Science. The Structuralist Program*. Dordrecht: Springer Netherlands.

Bauersfeld, H. (1983). Subjektive Erfahrungsbereiche als Grundlage einer Interaktionstheorie des Mathematiklernens und -lehrens. In: Bauersfeld, H., Bussmann, H., Krummheuer, G., Lorenz, J. H. & Voigt, J. (Eds.). *Lernen und Lehren von Mathematik. Analysen zum Unterrichtshandeln* II (pp.1–56). Köln: Aulis.

Einstein, A. (1921). Geometrie und Erfahrung. Erweiterte Fassung des Festvortrages gehalten an der *Preußischen Akademie der Wissenschaften* zu Berlin am 27. Januar 1921. Berlin: Springer.

Elschenbroich, H.-J. (2015). Die interaktive Funktionenlupe – Ein neuer Vorschlag zur visuellen Vermittlung von Grundvorstellungen der Analysis. *Beiträge zum Mathematikunterricht* 2015, 264-267.

Freudenthal, H. (1961). Die Grundlagen der Geometrie um die Wende des 19. *Jahrhunderts. Mathematisch-Physikalische Semesterberichte*, *7*, 2-25.

Garbe, A. (2001). *Die partiell konventional, partiell empirisch bestimmte Realität physikalischer RaumZeiten*. Würzburg: Königshausen & Neumann.

Gopnik, A. M., Meltzoff, A. N., & Kuhl, P. K. (2007). *The Scientist in the Crib*. New York: Harper Perennial. Dordrecht: Reidel.

Green, T. F. (1971). *The activities of teaching*. Columbus: McGraw-Hill.

Grigutsch, S., Raatz, U., & Törner, G. (1998). Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematik-Didaktik, 19* (1), 3-45.

Hefendehl-Hebeker, L. (2016). Mathematische Wissensbildung in Schule und Hochschule. In: Hoppenbrock, A. et al. (Eds.). *Lehren und Lernen von Mathematik in der Studieneingangsphase* (pp.15-24). Wiesbaden: Springer.

Heuser, H. (2009). *Lehrbuch der Analysis*. Wiesbaden: Vieweg + Teubner.

Hilbert, D. (1899). *Grundlagen der Geometrie*. Leipzig: B.G. Teubner.

Klein, F. (1908). *Elementarmathematik vom höheren Standpunkte aus* (3 Bände). Leipzig: B.G. Teubner.

Leibniz, G. (1693). Über die Analysis des Unendlichen. In: Kowalewski, G. (Ed., 1996). *Über die Analysis des Unendlichen / Abhandlung über die Quadratur von Kurven* (pp.1-72). Frankfurt am Main: Thun.

Pasch, M. (1982). *Vorlesungen über neuere Geometrie*. Leipzig: Teubner.

Pehkonen, E. & Pietilä, A. (2003). On relationships between beliefs and knowledge in mathematics education. Source: www.mathematik.uni-dortmund.de/~erme/CERME3/Groups/TG2/TG2_pehkonen_cerme3.pdf (Download: 08.07.2005).

Philipp, R. (2007). Mathematics Teacher's Beliefs and Affect. In: Lester, F.K. (Ed.). *Second handbook of research on mathematics teaching and learning* (pp.257–315). Charlotte: Information Age.

Rezat, S. (2009). *Das Mathematikbuch als Instrument des Schülers*. Wiesbaden: Vieweg + Teubner.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Orlando et al.: Academic Press.

Schoenfeld, A. H. (2011). *How we think*. New York et al.: Routledge.

Steiner, H.-G. (1987). Philosophical and Epistemological Aspects of Mathematics and Their Interaction with Theory and Practice in Mathematics Education. *For the Learning of Mathematics, 7*(1), 7-13.

Struve, H. (1990). *Grundlagen einer Geometriedidaktik*. Mannheim: BI-Verlag.

Winter, H. (2014). Mathematikunterricht und Allgemeinbildung. *Mitteilungen der Deutschen Mathematiker-Vereinigung 4*(2), 35-41.

Witzke, I. (2009). *Die Entwicklung des Leibnizschen Calculus*. Hildesheim, Berlin: Franzbecker.

Witzke, I. (2014). Zur Problematik der empirisch-gegenständlichen Analysis des Mathematikunterrichtes. *Der Mathematikunterricht*, *60*(2), 19–31.

Witzke, I., & Spies, S. (2016). Domain-Specific Beliefs of School Calculus. *Journal für Mathematik-Didaktik*, *37*(1), 131-161.

Witzke, I., Struve, H., Clark, K. & Stoffels, G. (2016). ÜberPro – A seminar constructed to confront the transition problem from school to university mathematics, based on epistemological and historical ideas of mathematics. *MENON – Journal of Educational Research, Special Issue: The Use of History of Mathematics in Mathematics Education* (pp.66-93).

Witzke, I. & Dilling, F. (2018). Vorschläge zum Einsatz der 3D-Druck-Technologie für den Analysisunterricht – Funktionen zum „Anfassen“. *Beiträge zum Mathematikunterricht 2018*, 2011-2014.

HISTORY, EPISTEMOLOGY AND TEACHING MATHEMATICS

A challenging partnership?

Caterina VICENTINI[1,2,8,9,13,14] **, Nathalie CHEVALARIAS**[3,4,10,15]**, Kathleen M. CLARK**[5,11,]**, Michel ROELENS**[6,7,12,16]

[1]Mathematics and Physics Teacher at Liceo Scientifico "Michelangelo Buonarroti", Monfalcone (GO), Italy
[2]Mathematics Teacher at Herschel Grammar School, Slough, London SL1 3BW, UK,

[3]Mathematics Teacher at Lycée Pilote Innovant International, Jaunay-Marigny,
[4]member of IREM Poitiers, France

[5]Associate Professor, Florida State University, Tallahassee, Florida, USA

[6]Lecturer of Geometry and Mathematics Didactics at University Colleges Leuven-Limburg, Faculty of Teacher Education, Diepenbeek, Belgium,
[7]Mathematics Teacher at Maria-Boodschaplyceum, Brussels, Belgium

[8]caterina.vicentini@goiss.it, [9]cvi@herschel.slough.sch.uk,
[10]nathaliechevalarias.irem@gmail.com, [11]kclark@fsu.edu, [12]michel.roelens@ucll.be

ABSTRACT

In this paper we approach an analysis of the partnership between history, epistemology and teaching in 'real maths classes'. We will do so by starting our consideration as much as possible under the 'real maths' teachers' perspective. The initial focus is on comparing and contrasting the situations in different countries. In the follow-up, however, we investigate a bit further, entering deeper into specific situations and suggesting good practices. To conclude, more theoretical features will be briefly discussed, as well.

1 Introduction by the panel coordinator

A typical danger in mathematics education research, and in my opinion the main reason why it has less effectiveness than it could have, is the general lack of open discussions taking into account teachers' feelings as well as "orthodox statistics" or academic literature on the subject. This field could benefit from a more humanistic and realistic setting if teachers with 30 or 40 years of classroom experience were respected when expressing their conceptions as much as the academic researchers usually are. I was fortunate enough to be able to learn this way of working by Nicolas Rouche at the Université Catholique de Louvain in Louvain-la-Neuve (BE) in the late 1980s. For this reason, I would like to dedicate this work to him. Moreover, I feel particularly thankful to Évelyne Barbin and all the French ESUs' participants for having let their inter-IREM conferences become European Universities in which the original spirit of Montpellier

---

[13] Panel coordinator.

[14] Also, Member of the British Society for the History of Mathematics; Professor on contract of History and Epistemology in Mathematics Education, Università degli studi di Udine, Italy; and Professor on contract of Teaching Maths in CLIL, UFR-FVG Trieste, Università "Ca' Foscari" di Venezia, Italy.

[15] Also, co-responsible of CII Epistemology and History of Maths, France.

[16] Also, Editor of *Uitwiskeling*, journal for mathematics teachers, Belgium.

1993 is still alive 25 years later.

I am particularly glad to also express my gratitude to the organizers who invited me. I am essentially a “general maths teacher” and therefore I could have been considered not entitled to coordinate a Plenary Panel dealing with general questions as the presence and the integration of History and Epistemology of Mathematics in the curricula. Furthermore, I have had the opportunity to invite at the table with me two other “general maths teachers”: Michel and Nathalie. The presence of Kathleen has contributed to balance the point of view that we share.

The main aim of this panel has been to give a voice to the teachers, trying to discover in which measure they are keen on letting history and epistemology of mathematics enter their classrooms, whether they do it, and why. Then we concentrated on comparing and contrasting the situations in different countries in order to set the right directions for our research.

We moved from the analysis of a questionnaire we implemented in our particular countries and regions. We originally took into account the Flanders in Belgium and the Netherlands, France, Italy, and Florida and Massachusetts in the USA. Ultimately, we also had representation from China, as well. I would like to thank very much indeed Yue Zengcheng, PhD, from East China Normal University in Shanghai, who wrote me kindly proposing himself as a collaborator.

Eventually, following to the suggestion given by Costas Tzanakis, we decided not to discuss in detail the results of the questionnaires in these proceedings. We would rather prefer to emphasize the possible reasons which led to such results, and discuss some steps towards ‘solutions’ considering the different teaching and teachers’ training situations, as well as the very recent changes in the programs or guidelines of some of the nations involved.

To conclude, I would like to express my gratitude to my dear friends Daniele Gouthier and Colette De Coster. Daniele is a very eclectic freelance mathematician, who teaches at the Master of Science in Communication at the Scuola Superiore di Studi Avanzati (SISSA) in Trieste. He helped me both with the questionnaires and with the review of this paper. Colette works as professor at Université de Valencienne. She recently came to Trieste as visiting professor at the local University. In such occasion I met her and while having a pleasant meal we informally discussed about the training courses for French young teachers held by her university.

2 The starting point of the panel: the questionnaire

As has already been stated, this panel report does not focus on the work of researchers in history of mathematics or in educational science. The main aim is to instead give a voice to the teachers, focussing on those who work with adolescent pupils. Therefore, we will discuss about the difficulties that teachers meet, the possible reasons of such difficulties and consequently propose some realistic steps which could help them to introduce history and epistemology in their lessons.

Our panel’s starting point was a questionnaire we adapted for the regions involved. The questionnaire can be found in the conference site, where it was published to allow ESU8’s prospective participants to express their opinion as well (https://esu8.edc.uoc.gr/esu-8-main-themes-plenary-sessions/).

The questions collected information about the teachers’ background in the history of mathematics, their epistemological awareness, the introduction of history of maths in their class practice, how and why they do it or why they do not.

Kathleen, with the assistance of two colleagues, was able to distribute the questionnaire to two different teacher populations professional networks. In total, 149 mathematics teachers, predominantly from Florida and Massachusetts, completed the questionnaire.

Nathalie analysed the situation in France. She first proposed the survey to all the teachers of Poitou-Charentes, the region where she teaches. She received approximately 230 responses. To get more feedback, she asked for help from colleagues in other regions. In the end, the questionnaire was given to teachers both in Paris and in Franche-Comté. In those two regions, she only managed to contact in-service training teachers; ultimately, the results of the survey were similar and she received 530 responses altogether.

As far as the Michel's work is concerned, he took into account the Flanders (the Flemish part of Belgium) and the Netherlands (with the collaboration of Jeanine Daems and Desiree van den Bogaart). In Flanders, the questionnaire was distributed on the website of *Uitwiskeling* (a journal for mathematics teachers that he has edited since 1984). In the Netherlands, the questionnaire was posted on the Facebook group of Dutch mathematics teachers. In total, 127 teachers responded.

Caterina submitted the questionnaire to 1500 teachers belonging to the national association "Mathesis", randomly spread around Italy with the help of Professor Salvatore Damantino, chair of the section of Udine. The response rate was very poor: only 3%. She underlines that it is very difficult to talk about "the Italian situation". Italy is a quite recent country that was dominated by cultures really distant from one another. It is neither as small as the Belgian Flanders or the Netherlands nor it has the shape of a hexagon like France. Even though it is one state, federal issues have been influencing the political choices and the personal feelings since its unification in 1861 and undoubtedly in a stronger way than before during the last 25 years. These facts are strongly related to quite a large number of different situations moving north, south and east or west within Italy. A sort of "average" is not really meaningful. However, even though Italy is neither as "systematic" as France, nor as clearly "split in parts" like Belgium or the USA, there are some features that are quite uniformly widespread and are worth being discussed.

In China, the questionnaire was implemented by Zengdeng through the WeChat app, a very popular social network in China (more than one billion people regularly use it). Furthermore, he also contacted the teachers he is linked to. Altogether he received 64 responses.

3 Evidence of the inquiry

We start our reflection from the general results from the questionnaire responses. However, since the sample of teachers interviewed was certainly not well representative of the entire population, and therefore our survey was certainly biased, we do not present the inquiry evidence as a statistic. We prefer to enlighten some interesting features that are nonetheless worth being extrapolated, because quite a strong core of similar answers around the occidental parts of the world involved. In China, due to a quite different cultural background, some differences arose.

In short, the common facts and feelings with very few exceptions are:

- Few teachers have followed a course in history of mathematics, either in their graduate program or in their initial training courses or master's; China was an exception, where a course in history of mathematics is usually present and specially oriented to the introduction of the history in classrooms;

- Very few teachers are able to convey an epistemological position and teachers generally do not know what epistemology is;
- The majority is keen on making use of history of mathematics in their daily teaching;
- There are general opinions that history will:
  - render mathematics more human and interesting,
  - underline that mathematics are an historically contextualized cultural product,
  - allow a deeper acquisition of the conveyed notions,
  - make both the learning and the teaching more fun;
- When teachers integrate history in their teaching it is most often in an anecdotal way (to present a new topic, or a mathematician), more rarely solving old problems and almost never working on original sources;
- The rather superficial presence of history is allocated to a lack of knowledge, of training, of resources or of time.

4 General observations

Even though we have examined different cultural contexts, there are some overall teaching features that seem meaningful to us in order to understand the homogeneous situation which generally emerged.

First of all, mathematics is not always taught by mathematicians. The reasons can be different according to the geographical region involved, however this is a general fact that has essentially to do with the worldwide shortage of mathematics teachers.

On the second hand, the presence of history of mathematics in textbooks is usually characterized by a sort of juxtaposition. History is not integrated and does rarely permeate the way of presenting mathematical evidences. It is often used to share anecdotes or to introduce a mathematician, and whenever the presence is more substantial, it is either exiled in single problems or stored in very small "boxes".

Sources directly coming from research are usually perceived as too difficult to be adapted to "real classes" by "real teachers". This is also likely due to the very trendy standardization to which teaching has increasingly undergone. As a consequence, a dangerous need is perceived: the necessity of what during the panel Michel defined with a Flemish expression "gesneden brood" that can be translated as "bread already cut in slices".

However, in China the link between research on history of mathematics in an educational perspective and teaching of mathematics appears to be quite stronger.

In summary, the importance of history of mathematics was not strongly emphasized, nor it was included in the national final exam of high schools in the countries in which a final exam exists; even though China seems to be an exception again. However, it must be said that especially in France, but also in a weaker way in Italy, the situation very recently has changed. New ministerial indications have been added, as we will see in the next section.

In China, although they do not emphasize explicitly the importance of history of mathematics, they strongly enhance mathematical culture, and history of mathematics is generally recognized as a rather important part of this.

5 Context-specific remarks

5.1 Specific features of French context

As far as the initial training in concerned, it has to be said that teachers in “college” (pupils aged 11 to 15 years) and “lycée” (pupils aged 15 to 18 years) are normally recruited through a competitive examination. In the last 10 years, in addition to the competition, teachers are required to have master’s degree, as well. Very often, but not always, they enrol in a master’s degree related to teaching in their subject and they are allowed to participate to the competition during the first year of their master’s studies. The competitive examination is national and even if each university organizes the master’s program autonomously, there exists official national guidelines to define the training.

The last reform of university in France introduced on average more history of mathematics in the training of the future teachers, but there can be many differences from one university to another. On the contrary, teachers recruited more than 10 years ago had had in general very little of history of mathematics in their courses, and some of them had no history at all.

Additionally, in order to face the shortage of mathematics teachers, numerous staff members are recruited without competitive examination to teach as temporary instructors just for 3-4 months, one year, or even more (depending on their status and the needs of the corresponding institution). In this case, teachers very rarely have had training in history of mathematics, and, if fortunate to do so, they have the possibility to undergo a training specially conceived for them; History of Mathematics is certainly not a priority.

What is more, there are also people who have a very different cultural background and “convert” to becoming mathematics teachers by taking from time to time some exams specially arranged for them. In this case, they may be interested in history because it allows them to understand a content they cannot cope with well enough, otherwise they are looking for didactical prompts (“bread already cut in slices”).

Nowadays, history of mathematics is present either in the first or the second year of the master’s degree. When introduced in the first year it brings to prospective teachers a minimum of culture in this field: names of mathematicians, their work, the fields of mathematics, the specificities and contributions of each period, but it is not very strongly linked with the programs and what can be taught to the pupils. However, if the training in the history of mathematics takes place in the second year of the master's degree, this is when students are interning with classes, and it usually starts from the examination of documents (e.g., primary sources) in order to construct teaching material to be tested in classrooms.

While the initial training in history of mathematics seems to have increased in recent years, in-service training has certainly decreased. In France it is quite easy to find online resources or books, especially edited by the *Instituts de Recherche sur L’Enseignement des Mathématiques* (IREMs), though not exclusively. However, it is not always simple to find the time to consult them and even more to understand them sufficiently while working independently. It is also not easy to “sort out” reliable from unreliable sources if someone does not have a minimal basic knowledge.

In textbooks, it is common for very little history of mathematics to be found and such content is often limited to an image of a mathematician, his birth and death date, a small remark on his work, usually written in few lines and collocated in the page margin. Very rarely there are class activities or exercises entirely built on a historical situation.[17]

Therefore, teachers need to decide to undergo some voluntary training if they want to use original sources and they have to do additional work in case they want to become able to create activities based on history of mathematics.

The French curriculum does not require to teach precise aspects of history of mathematics but it does encourage teachers to use history:

> Elements of epistemology and history of mathematics naturally fit into the implementation of the curriculum. The knowledge of the name of some famous mathematicians, the time when they used to live and their contribution is part of the cultural background of any student with scientific training. The presentation of historical texts helps to understand the genesis and the development of some concepts. (Ministère de l'Education, 2010)

Recently the Villani and Torossian report (2018) reaffirmed the importance of the history and epistemology of mathematics:

> First, epistemology and history of the construction of mathematical notions, which bring a real didactic richness, are little taught in initial training. […] By taking advantage of history of mathematics, teachers place their teaching in the evolution thought. In addition, students are often sensitive to the "mathematics legend". Narrative can play a motivating role here. On the other hand, the epistemological lessons that emerge from history as the role of problems, the entanglement of concepts and techniques, the need of abstraction, etc., are obviously likely to contribute to training, in particular by overcoming short-sighted utilitarianism.

Apparently, this report has influenced some of the ministry decisions. In January 2019, new curricula were published and history of mathematics indeed takes a real important new place. The introduction indicates:

> It may be useful to illuminate the course with historical or epistemological contextual elements. History can also be seen as a fertile source of problems that clarify the meaning of certain concepts. The "History of Mathematics" items identify some possibilities in this direction. To support them, the teacher can rely on the study of historical documents. (Ministère de l'Education, 2019)

For each part of the curricula, some examples and indications are given. For example, in the part "Numbers and calculus" of the curriculum it is explained:

> The seemingly familiar notion of number is not self-evident. Two examples: the crisis caused by the discovery of irrationals by Greek mathematicians, the difference between "real numbers" and "calculator numbers". It is also important to highlight the gain in efficiency and generality brought by literal calculus, by explaining that a large part of mathematics could only develop once this formalism had stabilized over the centuries. It is possible to study ancient texts by authors such as Diophantus, Euclid, Al-Khwarizmi, Fibonacci, Viète, Fermat, Descartes and highlight their algorithmic aspects. (ibid.)

We still do not know what will be introduced in the textbooks, but hopefully teachers will be given more resources and activities to be encouraged to introduce history of

[17] Although a separate subsection for the United States context is not presented in Section 5, this description is true of textbooks in the US, as well. See Smestad, Jankvist, and Clark (2014) for additional details.

mathematics in their classes.

5.2 Specific features of Flemish context

The Belgian Flanders have a strong tradition in mathematics education, and still produce good results in comparative studies as PISA. However, they are facing problems that could influence future issues. The main critical points are the lack of a master's degree in mathematics and future teachers who have to learn everything.

In Belgium, to teach to pupils aged 15 to 18 years, a university master's degree is required. In the last decades, Belgium has been facing a shortage of mathematics students, accompanied by a shortage of mathematics teachers. Therefore, people having a master's in other subjects such as economics, biology, engineering, etc. often teach mathematics. Usually they do not feel a deep interest in mathematics and its history, and often they are less creative in doing mathematics and teaching it than mathematicians. They are more comfortable with ready-to-use materials and, as a consequence, publishers have been starting to provide these prompts. These kind of resources include exercises and tests with all the solutions, pre-prepared slides for projection on smartboards and other similar means. As a consequence, their creativity is not enhanced and tends to be more and more reduced.

Alternatively, in order to teach to pupils aged 12 to 15 years, a professional bachelor's degree is needed. Many teachers start their bachelor courses with a poor background in mathematics. Their view of the subject is almost always limited to techniques and algorithms. Teacher trainers are consequently called upon to do a big work. They must: teach problem solving, spatial insight, reasoning, proving, speaking and writing about mathematics, recognizing mathematics in different contexts and something about the historical and cultural aspects of the mathematics they are supposed to use while teaching.

5.3 Specific features of Italian context

As already mentioned, in Italy mathematics is also not always taught by mathematicians. Teachers working in "scuola media inferiore", where pupils are aged between 11 and 14 years, can have a master's in biology, natural sciences, geology, chemistry, physics, and many other subjects since they are supposed to teach general sciences too. Professors working in "scuola media superiore", where students are aged between 14 and 19 years, on the contrary, need to have a master's in Mathematics, Physics, Engineering or Computer Sciences (and in rare cases, Economics).

What is more, as far as initial teacher training is concerned, the actual in-service teacher population is composed by essentially three different categories of people:

(i) Those who have been employed via a "concorso ordinario";

(ii) Those who have been employed via a "concorso riservato"; and

(iii) Those who have been employed after having had a university training followed by a thesis.

The teachers employed via a "concorso ordinario" only had to study their subjects and the laws that concern the school work; those who took part in a "concorso riservato", generally speaking, also had to attend some didactical courses; the youngest ones who had university training were supposed to study pedagogy, didactics and quite rarely, as already seen, history and epistemology courses, as well.

Another important remark to understand the Italian situation is related to the number of hours dedicated to mathematics instruction in schools. In the “Liceo Scientifico opzione Scienze Applicate”, the most scientific Italian school, students have only five hours of mathematics per week in the first year and only four hours in the following four years. To understand the historical reasons of this inadequate instructional time, the paper “Personal and Social Conscious and Unconscious Backgrounds in Mathematics Education (Vicentini, 1994) could be a useful reading.

Furthermore, in Italy the curricula are defined on a nationwide basis. However, since some years ago, the ministerial document has been named “indicazioni ministeriali”, which means ministry’s indications. Therefore, the ministry underlines the importance of certain contents, but does not oblige the teacher to deal with all the suggested topics. What is more, the “*independence*” and the “*full freedom*” of teachers’ choices are underlined. To sum up, these indications are rather general. (see MIUR, 2010) Therefore, as a matter of fact, in Italy a teacher legally has a large autonomy in choosing what to teach and how to teach it. Practically speaking however, not many teachers use this legal freedom and there are many reasons for such a choice. To mention just one, in the final State exam the mathematics exam paper is the same for all the different types of “liceo scientifico” and it is issued from Rome for the entire country. Moreover, teachers do not feel they are really entitled to choose what to teach; therefore, they tend to adapt the didactical tradition. It must also be said that it is compulsory to “adopt” a textbook and there are only two syllabi that reach all together the majority of the students of Italian high schools (students aged between 14 and 19 years): one edited by Zanichelli in Bologna, whose authors are Massimo Bergamini, Graziella Barozzi and Anna Trifone; and the other edited by Petrini-De Agostini in Novara, whose author is Leonardo Sasso (recently together with Claudio Zanone).

As a result, what really happens in Italian classes is a melting pot of didactical tradition, of what is present in the two textbooks quoted above, what is written in the guidelines together with a few ideas coming from research in mathematics education (that can be in several cases very good, but do not generally thrill the majority of the teachers).

Concerning the presence and integration of the history of mathematics in the guidelines, the ministry refers to history, but mainly in the general part of the document in which all the subjects are mentioned. It is written: “According to the Lisbon’s indicators the various subjects should be studied in a systematic, historical and critic way by reading, analysing and translating literary, philosophical, scientific texts as well as essays” (MIUR, 2010), but very few mathematics teachers usually read the general part. Therefore, the integration of the history of mathematics in the development of the regular curriculum is “de facto” not enhanced since the majority of teachers usually avoid a careful study of the entire document; instead, they concentrate on the section concerning their subject.

On the other hand, we must observe that very recently the ministry, which is about to change the structure of the State exam, has released the so called “quadri di riferimento” (landmark frameworks). In these frameworks it is clearly stated that “the problems may have an abstract or practical character and also contain references to classical texts or significant historical moments related to Maths” (MIUR, 2018); therefore, perhaps the importance granted to history of mathematics will increase in the near future (Rogers et al., 2015).

A final observation relates to the teaching materials other than textbooks. In Italy we do not generally have institutions like IREMs in France. We have the NRD instead, i.e. the

Nuclei di Ricerca Didattica, allocated in mathematics departments within some universities, in which "general maths teachers" together with mathematicians and researchers in mathematics education cooperate to enhance didactics of mathematics and to produce didactical materials they then disseminate. However, these institutions are not specifically devoted to history or epistemology of mathematics; therefore, good material is not easily available. This is especially true of learning how to use original sources, which is quite difficult. (Though, there exists the book by Dematté and Furinghetti (2004), whose title (in English) is *Doing Mathematics with Historical Documents*, and an important site edited by Giorgio Bagni, which was last updated in 2009, the year of Giorgio's death (http://www.syllogismos.it/).)

6 Suggestions in view of a more promising future

In our opinion, a course about history of mathematics is not enough and perhaps also not the most important despite being strongly recommended.

What really enables teachers to let history permeate their teaching is having a quite extensive experience of dealing with historical original sources in their pre-service and in-service training. Using them while working in groups on the design of workshops or for lessons in which historical aspects of mathematics are re-discovered; to solve together historical problems; or translating, interpreting and contextualizing original sources into a portion of the curriculum. Moreover, it is fundamental for the teacher to give such types of lessons firstly to their peers in order to test them and afterwards to pupils, with the guidance of an expert supervisor.

What is more, the work based on original sources is surely very interesting and rich, but it can be too challenging for prospective teachers or pupils who have to learn the mathematical content in parallel with navigating the original source material. Having to simultaneously combine various goals, we rarely have the time to focus on the analysis of the old texts. Therefore, it can be a good idea to do it from time to time as an optional work.

Alternatively, we might consider ourselves allowed to treat historical contents in an anachronistic way, e.g. explaining a historical method in modern language, using equations written in our modern way, visualizing with GeoGebra even though the software did not exist, etc. Of course, we have to warn students that it is anachronistic and emphasize the moments in which the anachronism is particularly strong, clarifying in which period the tools we are making use of are born and by which civilization they were introduced. In China, this way of working is already a reality: the purpose of integrating history of mathematics into teaching is to achieve certain teaching goals and a rather consistent relationship between history and teaching has already been built (Wang, Qi, & Wang, 2017).

7 Some "virtuous" examples?

7.1 How Nathalie usually integrates History of Mathematics in her courses

Nathalie considers herself a "normal" teacher; she feels she has no lessons to give to colleagues on an "ideal" integration of the history of mathematics. She is used to think about how to make use of history through discussions with the members of the IREM's commission on epistemology and history of mathematics, who mainly are researchers.

However, in her classes, she does as much she can, with the time and the constraints she has. Sometimes she takes the time to read a historical text with the pupils or to construct short exercises from a historical situation, but often she only presents the context and the historical interest of a notion or problem (Chevalarias, 2016; Chevalarias & Minet, 2012). On certain occasions, she only gives a few elements from history, but she believes that knowing a bit of history of mathematics allows her to approach the content she teaches in a different way and stimulates her to be more attentive to the difficulties students encounter in approaching notions that we know have taken centuries to be built up and still are not trivial.

7.2 What Michel and his colleagues try to do while teaching at the professional bachelor and what he does in high school in Brussels

As far as the professional bachelor is concerned, since students generally enter the training with a poor vision of what mathematics is, thinking of mathematics as a magic box to solve exercises, he and his colleagues decide to teach alternative methods: sometimes theoretical, often through problem-solving, visualizing, etc. Here and there they mention the origins and stress the differences between old and new concepts. As an example, they underline the distance between Greek notion of number and today's concept of number; or they show the Arabic origin of trigonometry and the role of astronomy in its invention… but in the end the context and the historical side notes are easily forgotten.

In the third and last year of the teacher training, they let students develop teaching material by themselves. They are meant to design a workshop lasting one hour about a historical mathematical topic, in which they have to zoom out (sketch the cultural context) and zoom in (let the participants work on a proof or calculation or a figure…).

Concerning the teaching in high school instead, an observation, which undoubtedly does not apply only to Belgium, is the very diverse, multicultural student population. Taking into account the also very diverse cultural background of the branches of mathematics taught, it seems very important to let the pupils know which part of mathematics comes from ancient Greece, or from the Arabic/Islamic tradition from the Middle Ages, and what we have inherited from other centuries and cultures, also in view of motivating students with roots in Islamic countries to work hard and consider the possibility of going ahead studying mathematics and science at university.

7.3 How Caterina usually treat history of Maths in her lessons

While teaching in high school Caterina acts in essentially in two parallel ways. On one side, she lets history permeate her courses, on the other side, she makes use of original sources. By "general" integration of history she means that the historical perspective allows her decide the order of topics, the grade of emphasis which has to be given to a notion or a problem, which type of language which is worthwhile to use in situations, and the degree of "hybridization" among different branches of mathematics which is functional to a deeper understanding of a concept (e.g. Euclidean geometry, Cartesian geometry, linear algebra).

The use of sources is done either during the normal timetable or during special extracurricular and volunteer seminars on history and epistemology of mathematics open to all the students of the school, according to the situation. Very often she also encourages students to produce something ‘concrete’: a mathematical social play to be presented in a festival, an exposition of sources in a gallery, a theatrical show to be acted in some theatre, etc. (Vicentini, 2001; 2004; 2007)

While teaching to future teachers at university, first Caterina introduces a few general elements of epistemology and history, presents some points of view on didactics of mathematics, and then she imitates what Nicolas Rouche did in his course of “Méthodologie de l’enseignement” in Louvain-la-Neuve (BE), in letting future teachers work in groups in order to examine an original source and build a series of lessons making use of this source (cf. Rouche et al., 2006; 2008).

Eventually, in order to pass the exam, prospective teachers are supposed to present and discuss a complete unit based on historical material. The main phases of this unit must include an introduction to the topic, the historical presentation of the source, the guided reading of the source, some exercises and problems built from the original text, some possible hermeneutics hints for the students, some guided social activities on the source, some individual activities on the text, a summary of the important notions acquired, and the contextualization of the notions acquired in the school program (Bagni & Vicentini, 2008).

8 Closing remarks by the panel coordinator

I think some observations of a different kind and relevance are worth making before closing this quite long (and I hope never boring) paper. Let’s sketch them in a very synthetic way.

1. Unusually, in a references list, one can find more items than those strictly quoted. This is due to the will of giving to those potential readers who are teachers the possibility to find ready-to-be-used classroom sources possibly in their mother tongue. What is more, some additional, theoretically important references are present to consent an enrichment of the teachers’ personal theoretical framework.
2. The situation in China appears to be somewhat different to the occidental one. Maybe this fact allows an explanation for why “more than one in four students in Beijing-Shanghai-Jiangsu-Guangdong (China), Hong Kong (China), Singapore and Chinese Taipei are top-performing students in mathematics, meaning that they can handle tasks that require the ability to formulate complex situations mathematically, using symbolic representations” (PISA 2015-OECD).
3. The introduction of history in classrooms can help teachers to ‘row against’ what Freudenthal called the widespread “anti-didactical inversions” present in the curricula all over the world (Barbin, 2015; Freudenthal, 1999).
4. We focused more on history than on epistemology since according to the inquiry results, historical issues are more explicitly felt by teachers compared with epistemological ones (D’Amore, 2004).
5. We neither addressed hermeneutical problems that naturally arise while using history in general and original sources in particular (Bagni, 2006; 2009; Rorty, 1979), nor provided a reflection under the semiotics’ point of view (Peirce, 1989). Nevertheless, I am strongly convinced that a thorough analysis should deal with this perspective.
6. We only lightly touched upon cultural issues (D’Amore, Radford, & Bagni, 2006) and we did not approach interdisciplinarity (Battisttutti & Vicentini, 1994).
7. A more massive political and institutional presence of teachers and researchers seems

suitable in view of amending the curricula.

To conclude, I kindly thank Alain Bernard from France for his participation in the discussion on this topic with an interesting remark. Quoting the paper, “No, I Don’t Use History of Mathematics in my Class. Why?” by Man Keung Siu, Alain criticized the term “use” in our questionnaire, proposing to substitute it with the verbs “integrate” or “permeate”. Obviously, under the point of view of the HPM research, I agree with him. Nonetheless, given the results we acquired, I wonder if the majority of teachers interviewed would have noticed the difference and answered in a different way.

While underlining that to improve the situation in real classes we have no need to be too demanding; meaning (among other things) that the entire HPM research community should not bother too much about a sort of ‘contamination’ of their results. The distance between the ‘ideal’ and the ‘real’ class being quite large, the first step has to be a sort of ‘breaking the ice’, and I would almost say ‘no matter how’. Let’s take as an example the widespread use of the English language: the more it is spoken, the less it is orthodox BBC English. This seems to be an unavoidable consequence.

Often in conferences, while talking about the effectiveness of research, a tacit assumption is silently implied, i.e. that teachers have to do something more to align with researchers’ work. In my opinion the converse is also important: researchers should worry a bit more about the large integration of their research, not being too afraid of ‘contaminations’ going in the sense of ‘adaptability’ to real situations (Artigue, 2014).

Man Keung Siu himself, in the introduction to the same paper quoted by Alain Bernard, suggests: “instead of harbouring a preconceived view one should join the company of school teachers and listen with an open mind to what they have to tell about their classroom experience” (2006, p. 268)

I am glad to thank all the audience members during the panel for their open, rich and very kind participation. Now I feel comfortable in loudly confessing that I was really worried by the embarrassing silence that could have occurred due to my decision to manage the situation much more as a reflection’s starting point than as a presentation of ready-made results. In my opinion a panel should always be only loosely structured, otherwise instead of being one of the rare brainstorming official occasions in a conference, it runs the risk of becoming something like a sequence of multiple talks, losing a bit its appeal. In Italy we call panels ‘round tables’ and in a round table there are no privileged seats.

Finally, I offer all participants of ESU with the wish that instructional practice and HPM research will be able to move one towards the other in the fastest and most harmonious way possible. The meeting of these two essential aspects will hopefully be able to strengthen each other, in the same way as the experience of teaching and learning of mathematics.

REFERENCES

Artigue, M. (2014). *The challenging relationship between fundamental research and actions in mathematics education*. In G. Kaiser (Ed.) *Proceedings of the 13th International Congress on Mathematical Education*, ICME 13 Monographs (pp. 145-163). Cham: Springer.

Bagni, G. T. (n.d.). *History and Hermeneutics for Mathematics Education*. http://www.syllogismos.it.

Bagni, G. T. (2006). *Linguaggio, Storia e Didattica della Matematica*. Bologna: Pitagora.

Bagni, G. T., & Vicentini. C. (2008). History and Epistemology of Calculus and Algebra, Celebrating Leonhard Euler's Tercentenary. Cooperative Learning and Effectiveness of Perspective Teacher Training. In É. Barbin, N. Stehlíková, & C. Tzanakis (Eds.). *History and Epistemology in Mathematics Education: Proceedings of the 5th ESU* (pp. 439-440). Prague: Vydavatelský servis.

Bagni, G. T. (2009). *Interpretazione e didattica della matematica. Una prospettiva ermeneutica.* Bologna: Pitagora.

Barbin, É. (1994). Het belang van geschiedenis voor de wiskundige vorming. *Uitwiskeling 10*(3), 1-7.

Barbin, É. (2015). Philosophies and theories behind history and education: thirty years after Hans Freudenthal. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 21-39). Copenhagen: Danish School of Education, Aarhus University.

Battistutti, M., & Vicentini, C. (1995). Achille, la tartaruga e l'interdisciplinarità, *Quaderno 21 del NRD dell'Università di Trieste,* Trieste.

Bergamini, M., Trifone, A., & Barozzi, G. (from 1999 to 2019). Various Maths Textbooks. Bologna: Zanichelli.

Berlinghoff, W. P., & Gouvêa F., Q. (2016). *Wortels van de Wiskunde. Een historisch overzicht voor leraren en anderen*. Amsterdam: Epsilon Uitgaven.

Bkouche, R., Charlot, B., & Rouche, N. (1991). *Faire des mathématiques: le plaisir du sens*. Paris: Armand Colin.

Buchini, M., & Vicentini, C. (2017). Le equazioni di secondo grado con Lagrange, un insegnamento per competenze. *Periodico di matematiche,* vol. *9*, serie XIII anno CXXVII, 71-81.

Buchini, M., & Vicentini, C. (2018). Tweedegraadsvergelijkingen bij Lagrange, *Uitwiskeling 34*(2), 11-17.

Chevalarias, N. (2016). Comment calculer ses chances de gagner ? ... réponses historiques. In *Enseigner les mathématiques en seconde. Statistiques et probabilités : quatre parcours*. IREM de Poitiers, groupe lycée, 95-115.

Chevalarias, N., & Minet, N. (2012). Des séances "Maths-Histoire" en classe de seconde; *REPÈRES - IREM. 86*. http://www.univ-irem.fr/exemple/reperes/articles/86_article_579.pdf.

Chevallard, Y. (1985). *La transposition didactique: du savoir savant au savoir enseigné*. Grenoble: La pensée sauvage.

D'Amore, B. (2004). *Le basi filosofiche, epistemologiche e concettuali della didattica della matematica*. Bologna: Pitagora Edizioni.

D'Amore, B., Radford, L., & Bagni, G. T. (2006). Ostacoli epistemologici e prospettive socioculturali, *L'insegnamento della matematica e delle scienze integrate, 29B*(1), 11-40.

Demattè, A. (2015). History in the classroom: educational opportunities and open questions. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 335-352). Copenhagen: Danish School of Education, Aarhus University.

Demattè, A. (2015). Yes, I do use history of Mathematics in my class. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 511-525). Copenhagen: Danish School of Education, Aarhus University.

Demattè, A., & Furinghetti, F. (2006). *Fare matematica con i documenti storici.* Trento: Iprase.

ESU 8 website https://esu8.edc.uoc.gr/esu-8-main-themes-plenary-sessions/

Freudenthal, H. (1958). Sur la responsabilité humaine du mathématicien. *Mathematica & Pedagogia 15*, 38-40.

Freudenthal, H. (1999). *Didactical Phenomenology of Mathematical Structures.* New York: Kluwer Academic Publishers.

GEM (from 1993 to 1996). *De question en question 1,2,3*. Bruxelles: Didier-Hatier.

Gilbert, T., & Rouche, N. (2001). *La notion d'infini, L'infini mathématique entre mystère et raison*. Paris: Ellipses.

Gilbert, T., & Rouche, N. (2004). *L'infinito matematico fra mistero e ragione. Intuizioni, paradossi, rigore*. Bologna: Pitagora edizioni.

Hauchart, C., & Rouche, N. (1987). Apprivoiser l'infini, Louvain-la-Neuve: CIACO.

Jankvist, U. T. (2009). A categorisation of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics 71*, 235-261.

Kaenders, R., & Weiss, Y. (2018). Algebra without Context is empty, Visualizations without Concepts are Blind, In K. M. Clark, T. H. Kjeldsen, S. Schorcht, & C. Tzanakis, (Eds.), *Mathematics, Education and History toward a Harmonious Partnership* (pp. 121-141). ICME 13 Monograph Series. Cham: Springer.

Lakoff, G., & Nunez, R.E., (2000), *Where Mathematics Comes From. How the Embodied Mind Brings Mathematics into Being*. New York: Basic Books (Perseus Books Group).

Lakoff, G., & Nunez, R. E. (2005), *Da dove viene la matematica. Come la mente embodied dà origine alla matematica.* Torino: Bollati Boringhieri.

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2010). *Bulletin officiel spécial n° 9 du 30 septembre 2010*. http://cache.media.education.gouv.fr/file/special_9/21/1/mathsS_155211.pdf. Accessed: 22 November 2018.

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2019). *Bulletin officiel spécial n°1 du 22 janvier 2019.* https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/95/7/spe631_annexe_1062957.pdf. Accessed: 10 March 2019.

MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca). (2010). *Indicazioni Nazionali riguardanti gli obiettivi specifici di apprendimento per il liceo scientifico* http://www.indire.it/lucabas/lkmw_file/licei2010///indicazioni_nuovo_impaginato/_Liceo%20scientifico%20opzione%20Scienze%20Applicate.pdf

MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca). (2018). *Quadro di riferimento per la redazione e lo svolgimento della seconda prova scritta dell'esame di Stato. Percorsi liceali: liceo scientifico opzioni scienze applicate* http://blog.deascuola.it/uploads/sites/5/2018/11/QDR-liceo-scientifico-LI02-22-11.pdf.

Peirce, C. S. (1989). *Semiotica*. Torino: Einaudi Paperbacks 115.

PISA 2015-OECD https://www.oecd.org/pisa/pisa-2015-results-in-focus.pdf

Roelens, M. (2008). The volume of a pyramid through the ages. To slice or not to slice, that's the question. In É. Barbin, N. Stehlíková, & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 5th ESU*. Plzen: Vydavetelsky Servis.

Roelens, M. (2018). Archimedes en het volume van de bol. *Pythagoras 58*(2), 22-26.

Roelens, M., & de Graaf, W. (2015). Workshop on the Use and Mathematics of the Astrolabe. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU.* Copenhagen: Danish School of Education, Aarhus University Denmark.

Roelens, M., & Van den Broeck, L. (2015). Veeltermvergelijkingen van vroeger tot nu. *Uitwiskeling 31*(4), 12-38

Roelens, M. (2018). Archimedes en het volume van de bol. *Pythagoras 58*(2), 22-26.

Rogers, L., Weiss-Pidstrygach, Y., Barnett, J. H.; Guillemette, D., & Metin, F. (2015). The question of evaluation and assessment of experiences with introducing history of mathematics in the classroom, Panel Discussion 2. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 59-76). Copenhagen: Danish School of Education, Aarhus University.

Rorty, R., (1979), *The Philosophy and the Mirror of Nature*. Princeton: Princeton University Press.

Rouche, N., in collaboration with De Laet, L., Docq, C., Gantois, J. Y., Hauchart, C., Tancré, M., & Tossut, R. (2006). *Du quotidien aux mathématiques, Nombres, grandeurs, proportions*. Paris: Ellipses.

Rouche, N., in collaboration with Cuisinier, G., De Laet, L., Docq, C., Gantois, J. Y., Hauchart, C., Tancré, M., & Tossut, R. (2008). *Du quotidien aux mathématiques, Géométrie*. Paris: Ellipses.

Sasso, L. (in some cases together with Zanone, C.). (from 2006 to 2019). Various Maths textbooks, Novara: De Agostini.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coins. *Educational Studies in Mathematics 22*, 1-36.

Sfard, A. (2008), *Thinking as communicating: Human development, the growth of discourses, and mathematizing,* Cambridge: Cambridge University Press.

Sfard, A. (2009). *Psicologia del pensiero matematico, il ruolo della comunicazione nel processo cognitivo*, Trento: Erickson.

Siu, M. K. (2005). No, I don't use history of mathematics in my class. Why? In F. Furinghetti, S. Kaisjer, & C. Tzanakis (Eds.), *Proceedings of HPM2004 & ESU4* (pp. 268-277). Iraklion: University of Crete.

Smestad, B., Jankvist, U. T., & Clark, K. (2014). Teachers' mathematical knowledge for teaching in relation to the inclusion of history of mathematics in teaching. *Nordic Studies in Mathematics Education*, *19*(3-4), 169-183.

Tall, D. (2013). *How humans learn to think mathematically. Exploring the three worlds of mathematics.* Learning in Doing: Social, Cognitive and Computational Perspectives. Cambridge: Cambridge University Press.

Vicentini, C. (1994). Personal and social conscious and unconscious backgrounds in mathematics Education, theory and practice in mathematics education. In L. Bazzini (Ed.), *Proceedings of the Fifth International Conference on Systematic Cooperation Between Theory and Practice in Mathematics Education* (pp. 231-240). Pavia: ISDAF.

Vicentini, C. (1998). Percorso didattico alla riscoperta della trigonometria. In *Matematica é/e Cultura, la bellezza della teoria, il sapore delle applicazioni, Atti del Congresso nazionale Mathesis 1998* (pp. 301-310). Bologna: Pitagora editrice.

Vicentini, C. (2001). Si les mathématiques m'étaient contées... In P. Radelet-De Grave & R. Brichard (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 3rd ESU* (pp. 355-366). Leuven and Louvain-la-Neuve: Université catholique de Louvain.

Vicentini, C. (2003). Si può 'fare matematica' in prima media? *Progetto Alice 2003-III*, *4*(12), 597-615, Roma: Pagina.

Vicentini C. (2004). Once upon a time mathematics... (part 2). In F. Furinghetti, S. Kaisjer, & C. Tzanakis (Eds.), *Proceedings of HPM2004 & ESU4* (pp. 296 -305). Iraklion: University of Crete.

Vicentini, C. (2007). From π to aleph through the theatre: a way to avoid didactical frauds and pseudo-structured assurances, *La matematica e la sua didattica, 21*(1), 111-115.

Vicentini, C. (2008). 'Echte wiskunde' in de eerste graad, *Uitwiskeling 34*(3), 8-18.

Vicentini, C. (2012). *Ocalogik*, https://www.premiocesarecancellieri.it/2012-lavori-premiati-concorso-cancellieri/

Vicentini, C. (2012). Si può migliorare il ragionamento giocando? In *Progetto Alice 13*(38), 337-351, Roma: Pagina.

Vicentini, C. (2014). Ocalogik: al spelend je redeneervermogen verbeteren, *Uitwiskeling 30*(3), 16-21.

Vicentini, C. (2015). Playing with Euler. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 373-387). Copenhagen: Danish School of Education, Aarhus University.

Vicentini, C (in progress). *Storia ed epistemologia della matematica, CLIL e il gioco della 'briscola* *pitagorica'*.

Villani, C., & Torossian, C. (2018). *21 mesures pour l'enseignement des mathématiques.* http://cache.media.education.gouv.fr/file/Fevrier/19/0/Rapport_Villani_Torossian_21_mesures_pour_enseignement_des_mathematiques_896190.pdf. Accessed: 22 November 2018.

Wang, X., Qi, C., & Wang, K. (2017) A categorization model for educational values of history of mathematics: an empirical study. *Science & Education*, *26*(7-9), 1029-1052.

LEONHARD EULER'S DIFFERENTIALS

An Attempt to Restructure Teaching of the Derivative Concept

Cihan CAN[1,4], Mehmet AKTAS[2,5], Janet BARNETT[3,6], Kathleen M. CLARK[1,7]

[1]Florida State University, USA, [2]University of Central Oklahoma, USA

[3]Colorado State University, Pueblo, USA

[4]cc15m@my.fsu.edu, [5]maktas@uco.edu, [6]janet.barnett@csupueblo.edu, [7]kclark@fsu.edu

ABSTRACT

This workshop will present a strategy for teaching the derivative concept based on Leonhard Euler's text *Foundations of Differential Calculus.* The idea of using Euler's approach in a modern classroom is an extension of the primary source project (PSP) *The Derivatives of the Sines and Cosine Functions* (Klyve, 2017), itself developed as part of a larger effort to incorporate primary sources into the teaching of standard mathematics topics currently supported by major funding from the US National Science Foundation. In this particular PSP, Euler's method of determining differentials for the sine and cosine functions is shared with students through guided reading of selected excerpts from an English translation of Euler's original text. However, Euler's use of differentials is not limited to these two functions but is developed and applied to other functions throughout his text. In other words, the symbols *dy, dx* and the mathematical concepts these symbols represented had a central role in Euler's calculations.

In contrast to Euler's approach, the topic "differentials" is placed at the end of the chapter on derivatives in today's calculus textbooks. When the symbols *dy* and *dx* appear in calculations prior to this point in the course, they are used to represent the derivative function: $f'(x)=\mathrm{d}y/\mathrm{d}x$. This raises a number of perplexing questions for students and instructors alike:

Is [$\mathrm{d}y/\mathrm{d}x$] *a fraction, or a single indivisible symbol? What is the relationship between the* $\mathrm{d}x$ *in* $\mathrm{d}y/\mathrm{d}x$ *and the* $\mathrm{d}x$ *in* $\int f(x)\,\mathrm{d}x$*? Can the* $\mathrm{d}u$ *be cancelled in the equation* $\mathrm{d}y/\mathrm{d}x = (\mathrm{d}y/\mathrm{d}u)\cdot(\mathrm{d}u/\mathrm{d}x)$*?* (Tall, 1993, p. 5)

The conceptual challenges encountered by students as a result of the current approach to teaching and learning calculus suggest that placing differentials at the center of the subject – as Euler did – might inform (and reform) our pedagogical approach to teaching not just about the derivative, but about calculus more generally. Such a pedagogical approach is currently being tested in a university-level first-year calculus course in the United States.

During the workshop, participants will consider several of its student tasks, the primary goal of which is to use Euler's perspective on differentials as the building block for student learning of the derivative concept. Following a brief introduction to a study conducted in Fall 2017 study, participants will first be asked to work together in small groups on these problems as calculus students might (approximately 15 minutes). They will then be asked to reflect on their problem-solving experience as "students" from the perspective of a teacher, and discuss pedagogical issues and concerns in these same small groups (approximately 20 minutes). This will be followed by a whole group discussion during which participants will share their reflections and consider issues related to

classroom implementation of the approach (approximately 30 minutes). Finally, I will share some findings from the actual classroom implementation of the PSP and other Euler-based differential tasks, and invite questions and reflections from the workshop participants (approximately 25 minutes).

REFERENCES

Euler, L. (1755). *Foundations of Differential Calculus*. (J. D. Blanton, Trans.). New York, NY: Springer-Verlag New York, Inc. (Original work published in 1755)

Klyve, D. (2017). *The derivatives of the sine and cosine functions*. Retrieved from http://digitalcommons.ursinus.edu/triumphs_calculus

Tall, D. (1993). Students' difficulties in calculus. In C. Cauglin et al. (Eds.), *Proceedings of Working Group on Students' Difficulties in Calculus, ICME-7* (pp. 13–28). Québec, Canada.

CAPTURING STUDENT BELIEFS AT, DURING, AND BECAUSE OF THE TRANSITION FROM SCHOOL TO UNIVERSITY MATHEMATICS

Evidence of Influence of the Historical Development of Geometry

Kathleen M. CLARK[1,4], Gero STOFFELS[2,5], Ingo WITZKE[2,6], Horst STRUVE[3,7]

[1]Florida State University, USA, [2]University of Siegen, Germany,
[3]University of Cologne, Germany

[4]kclark@fsu.edu, [5]stoffels@mathematik.uni-siegen.de,
[6]witzke@mathematik.uni-siegen.de [7]h.struve@uni-koeln.de

ABSTRACT

In 2015, we began work on the design, development, and implementation of a seminar for undergraduate mathematics students who were preparing to teach mathematics. The theoretical underpinning of the seminar included our hypothesis that:

> The change from an empirical-object oriented to a formal-abstract belief system of mathematics constitutes a crucial obstacle for the transition from school to university. And, on epistemological grounds, similar changes regarding different natures of mathematics can be described for the history of mathematics. The explicit analysis of the historical genesis provides support for students dealing with their individual transition processes.

From this hypothesis we also highlight our notion of the two extremities of the belief system continuum. As we previously asserted (Witzke, Struve, Clark, & Stoffels, 2016), "an empirical belief system on the one hand describes a set of beliefs in which mathematics is understood as an experimental natural science, which includes deductive reasoning about empirical objects" (p.71); whereas, a formalistic belief system "describes a set of beliefs in which mathematics is understood as a system of un-interpreted concepts and their connections in propositional functions …, which can be established using axioms, (implicit) definitions, and proofs" (p.72).

Thus, driven by the stated hypothesis, the aim of the seminar was to promote students' awareness of the changes regarding the nature of mathematics from school to university. We used geometry as the topic of the seminar's mathematical content and the seminar activities included engaging students in reading and discussing excerpts, task transcripts, textbooks, standards, and historical resources, as well as working on various tasks prompted by historical sources and content. The seminar, "Addressing the Transition Problem from School to University Mathematics" (which we refer to as the ÜberPro Seminar, from the German, Übergangs Problematik, and in English, "transition problem") was first implemented as a three-day intensive seminar in Spring 2015. In relation to literature focused on the transition from school to university mathematics contexts, Gueudet et al (2017) have described certain boundary objects which may play a significant role in helping students to "make this transition."[1] In particular, Bosch noted, "in the case

[1] To clarify, we are concerned more with working with students to recognize and address said transition, and less on "making the transition" in the research on the design and implementation of ÜberPro.

of the transition between secondary and tertiary education, an interesting boundary object can be the so-called bridging courses organized in different universities to smooth the gap between upper-secondary school and university" (*ibid* p.109). In this way, ÜberPro may be considered in this classification of such bridging courses; however, the unique attribute of our course is in the potential of utilizing the case of the historical development of geometry. We have described the first implementation of ÜberPro elsewhere (Witzke et al., 2016), as well as the modifications to extend the initial intensive seminar into a semester-long seminar experience (Witzke, Clark, Struve, & Stoffels, 2018). Furthermore, we propose that in light of the instructional materials (e.g., textbooks) that students face in school mathematics, they are more likely to acquire an empirical belief system. Yet, at university, students are likely to obtain a formalistic belief system based upon the instructional materials found there. Epistemologically, both of these experiences provide parallels to specific historical understandings of mathematics, which we sought to highlight as the fundamental components for the design of our "transition problem" seminar for students.

In this presentation, we will describe one aspect of the research conducted during the third implementation of ÜberPro in Summer 2017. We sought to address the research question: In what ways do ÜberPro Seminar students, confronted with the historical development of mathematics, recognize their own transition? Data sources included recordings of and field notes from 12 class sessions, seminar materials, and "reflection learning diaries" (RLDs) from approximately 15 seminar students. We conjectured that the RLDs would provide important evidence about whether the explicit analysis of the historical genesis addressed in the seminar supported students in dealing with their individual transition processes. In our analysis of the students' seminar experience – especially with a focus on the way in which the seminar served to promote students' awareness of changes in the nature of mathematics – we searched for evidence of the ways in which students responded to weekly research diary questions. In particular, we sought to (1) determine whether the RLD questions were addressed or not by the student, and (2) classify whether there was explicit attention to an emotional, content, or opinion orientation within their responses. Since a key outcome of the seminar was to understand how students were recognizing and dealing with their own transition process, the first author was particularly interested in the emotionality expressed as students were confronted by elements of the transition from school to university mathematics as presented during the seminar. In the presentation, we will present cases that represent students at different points of recognition and awareness of their own transition as prompted by an examination of the historical development of geometry.

REFERENCES

Gueudet, G., Bosch, M., di Sessa, A. A., Kwon, O. N., & Verschaffel, L. (2017). Transitions in mathematics education: The panel debate. In G. Kaiser (Ed.), *Proceedings of the 12th International Congress on Mathematical Education,* ICME-13 Monographs (pp. 101-117). Cham, Switzerland: Springer Nature.

Witzke, I., Clark, K. M., Struve, H., & Stoffels, G. (2018). ÜberPro: Addressing the transition from school to university. Evolution of a seminar emphasizing historical sources and student reflections. In K. M. Clark, T. H. Kjeldsen, S. Schorcht, & C. Tzanakis (Eds.), *Mathematics, education and history: Towards a harmonious partnership* (ICME-13 Monograph Series) (pp.61-82). Cham: Springer.

Witzke, I., Struve, H., Clark, K., & Stoffels, G. (2016). ÜberPro – A seminar constructed to confront the transition problem from school to university mathematics, based on epistemological and historical ideas of mathematics. *MENON: Journal of Educational Research*, *2nd Thematic Issue*, 66-93. Retrieved from http://www.edu.uowm.gr/site/system/files/uberpro_-_a_seminar_constructe_scienc_en.pdf

MATHEMATICS AND EXPERIMENT

How to calculate areas without formulas?

Carène GUILLET[1], Marie-Line MOUREAU[2], Isabelle VOILLEQUIN[3]
IREM des pays de Loire, Faculté des sciences et des techniques, 2 rue de la Houssinière, Nantes, France
[1]carene.guillet@gmail.com, [2]marie-line.moureau@ac-nantes.fr,
[3]isabelle.voillequin@hotmail.fr

ABSTRACT

In this paper, we are presenting a work carried out by a research group on the history of mathematics for secondary school teaching in France. The goal of this work is to introduce the notion of area from a historical perspective, using various media as ancient texts, software and concrete objects to manipulate. We show how this original approach increases the student's motivation and understanding. To do this, we present the historical texts worked on in our group and then explain how we have integrated them into our teaching at different levels and as transitions between the levels. In particular, we present some "integration machines", whose manipulation aroused our audience surprise, whether students' as well as teachers'.

Introduction

In our IREM[1] research group on history of mathematics, we have been working on the issue of experiment in mathematics since 2014. After studying the use of the balance in mathematics (Equipe GRHEM, 2018), we started to work on the notion of areas throughout history[2].

That's how we came up with the idea of introducing the notion of area and the associated calculations, based on historical methods. This approach has a cultural purpose since it helps to familiarize our students with elements of the history of mathematics. In addition, it seems useful to us for a better understanding of the students for three main reasons:

- They often calculate areas with formulas they have learnt without having understood their meaning, as a kind of application of a "magic formula". The historical method allows them to obtain results with the sole force of their reasoning and makes more concrete formulas learned by heart in the past.
- The curriculum in France does not make any link between the different approaches to the concept of areas: 11 to 15 year old students[3] learn it in the context of geometry, high school[4] students use integrations in the context of calculus. The history of mathematics makes it possible to connect those different points of view as it provides a real transition in the curriculum between successive levels.
- Initiating students to ancient ideas and techniques can improve their motivation in mathematics. For example, the use of machines like planimeters and integrators stimulates their curiosity and makes them connect the current methods to the oldest.

[1] Since the late 1860s, the *Instituts de recherche sur l'enseignement des mathématiques* (Research institutes on mathematics education) bring together primary, secondary and higher education teachers to conduct research on mathematics education and thus participate in teacher training. The authors of this article are high school teachers.

[2] The link between these two subjects was the study of Archimedes' text about his "mechanical method", which will be discussed later.

[3] Secondary school is called "collège" in France.

[4] High school is called "lycée" in France.

In this paper, we present a possible progression from the level "Seconde" (first grade in high school, with students aged from 15 to 16 years) to the level "Terminale" (last grade in high school, with students between 17 and 18 years old). In French curriculum, rectilinear areas are studied in primary school and in "collège". The areas then only reappear in "Terminale" with the integral calculation. In between, there is a gap in the school programs that we are trying to fill.

This progression is based on well-known historical supports, extending over three distinct periods:

- In Antiquity: *The Elements* of Euclid, *Measurement of a circle* and the "mechanical method" of Archimedes.
- In the 17th century: *Geometria Indivisibilus* by Cavalieri and the *Treatise on the Indivisible* by Roberval.
- In the 19th century: integration machines such as planimeters and integrators.

Our work takes place in several stages: first of all, reading and analysis of texts in a research group, then reflection on how to apply it in class, followed by experimentation with our students, and finally feedback to fellow teachers via presentations in workshops and training courses.

The pedagogical work we present to you followed this process and was presented at a workshop at ESU-8 in Oslo.

1 Greek quadratures in "Seconde"

1.1 Areas in Euclid's way

1.1.1 Text presentation

In Euclid's *Elements*, a plane figure is a shape and a magnitude (its area). Contrary to what seems familiar to us today, however, the areas were not evaluated by numerical units of measurement. To measure areas consists in equaling them geometrically by comparison or by addition for example. Indeed, common notion 4 states that "things which coincide with one another are equal to one another" and common notion 2 indicates: "if equals be added to equals, the wholes are equal" (Heath, 1956, p. 224). To square a figure is to construct (with ruler and compasses) the side of a square of the same area.

In his *Elements*, Euclid states theorems that consist in comparing areas of rectilinear figures:

"Book I – Proposition 35 – Parallelograms which are on the same base and in the same parallels are equal to one another. (Heath, 1956, p. 326) […]

Euclid also gives construction methods for moving from one geometric figure to another geometric figure.

Proposition 44 – To a given straight line to apply, in a given rectilinear angle, a parallelogram equal to a given triangle. (Heath, 1956, p. 341) […]

Proposition 45 – To construct, in a given rectilinear angle, a parallelogram equal to a given rectilinear figure." (Heath 1956, p. 345) (fig. 1.1)

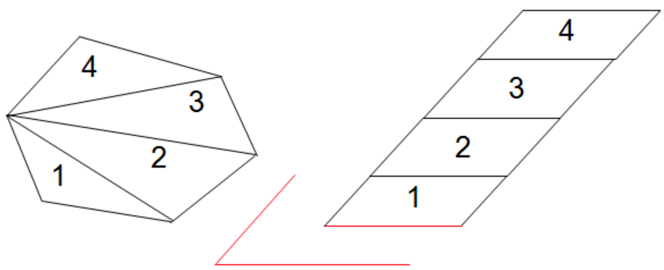


Figure 1.1: Euclid-Book I-Proposition 45[5]

Quadratures are therefore fundamental in this geometry since they make it possible to reduce any figure to a square, which can then be easily compared to other squares. The last proposal in Book II in fact gives the quadrature of any rectilinear figure:

"Book II – Proposition 14 – To construct a square equal to a given rectilinear figure." (Heath, 1956, p. 409)

To prove this last proposition, Euclid begins to construct a rectangle equal to the rectilinear figure, and then squares this rectangle.

In that way, one can square any rectilinear figure by decomposing it into triangles, each triangle being equal to a parallelogram, itself equal to a rectangle, finally equal to a square.

1.1.2 Pedagogical application

Without presenting the text to the students, we proposed them exercises in which this method could be applied. They had to determine areas without calculating, but rather by counting tiles, making decompositions or puzzles (see exercises 1 and 2 in the appendix).

In our workshop, we presented these problems to the participants, and we asked them which level of high school students they could give it to. First of all, they were surprised that we proposed this kind of exercises to high school students, because these notions are taught in "collège" and even in elementary school. In our classroom experiments, solving those problems wasn't so easy for our fifteen years old pupils (see works 1 and 2 in the appendix).

One of us also gave exercise 1 to her students in "prépa ECT"[6]. That class includes students from technological education[7], aged from 18 to 20 years. While they were in high school, they didn't study a lot of mathematics (about three hours per week) and so it wasn't their main subject at school, and nor their favorite one! The teacher was surprised at the reaction of students to this exercise. First, they counted each tile of the rectangle, and after counting, realized it was only the multiplication of the length by the width, and remembering the formula, lastly said: "That's why!"[8]

Finally, even an exercise that may look simple is not necessarily so for students, when it uses reasoning and unusual figure manipulations instead of the implementation of formulas. Coming back to the source of the calculation of areas by a method related to history has allowed students to rediscover the formulas learned long ago by understanding

[5] This figure has been realized with a geometry software as well as figures # 2, 4, 5, 6, 7 and 8.
[6] In this two years course, students prepare entrance contests to major business and management schools.
[7] STMG series in France "Science et technologie du management et de la gestion" (Financial management science and technology).
[8] In "ECT" class, this work started the "integration" chapter. The other experiments were conducted in "Seconde" in half classes, over time devoted to research, and at any time of the year, since these concepts do not appear in the official program of this level.

them. This leaves us thinking about the all-powerful place of calculations in the programs, especially when they are meaningless for students.

1.2 Squaring parabola with Archimedes' method

1.2.1 Texts presentation

After having squared all rectilinear figures by Euclid's method, one may wonder how the Ancients were able to square curvilinear figures.

In his treatise *Measurement of a circle* (Heath, 1897), Archimedes solves the quadrature of the circle in that way:

"Proposition 1 – The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle" (Heath, 1897, p. 91) (fig. 1.2).

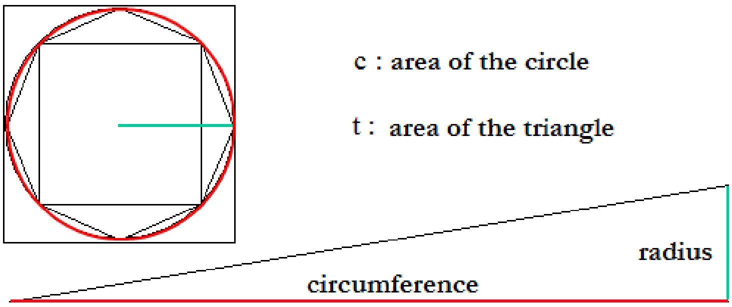


Figure 1.2: Squaring the Circle

In this text, Archimedes proposes a proof of the theorem based on a double *reductio ad absurdum*, which is a classical method in ancient Greek texts. Thus, the method of finding the result is not indicated.

In the 1630s, mathematicians made many reproaches to the Ancients for having hidden their method to find their results (Barbin, 1987, pp. 125-159). They tried to find a new "method of invention", which could solve any quadrature by giving the way they obtained them, and also for new curves, like the cycloid, defined by a kinematic process[9].

In this period, they didn't know that Archimedes gave one in a letter that had been lost for centuries, and discovered in 1906. The original Archimedes' text was copied around the 10th century on a parchment, which had been reused in the 12th century. The hidden text of the palimpsest had been entirely revealed by technological methods in the 2000s (Noel, 2008).

In this text, Archimedes describes his "mechanical method" and applies it among others to the quadrature of parabola (Heath, 1912).

"Proposition 1 – Let *ABC* be a segment of a parabola bounded by the straight line *AC* and the parabola *ABC*, and let *D* be the middle point of *AC*. Draw the straight line *DBE* parallel to the axis of the parabola and join *AB*, *BC*.

Then shall the segment *ABC* be $\frac{4}{3}$ of the triangle *ABC*." (Heath 1912, pp. 15-16) (fig. 1.3)

[9] See "Indivisibles Method" in section 2.

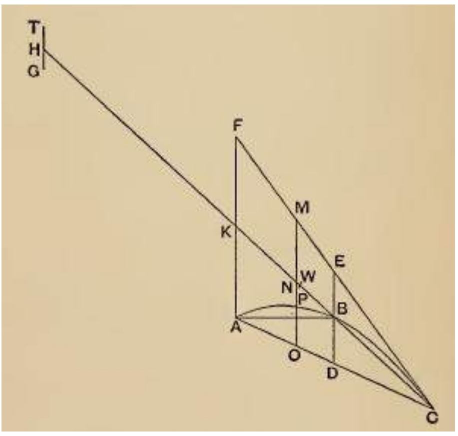


Figure 1.3: Archimedes' Mechanical Method (Heath 1912, p. 16)

He makes a both mechanical and mathematical proof, using the straight line *CK* as a lever around which he balanced the areas to be compared.

1.2.2 Pedagogical application

For teaching, the issues are the same: we can prove the result to our students in rigorous ways, but they won't understand the result if we don't give them a way to catch the "invention method" with which we found it. In this sense, working on Archimedes' mechanical method could be very instructive for our students.

We did not work on Archimedes' text with our students, because it seemed too difficult to address, especially because of the underlying properties of conics, which would require prior work to be introduced. On the other hand, we proposed an open problem to our students in "première" (aged from 16 to 17 years) aimed at conjecturing the area under a portion of a parabola.[10]

You can see the exercise and their works in the appendix (works 6 and 7). That area (below the parabola) represents a third of the area of the rectangle. Indeed, the area above the parabola covers four third of the triangle *ABC* (according to Archimedes' result), i.e. two third of the rectangle *ABCD*. This reasoning could already be the subject of an exercise in itself. Some students found the good ratio and others weren't too far from the result. Their methods were varied and imaginative.

In our workshop, we presented these exercises to participants, and asked them to think about a way to study Archimedes' "mechanical" proof with students. Indeed, we are still looking for some relevant and efficient schoolwork to propose on this subject in our class. We created a GeoGebra animation which details all the steps of the proof. That could be a good tool to build an educational sequence[11].

This pedagogical sequence is still a "work in progress" to be achieved but we are sure it might be rewarding for pupils to study that method. That exercise refers to the classical theme of the Archimedean lever in a particular way, and in a mathematical proof. It also appeals to the intuitive notion of balance while requiring strong geometric knowledge. It could show to students that mixing mathematics and mechanics is original and can be effective.

[10] The experiment was carried out in half classes, over time devoted to research, and at any time of the year, since these concepts do not appear in the official program of this level.
[11] The GeoGebra figure is available in following the link: https://www.geogebra.org/classic/chqspxrg.

2 In "Première"[12]: Cavalieri's Indivisibles Method

2.1 Text presentation

In 17th century, the Italian mathematician Bonaventura Cavalieri publishes *Geometria indivisibilibus continuorum nova quadam ratione promota* (1635) (Cavalieri, 1653). In this treatise, he presents his new method called "Geometry of indivisibles" in these terms:

"If, between the same parallels, any two figures are constructed, if inside them, any straights lines are drawn equidistant from the parallels, and if the portions included in any one of these lines are equal, then the plane figures are also equals to each other" (fig. 2.1)

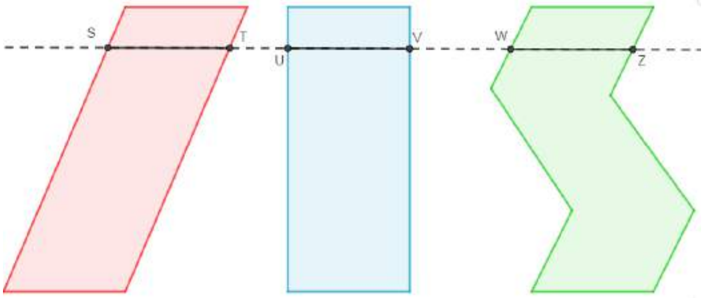


Figure 2.1: Cavalieri's Indivisibles

This new process is quite simple to understand and is easily applicable to students.

2.2 Pedagogical application

We proposed an exercise, based on that principle, to our classes in level "première" (students aged from 16 to 17 years). Its statement is contained in the appendix as "Exercise 3", as well as one student's work (work 3). As you can see in question 1, this young student didn't notice that the three areas were the same. All the other students made the same mistake. In question 2, a few pupils realized that the figures had the same area.

Students made also a construction on GeoGebra software to visualize the proof of the circle area with indivisibles method, proposed by Cavalieri (fig. 2.2):

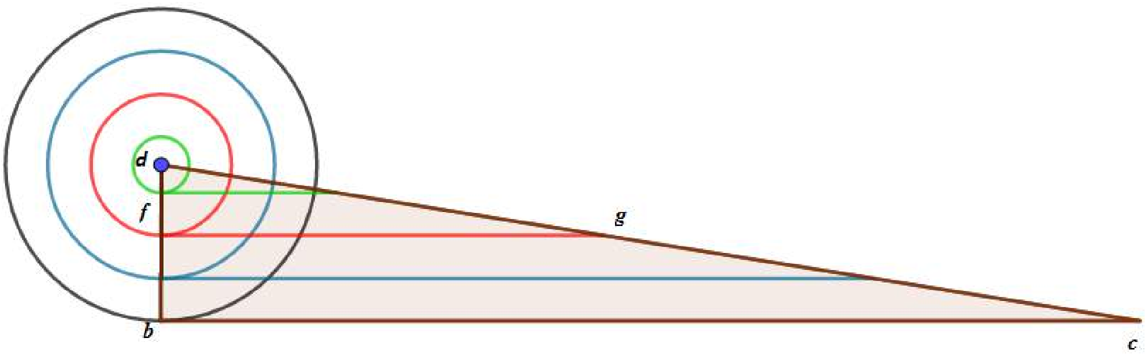


Figure 2.2: Squaring a Circle by Cavalieri's Method

Our students easily built it on GeoGebra and were able to experience the indivisible thanks to the "trace" function of the software. They were thus able to understand visually the formula of the area of a disc.

We showed exercises and student works to the participants of the workshop. They could also watch and manipulate the GeoGebra animation about the circle area. We discussed about the relevance of that kind of exercise and the contribution of such a

[12] Second grade in high school, with students aged from 16 to 17 years.

method to the training of pupils.

Compared to the proof of the same theorem by Archimedes, this way of proving ~~is~~ could be more intuitive and understandable for the students. A double proof by the absurd does probably not have a meaning for them, contrary to this very visual evidence, which allows them to understand the profound reason for this result. We plan to build a new learning sequence in this sense, with a parallel study of Archimedes' and Cavalieri's proofs, in order to verify this theory.

2.3 To go further

To be complete on that subject, and go further with certain students, it is also possible to talk to them about the criticisms encountered by indivisibles method in the 17^{th} century. Indeed, Cavalieri’s contemporaries like Guldin[13] criticized his method because it generated paradoxes. The two most famous ones are the followings (fig. 2.3, 2.4 and 2.5):

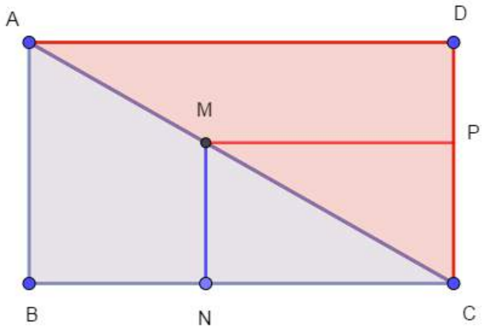


Figure 2.3: Paradox 1

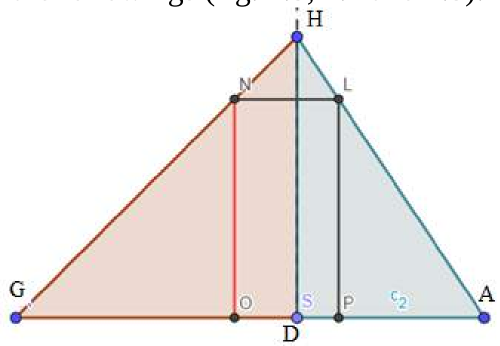


Figure 2.4: Paradox 2

“Do blue and red triangles have the same area?”

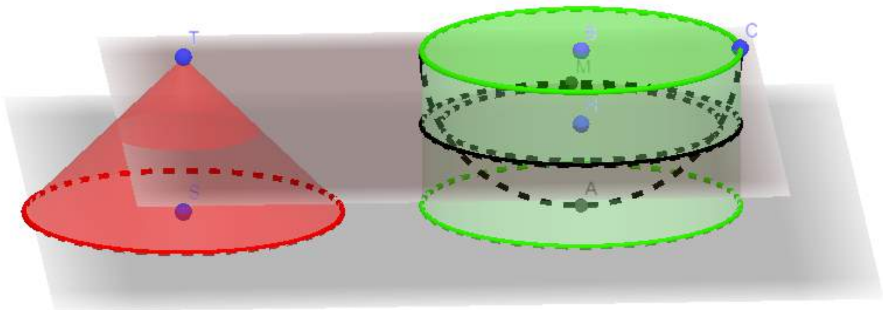


Figure 2.5: The Bowl’s Paradox: “Is a point equal to a circle?”[14]

On these few examples, we can see how Indivisibles’ method led mathematicians to ask questions about infinity and the infinitely small. That kind of queries can arouse students’ curiosity, make them think about the way science works and progresses, the barriers people have had to overcome to build mathematics as we know them today. A good source to understand how Cavalieri improved his method to solve this kind of paradoxes is his letter to Torricelli of April 5th 1693 (Roberval, 1693, pp. 283-302).

[13] Paul Guldin (1577 – 1643) was a Swiss Jesuit, mathematician and astronomer.

[14] Paradox treated by Galileo (Galileo 1914). He proved that the cone and the bowl have the same volume, which gave, passing to the limit, the equality between the disc and the point.

3 Transition from “Première” to “Terminale”: Roberval’s indivisibles method

3.1 Text presentation

The last step in our progression in high school curriculum is the presentation of Roberval’s proof for squaring the parabola.

Still in the 17^{th} century, Gilles Personne de Roberval[15] developed his own indivisibles method, in response to a problem put by Marin Mersenne[16]. Indeed, in the 1630s, Mersenne asked Roberval to determine the area of a portion of a cycloid. To answer him, Roberval devised a method of invention of the tangents and a method of quadratures with the help of the indivisibles. He applied this method to “all the curves” known at the time, solved the problem of the cycloid posed by Mersenne, and then obtained the quadrature of the parabola. He showed that the area of the portion of the parabola is equal to two-thirds of the area of the rectangle. (Walker, 1932, pp. 181-182) (fig. 3.1)

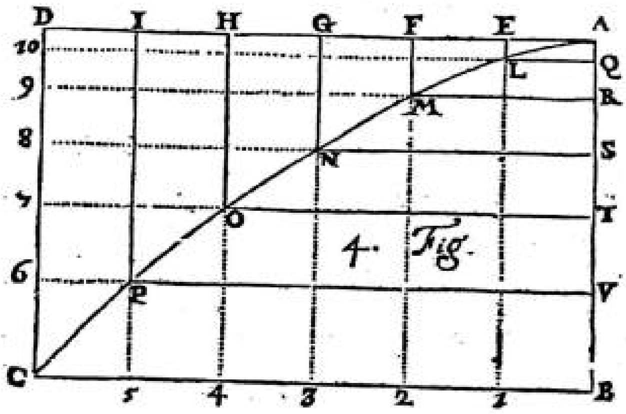


Figure 3.1: Roberval’s Quadrature of Parabola (Walker, 1932, p. 181)

His proof was based on the Indivisibles principle and a formula of sum, which he established a little before in his treatise.

3.2 Pedagogical application

We introduced this to our students in three steps:

- Firstly: they had to work on the sum of squares formula. They had at their disposal wood pyramids to assemble in order to form a rectangle parallelepiped. From this 3D puzzle, they had to guess the formula of a sum of squares (pict.3.1 and 3.2).

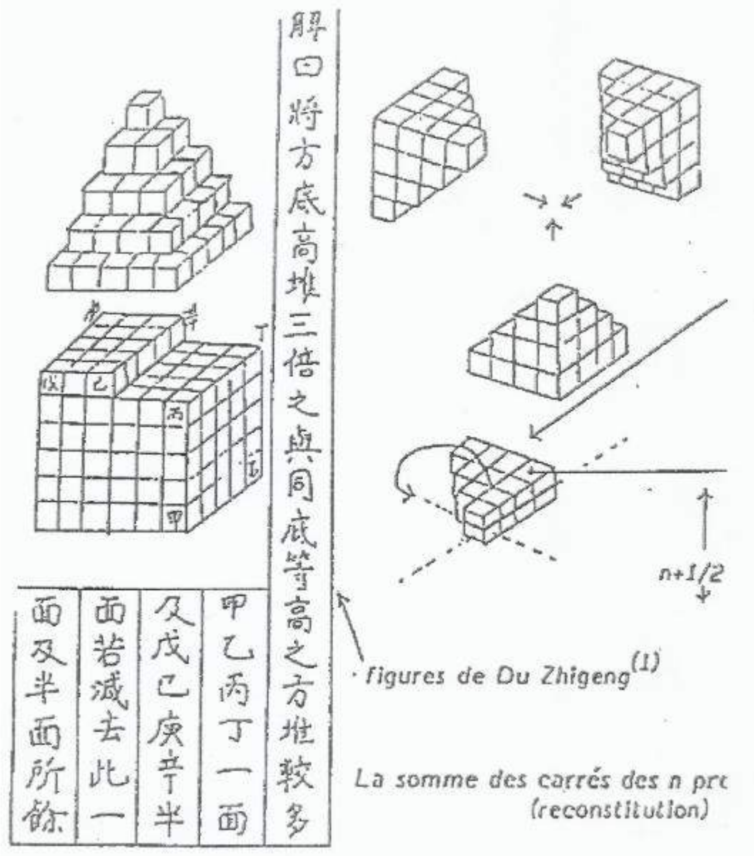
Picture 3.1: Students at work

Picture 3.2: Students have found

[15] Gilles Personne de Roberval (1602 – 1675) was a French mathematician.

[16] Marin Mersenne (1588 – 1648) was a French cleric, scholar, mathematician and philosopher.

They could also watch a video about this “Chinese puzzle”[17]. Furthermore, they had an iconographic document to refer to (doc. 3.1).



Document 3.1: Sum of Square

To further study this formula, they had an exercise to do on a spreadsheet[18]. The point was to see that a sum of squares is not so far from the third of the cube of the highest number of the sum, and that the difference is all the smaller as the number of terms is big.

- Secondly: they had to read and try to understand Roberval’s text (Walker, 1932; Roberval, 1693) and to illustrate it by completing a GeoGebra animation[19]. The idea was to build and see the indivisibles with the help of dynamic geometry, from the figure of the text.
- Thirdly: they had to read and try to catch the idea of the end of Roberval’s text, in which he extended his result to other power functions. Exercises were thus proposed to train students to calculate areas below curves in different configurations.

This sequence is the opportunity to make a transition between the “Première” curriculum (students aged from 16 and 17 years), and the “Terminale” one (a year later)[20]. Indeed, in “Terminale” class, students work a lot on integral calculation. These activities were intended to prepare students to develop reasoning on integrals, without resorting to calculate primitives.

---

[17] [Math Help]. (2015, Jan 14). *Sum of squares* (video file). Retrieved from: https://www.youtube.com/watch?v=kZTFrv3vRgg.

[18] You can see the spreadsheet following the link: https://docs.google.com/spreadsheets/d/1-dNkYiK_16IXT1Aa7yLitB9R_ZxxMlAAzq4gxmA90Z0/edit?usp=sharing.

[19] You can see the GeoGebra file following the link: https://www.geogebra.org/classic/a5436kdu.

[20] That’s why this sequence was experimented at the end of “première” and ended in some classes at the beginning of “Terminale” (the third step).

Participants of the workshop had access to all the pedagogical material: pyramids, videos, documents, GeoGebra and spreadsheets animations. For example, they could experience the difficulty to make the puzzle of the Chinese pyramids (pic. 3.3).

Picture 3.3: Making the puzzle at the workshop

They also could notice that, working on these exercises, some students had found by themselves the primitive formula of power functions. Some works are shown it in the appendix (works 4 and 5).

This part of our work is at the margins of our theme “how to calculate areas without formulas”, because Roberval’s proof requires to use the sum of squares formula, and is definitely more computational than Cavalieri’s or Archimedes’ approaches. However, we wanted to introduce it because it is a very rich teaching sequence for our students. It mixes the reading and understanding of historical texts with activities on spreadsheets and GeoGebra software as well as manipulation of concrete material. We had obtained results beyond our expectations, from the point of view of acquisition as well as the one of students’ motivation and interest.

4 In all classes: squaring with instruments and machines

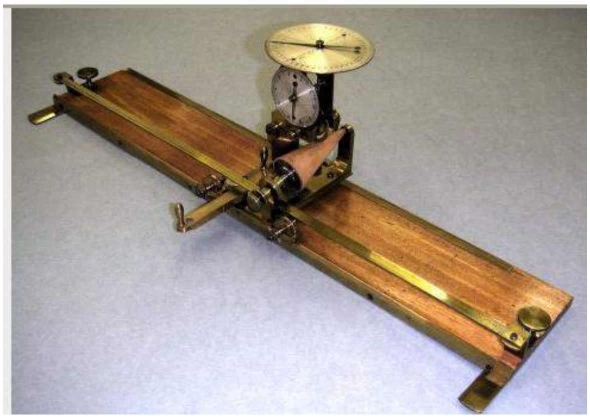
4.1 Machines presentation

Squaring with machines is a method almost forgotten nowadays. The use of computers has made it obsolete. But it was a fairly common way in the 19th century. The need was great in many fields like: computing stress in civil engineering, evaluation of the average power of a machine, calculation of the property tax, measure of the extent of a forest from a cadastral map. All those operations require the determination of the area of a surface.

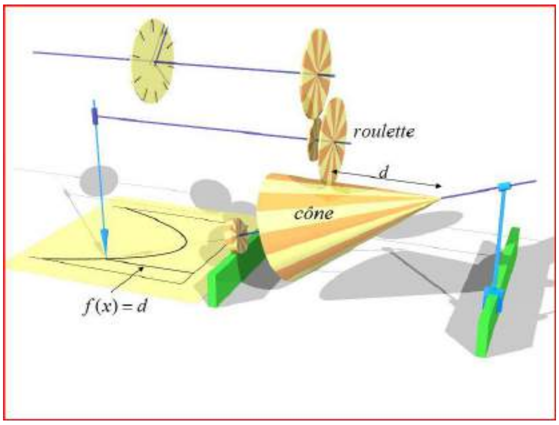
Economic, technical and industrial issues were therefore important.

As early as 1814 engineers invented machines like planimeters to measure the area of a surface by going along its contour and like integrators to draw an integral curve. These machines were used until the 1970s (Tournès, 2003; Gatterdam 1981).

One machine has been invented in 1814, by Johann Hermann in Germany. It had been named a "Cone planimeter" (pic. 4.1 and 4.2):



Picture 4.1: Cone Planimeter



Picture 4.2: Cone Planimeter's Scheme

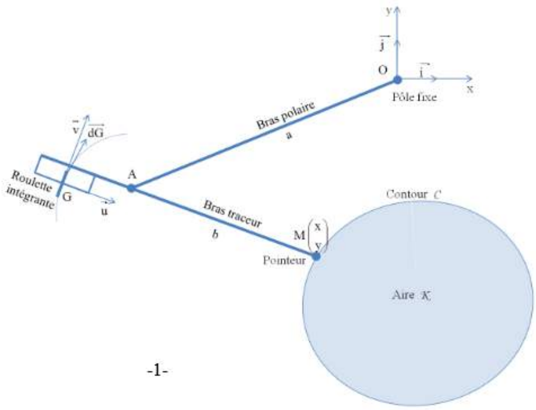
A stylus follows the curve, whose equation is $y = f(x)$, that delimits the surface, whose area one wants to measure. When the stylus holder moves with respect to the carriage in the direction (Ox), it causes an identical displacement of the carriage, thanks to a small disk. When the stylus holder moves with respect to the carriage in the (Oy) direction, it drives a disk, secured to a small toothed wheel, which rolls without slipping on the cone. The set of two toothed wheels is a speed reducer and drives the dial hand. For an elemental shift along the x-axis, the wheel rotates (with a coefficient close depending on the dimensions of the device) at an angle $f(x)dx$ and, for a movement along a segment [x_0, x], it rotates from a total angle: $\int_{x_0}^{x} f(t)dt$.

You can see how it works on the video linked here :

http://images.math.cnrs.fr/Un-planimetre-a-cone.html (Ghys & Leys 2009)

Another one was called "polar planimeter", invented by Johan Amsler in Swiss in 1854 (pic. 4.3 and 4.4):

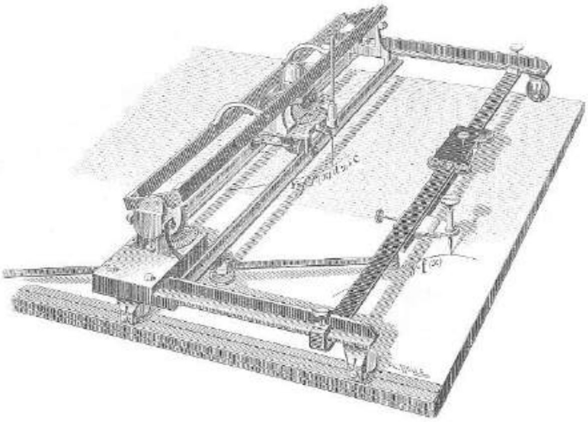
Picture 4.3: Polar Planimeter



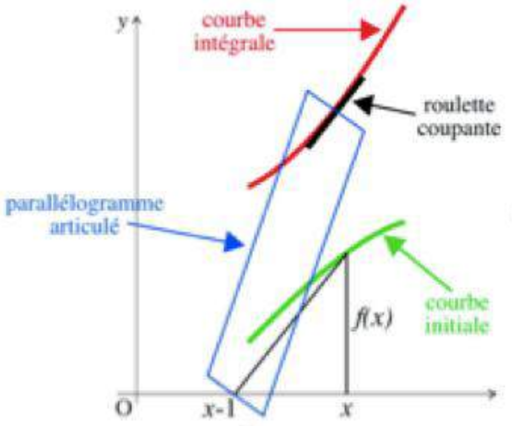
Picture 4.4: Polar Planimeter's Scheme

When the pointer moves, the wheel rotates. The number of turns is proportional to the component perpendicular to AM of the pointer movement. When the curve is traveled, the number of turns of the wheel is proportional to the length of the curve. Finally, by a calculation in which we use Green's theorem which makes it possible to transform a curvilinear integral into a double integral, we show that the number of turns of the wheel is proportional to the area of the domain. (Gatterdam, 1981).

There is also another one, made by Abdank-Abakanowicz in Poland in 1876, baptized "integrator" (pic. 4.5 and 4.6) (Abdank-Abakanowicz, 1886):



Picture 4.5: Integrator



Picture 4.6: Integrator's Scheme

This instrument draws an "integral" curve of the curve whose area it delimits. The tangent of the angle alpha is the ordinate $f(x)$. The other side of the triangle is orthogonal to the side of the articulated parallelogram. Parallelism "preserves" orthogonality, the inclination of the tangent to the integral curve is equal to the ordinate $f(x)$.

You can see one copy there made in Lego technics :

http://www.nico71.fr/integraph-graphic-planimeter/ (Nico71'Creations 2017)

4.2 Pedagogical application

In our classrooms, we intended to educate our students to the difficulty of determining non-geometric form areas. So we proposed them this exercise (doc. 4.1)[21].

The method of the grid works, but is not very practical to apply, and gives approximative results, hence the need for a more efficient technique that are the machines.

We have bought a polar planimeter to use with our students in our classes. It has been tested to determine the area of a square, and then of a circle. As these areas can be obtained by calculation, it was possible to check the result after manipulation. It was reliable to the nearest ten-thousandth.

Participants of the workshop in Oslo really enjoyed manipulating this material, and were also surprised at the accuracy of the result (pic. 4.7).

The use of a simple machine to get the result of an area precisely is surprising for the students. It is an original and practical method.

Thereafter, it is possible to deepen this exercise with the best students by proving the mathematical principles underlying these machines.

[21] This sequence was experimented in "ECT" class.

IV. How to calculate area of a non-rectilinear figure

1) by a finer and finer grid?

In Figure 1, area unit is a tile. Give a frame of area S delimited by the curve.

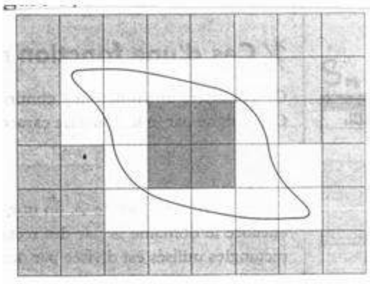
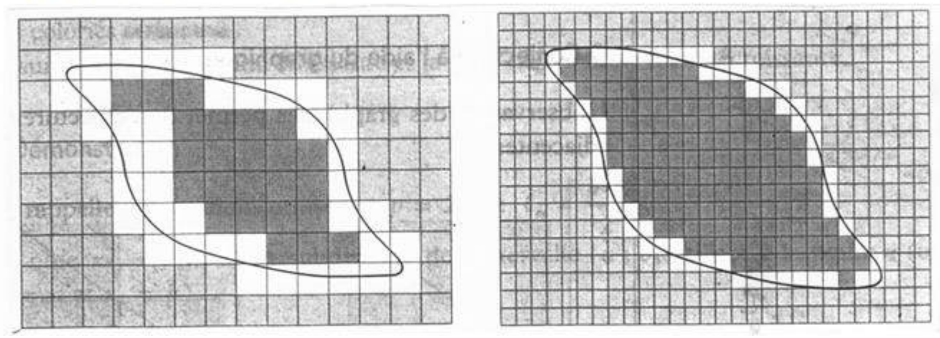


Figure 1

In Figures 2 and 3, we have squared the plan more and more finely:
In Figure 2, each square represents ¼ area unit, then 1/16 area unit for Figure 3.
Then give two more surroundings of S.



Document 4.1: Non Rectilinear Areas

Picture 4.7: Using Polar Planimeter at the workshop

5 Conclusion

Throughout the progression we have presented, the most important feature is the fact that the students were able to grasp what an area is, beyond the simple formulas applied hitherto without understanding them. Showing them historical methods, in their great diversity, is a way to achieve this. The originality and the beauty of the reasoning put into effect can more win their adhesion to mathematics in a more general way.

In addition, these activities can be very well integrated into French high school curricula: from elementary geometry taught in "Seconde", to function and sums concepts seen in "¨Premiere", until integrations learned in "Terminale". They also create links between the "collège" and high school, as well as being transitions between the different levels of high school.

As they were often carried out in small groups, sometimes in the computer room, the sessions were well experienced by the students, they motivated and interested them. They appreciated the manipulation of objects that allowed them to anchor their knowledge on concrete experience, which is rather rare in the current teaching. The diversity of media (historical texts, objects, videos, software) gave them opportunities to express broader skills than in traditional mathematical activities. We were able to see the main benefits of these sequences: a more concrete meaning given to the areas, a better understanding of the formulas and calculations, and an easier access to the integral calculation.

For the coming periods, we plan to continue this approach on another subject, that of tangents. Here again, it is a theme that closely combines geometry and analysis, that poses real difficulties of understanding in high school and that would benefit from the contribution of history.

We hope that the participants of the workshop and our readers will find this project inspirational, and will have as much pleasure in transmitting these notions as we had.

REFERENCES

Abdank-Abakanowicz, B. (1886). *Les Intégraphes*, Paris: Gauthier-Villars.

Équipe GRHEM (2018). *Expériences et théorie : la balance et le poids suspendu d'Archimède à Camus*. In É. Barbin, D. Bénard, G. Moussard. *Les mathématiques et le réel. Expériences, instruments, investigations* (ch. 1, pp. 15-32). Rennes: Presses Universitaires de Rennes.

Barbin, É. (1987). Heuristique et démonstration en mathématiques: la méthode des indivisibles au XVIIe siècle. *Fragments d'histoire des mathématiques tome II* (pp.125–159). Paris: APMEP.

Cavalieri, B. (1653). *Geometria indivisibilibus continuorum nova quadam ratione promota.* Bologna: Ex-typo-graphia de Duciis.

Chevalier A. (1995). Introduction aux aires et volumes dans une perspective historique. In F. Lalande, F. Jaboeuf, & Y. Nouazé (Eds.), *Actes de la première Université d'Été Européenne sur l'Histoire et Épistémologie dans l'Éducation Mathématique*. (pp. 143-155). Montpellier: IREM de Montpellier, Université Montpellier II.

Galileo, G. (1914). *Dialogues concerning two new sciences*, trans. Crew, H., De Salvio, A. New York : Macmillan.

Gatterdam, R. W. (1981). The planimeter as an Exemple of Green's Theorem. *The American Mathematical Monthly, 88*(9), 701-704.

Heath, T. L., (1912). *The method of Archimedes*. Cambridge: Cambridge University Press.

Heath, T. L. (1956). *The thirteen books of Euclid's Elements* (p.326). New York: Dover Publications.

Heath, T. L., (1897). *The work of Archimedes*. New York: Cosimo classics.

IREM de Basse-Normandie Cercle d'Histoire des sciences (1999). *Aux origines du calcul infinitésimal*. Paris: Ellipses.

Mercier, J-P. (2012). Le volume de la pyramide chez Euclide, Liu Hui, Cavalieri et Legendre. In É. Barbin (Éd.), *Les mathématiques éclairées par l'histoire* (pp. 91-112). Paris: Vuibert.

Merker C. (2004). La méthode des indivisibles racontée lors d'un stage. *Repères-IREM*, N°54, pp. 57-76.

Noel, W., & Netz, R. (2008). *Le codex d'Archimède – Les secrets du manuscrit le plus célèbre de la science*. Paris: JC Lattès.

Noel, W., & Netz, R. (2007). *Archimedes Codex : How a Medieval Prayer Book is Revealing the True Genius of Antiquity's Greatest Scientist.* Boston: Da Capo Press.

Nouet, M. (1992). Histoire des mathématiques en classe de terminale, *Repères-IREM*, N°9, p.15-33.

Roberval, G. P., (1693). *Lettre à Torricelli*, in *Divers Ouvrages mathématiques et de physique par MMrs les savants de l'Académie Royale des Sciences*) (pp. 283–302). Paris: Imprimerie Royale.

Tournès, D. (2003). *Du compas aux intégraphes: les instruments du calcul graphique*. Paris: Repères IREM.

Walker, E. (1932). *Quadrature of the Parabola, A study of the Traité des Indivisibles of Gilles Personne de Roberval*. New York: Bureau of Publications, Teachers College, Columbia University.

WEBOGRAPHY

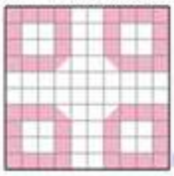
Nico71'Creations (2017, Feb 11). *Integraph: Graphic Planimeter.* http://www.nico71.fr/integraph-graphic-planimeter/

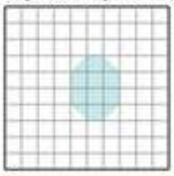
Ghys, E., & Leys, J. (2009, Nov 4). *Un planimètre à cône.* http://images.math.cnrs.fr/Un-planimetre-a-cone.html

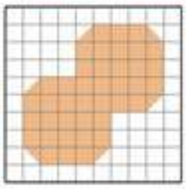
[Math Help]. (2015, Jan 14). *Sum of squares* (video file). Retrieved from: https://www.youtube.com/watch?v=kZTFrv3vRgg.

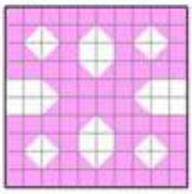
ACTIVITE 1 CALCULS D'AIRES SANS FORMULE
(EXERCISE 1) CALCULATIONS OF AREAS WITHOUT FORMULAS)

I. Calculez l'aire des zones colorées de chaque grille. Un bloc représente 1 unité carrée (u^2).
(Calculate area of colored areas of each grid. One block represents 1 square unit (u^2)).

1)


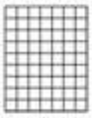
2)


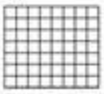
3)


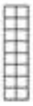
4)


II. Calcul de l'aire d'un rectangle (Calculating area of a rectangle)

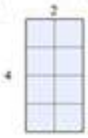
a) Calculez l'Aire. Chaque □ = 1 unité (u^2).
(Calculate area. Each □ = 1 unit (u^2))

1)


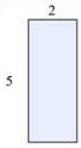
2)


3)


b) Calculez l'aire du rectangle en le décomposant par carreaux. Comme le premier exercice.
(Calculate area by breaking it down by tiles. Like the first exercise.)

1)


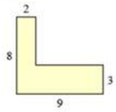
2)

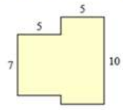

3)


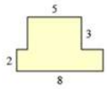
4)

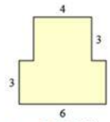

III. Calcul de l'aire d'une figure rectiligne
(Calculation of area of a rectilinear figure)

a) Calculez l'aire totale de chaque figure. Les mesures sont en millimètres (mm).
(Calculate total area of each figure. Measurements are in millimeters (mm))

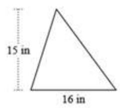
1)


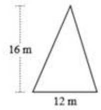
2)


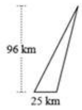
3)


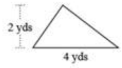
4)


b) Calculez l'aire de chaque triangle. Les triangles ne sont pas à l'échelle.
(Calculate area of each triangle. Triangles are not to scale.)

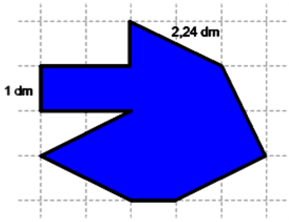
1)


2)


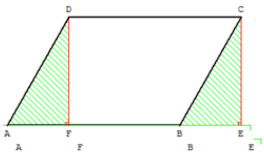
3)


4)


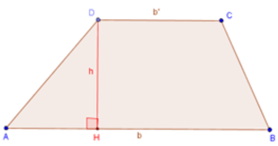
c) Calculez l'aire de cette figure en la décomposant
(Calculate area of this figure by breaking it down)



d) Calculez l'aire du parallélogramme ABCD
(Calculate area of parallelogram ABCD)



e) D'une manière analogue, calculez l'aire du trapèze ABCD
(In a similar way, calculate area of trapezoid ABCD)



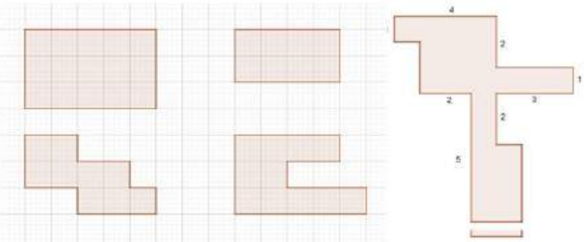
f) Aire sous une courbe rectiligne (Area under a straight curve) :

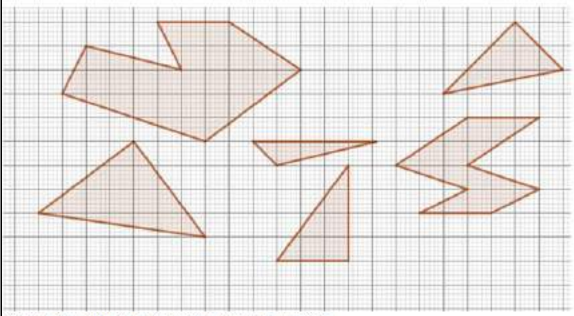
Calculez l'aire sous la courbe représentative de la fonction f définie par $f(x)=\begin{cases} x & si\ 0\le x\le 2 \\ 2 & si\ 2\le x\le 5 \\ \frac{x-1}{2} & si\ 5\le x\le 7 \\ -x+10 & si\ 7\le x\le 10 \end{cases}$

(Calculate area under representative curve of function f defined by $f(x)=\begin{cases} x & \text{if } 0\le x\le 2 \\ 2 & \text{if } 2\le x\le 5 \\ \frac{x-1}{2} & \text{if } 5\le x\le 7 \\ -x+10 & \text{if } 7\le x\le 10 \end{cases}$)

Activité 2 (Exercise 2): Aires rectilignes (Rectilinear areas)

Déterminer les aires de chacune de ces surfaces, en unité de quadrillage
(Determine areas of each of these surfaces in a grid unit.)

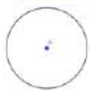




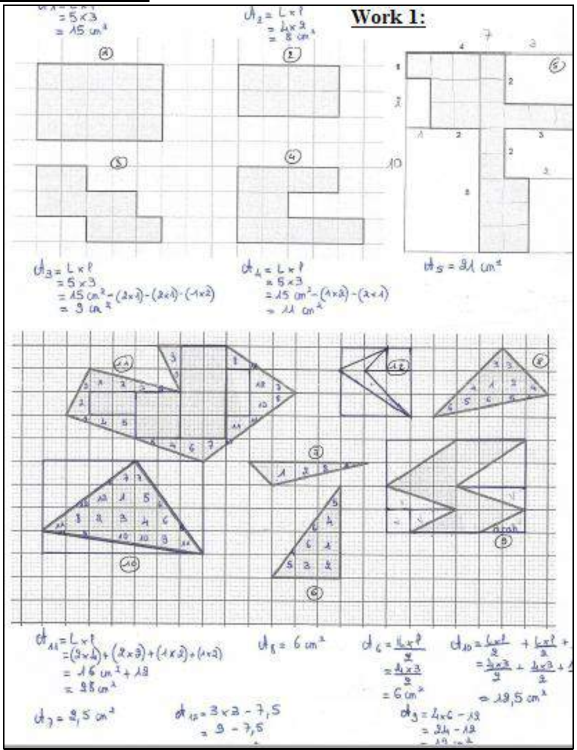
Exercise 3: To compare areas without computing

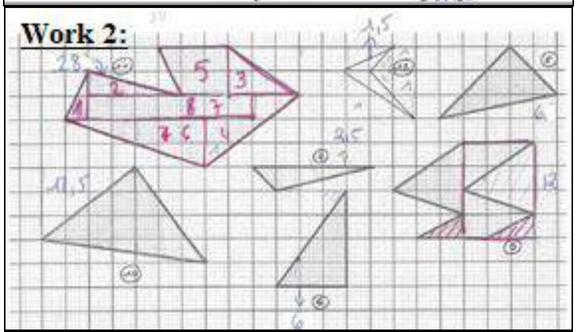
1. Sort these surfaces in ascending order of their areas

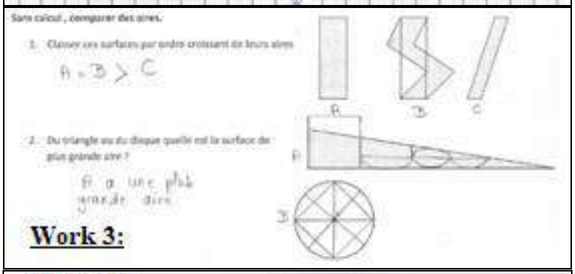
2. Of this triangle and disk what is the surface of bigger area?

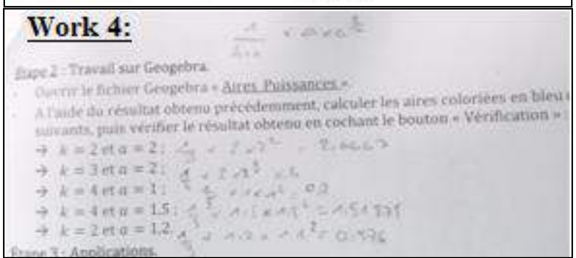


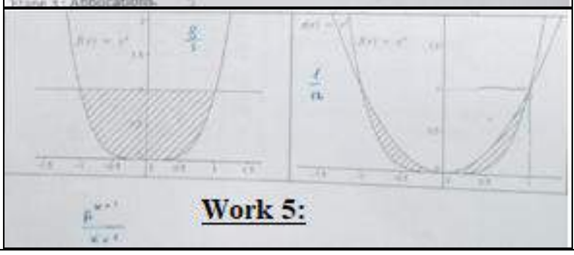
Students' works :

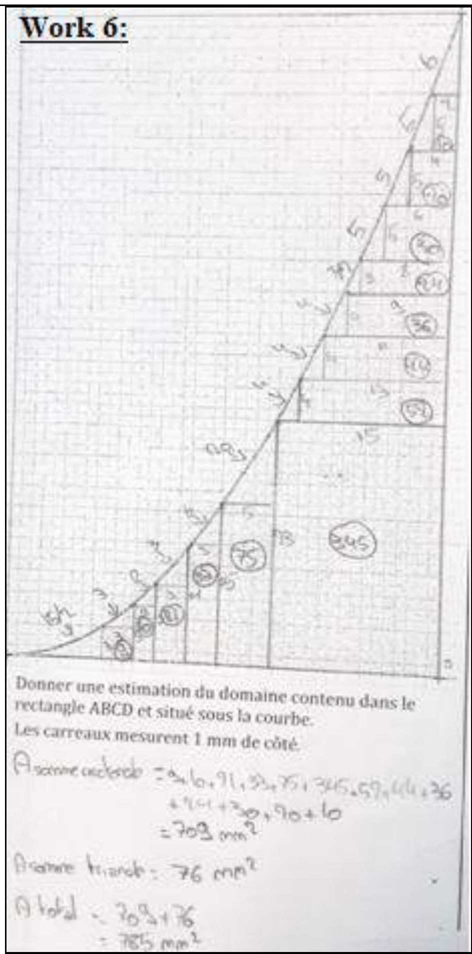


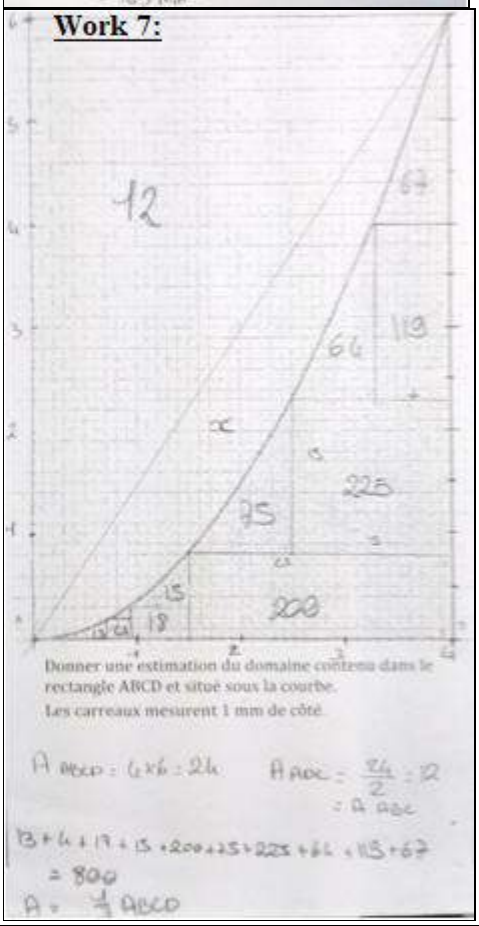












USING FRENCH WEBSITES TO FIND USEFUL ONLINE MATERIAL TO INTEGRATE THE HISTORY AND EPISTEMOLOGY OF MATHEMATICS INTO OUR TEACHING

Hombeline LANGUEREAU[1,3], Anne MICHEL-PAJUS[2,4]
[1]IREM-Université de Franche-Comté, route de Gray, 25030 Besançon Cedex, France
[2]IREM-Université Paris Diderot, Case 7018 Batiment Sophie Germain 75205, Paris Cedex 13, France
[3]hombeline.languereau@univ-fcomte.fr, [4]annie.pajus@orange.fr

ABSTRACT

This paper (based on a workshop which will not be described here) mainly aims to present the bibliographic database Publimath which contains in particular the works of IREM (Research Institutes in Mathematical Education) and APMEP (Association of French Professors of Mathematics), as well as numerous resources in different languages that the editorial team finds relevant: for instance, proceedings of earlier conferences. More than 8000 items associated with the history of mathematics, both original historical sources and accounts of classroom experiments, are included in the database.
The first part of this paper is an answer to the question: who manages Publimath?
The second part provides a first orientation to the database itself.

1 Two important actors in French mathematical education and their websites

The **IREM** Network consists of 28 regional IREMs (Institutes for Research on Mathematics Education) in France. It has been funded by the Ministry of Education and the universities since 1969. National committees (called CII = Commission Inter Irem) are organized either around a level of education or a theme. Two of these CII, *Repères IREM* and *Publimath*, are intended to disseminate the work of IREM. The CII *History and Epistemology of Mathematics* coordinates the regional groups that work with history, philosophy and/or epistemology of mathematics.

The APMEP (Association des Professeurs de Mathématiques de l'Enseignement Public) was created in 1910. It is a federation of 26 regional associations of mathematics teachers, with regional activities and national committees, funded by membership fees and sale of productions.

Both organizations act as an interface between research (in history and/or didactics) and teaching/training, bringing together teachers of mathematics from kindergarten to university. They produce paper and digital working documents, published by the IREMs or APMEP, as well as conference proceedings, journal articles, and books for private publishers.

Both also have websites which focus in part on the History of Mathematics:

- The APMEP website is: https://www.apmep.fr/. The page dedicated to the group working about History of Mathematics is https://www.apmep.fr/-Histoire-des-maths-

- The national IREM website is “Le Portail des IREM”: http://www.univ-irem.fr/.

- The page dedicated to the CII “Histoire et Epistémologie des mathématiques” is http://www.univ-irem.fr/spip.php?rubrique15

2 Publimath: a database and a search engine

Publimath is a joint operation of IREM and APMEP. It is a database that is intended to identify not only the work resulting from research carried out in the IREM network or APMEP, but also any publication useful to the mathematics teacher, as this is the easiest way to offer access to all their in-house materials productions alongside others related to teaching, learning, and scientific culture. These include:

- the working documents posted on the IREM websites;
- books and journals of the IREM network and APMEP ;
- proceedings of conferences such as HPM (History and Pedagogy of Mathematics and ESU (European Summer School) ;
- papers published by learned societies of mathematics ;
- books produced by private publishers;
- websites, videos and other educational materials.

Publimath is freely accessible to anyone, as is the associated database “Bibliothèque numérique des IREM” (IREM Digital Library). These resources are useful to the entire educational community of mathematics teachers, from school to university, teacher trainers, and students training to become math teachers., They are also of interest to researchers in didactics in history, and in the history of mathematics education, as the majority of these resources have been used in France for a long time by teachers and thus offer valuable information on French curricula and didactical practice.

2.1 Les *fiches*

Each item in the Publimath database is described in a "*fiche*". There are 28,759 *fiches* as of March 19, 2018. The following example is: http://publimath.univ-irem.fr/biblio/ACF11023.htm

Auteur(s): Smestad Bjorn; Nikolantonakis Konstantinos

Titre : History and Epistemology in Mathematics Education: Proceedings of the Sixth European Summer University (ESU 6). Historical methods for multiplication. p p. 235-244. (Méthodes historiques de multiplication.)

Editeur : Verlag Holzhausen GmbH Vienne, 2011, Autriche
Format : p. 235-244 Bibliogr. p. 244-244
ISBN : 3-85493-208-1 *EAN :* 9783854932086

Type : chapitre d'un ouvrage *Langue :* Anglais *Support :* papier

Public visé : chercheur, enseignant, formateur

Classification : F40

Résumé : Cet article résume le contenu de l'atelier, dans lequel les auteurs ont étudié la multiplication "grecque", donnée par Eutocios d'Ascalon dans son commentaire sur "La mesure du Cercle" dont ils étudient une partie. Ils partent de l'hypothèse que cet algorithme historique de multiplication est d'accès facile car les stratégies mentales informelles mises en oeuvre intuitivement par les enfants font appel aux mêmes processus de calcul. L'idée importante est que la numération de position est fondamentale et que les élèves agissent avec des quantités et non avec des symboles isolés comme cela arrive avec l'algorithme classique. Cela aide les élèves à contrôler leur pensée à chaque étape du calcul. Ils ont également discuté de la méthode russe et de la méthode "par croix" (fondamentalement la même que "la preuve par neuf") pour contrôler l'exécution des opérations.

Notes :

Chapitre des Actes de la sixième université d'été (ESU 6). @

Mots clés :

"calcul algorithmique"
"histoire des mathématiques"
"multiplication de nombres entiers"
"preuve par 9"
"réflexion sur l'enseignement des mathématiques"
"technique de multiplication"

Some comments :

The button gives access to an abstract in English:

This paper summarizes the contents of our workshop. In this workshop, we presented and discussed the "Greek" multiplication, given by Eutokios of Ascalon in his commentary on The Measurement of a Circle.
We discussed part of the text from the treatise of Eutokios. Our basic thesis is that we think that this historical method for multiplication is part of the algorithms friendly to the user (based on the ideas that the children use in their informal mental strategies). The important idea is that the place value of numbers is maintained and the students act with quantities and not with isolated symbols as it happens with the classic algorithm. This helps students to control their thought at every stage of calculation. We also discussed the Russian method and the method by the cross (basically the same as "Casting out nines") to control the execution of the operations.

The phrase "langue: anglais" indicates that the paper is written in English. If you want only items written in a specific language, go to the page **recherche avancée,** click on **Langue**, then choose "anglais", or "espagnol", etc. The database contains several hundreds of papers in English.

The symbol @ means that you can find the pdf online, either with direct access or through a link to another site. There are more than 10,000 documents of this type, including those in the *bibliothèque numérique* for which you can get the paper with 1 click on the sentence "Bibliothèque numérique des IREM et de l'APMEP". If you want to see only the items available on line, go to the page **Recherche avancée,** then click on **Ressource(s) en ligne@.** If you use the simple search engine, papers available on line have the symbol @ after the title.

2.2 Two modes for searching

2.2.1 Simple search

Let's try it: open the homepage here http://publimath.univ-irem.fr/. This gives access to "recherche simple" (simple search).

In the first box, you can put in all the key words that you like, in any language you like.

In the second box, you have a choice between *les fiches*, *la liste des mots-clefs* (keywords), *la liste des auteurs* (authors), *les notices du glossaire* (glossary). A glossary notice is intended to provide quick information (e.g., definition of an object or a concept, biographical details of mathematicians) and references for further research.

To start your search, click ✔**Valider.**

These two lists (*liste des mots-clefs*, *notices du glossaire)* are very useful for people who don't know French mathematical or teaching vocabulary well. If you put in an English word, you'll get the "fiches" in which the word appears in the abstract in English, even if the paper is written in another language.

As in all the basic search engines, you'll get all the items in which any one of your search terms appears (full text research). The tag "bibliothèque numérique" allows for searching **only** the items in the "bibliothèque numérique."

2.2.2 Advanced search

Now, if you know precisely what you are looking for, you can use the "recherche avancée et dans les revues" (advanced and journals search) by clicking on that tag; it opens a second page: http://publimath.univ-irem.fr/avancee.php

If you look for a specific kind of document, as a website, a video , click on the button "Type" and choose in the drop-down menu : site internet website) or film (video)

"Public visé" means "Target audience".

If you are wondering what "CDI Litteramath" is, it is a collection of books available in every french school library (CDI) for students in the first 4 years of secondary schools.

A useful tip: clicking on the logo Publimath on any page will take you back to the homepage: http://publimath.univ-irem.fr/

2.3 How to contribute to the improvement of the Publimath database?

First, a little experiment: on the first page, try this search: Put your last name in the line:

Chercher $\boxed{\text{My last name}}$ in $\boxed{\text{liste des auteurs}}$ ✔**Valider**

Do you find your name? If yes, click it!

Do you find all your papers? If yes, do you like the "fiches" about them?

If not, you can help us to improve Publimath!

Click on:

Aidez-nous à améliorer cette fiche (help us to improve this fiche). A small video will show you how to do it.

Ideally, the author or publisher of a document fills out the review form available on the homepage by clicking on **Proposer une fiche**

But you can also provide information on any document that you think is worth it. If some fields are not completed, the team members will do it.

Anyone can contribute to enriching the database by translating an abstract, proposing an improvement for the summary, proposing a new summary, or correcting existing data. Non-French associations and journals are welcome! We especially need your help in increasing the references for papers written in English.

How to contact us? By clicking on the button **Contactez-nous** on the home page.

2.4 More about Publimath

Usage statistics are available at**:** **http://publimath.univ-irem.fr/usage/**

This database, born in 1996, is maintained by a team of about ten people who, in addition to writing and improving the files, each have a specific task, such as
- ensuring the architecture of the database,- following the census of publications of interest to our public (in French)
- maintaining connections with
  • the CII History and Epistemology of Mathematics;
  • APMEP publications;
  • international publications.
- performing English-French translations.
- populating the glossary.

For the last 5 years, the two big new tasks have been
-the creation of a classification based on the ZDM ontology which is in the testing phase; in the “fiches,” you find this in the field “Classification”
- the digitalization of IREM and APMEP’s documents.

Hombeline Languereau is currently co-responsible for CII Publimath (with Michèle Bechler, who has served in this role since the creation of the CII in 1996). Anne Michel-Pajus ensures in particular the connections with the CII History of Mathematics and international publications.

THE CONTRIBUTION OF THE CHINESE ABACUS TO THE DEVELOPMENT OF THE NUMBER SENSE

Apostolos POPOTIS[1], Konstantinos NIKOLANTONAKIS[2]
University of Western Macedonia, 3rd km of National Road Florina-Niki, Florina, Greece
[1]apopotis@gmail.com, [2]knikolantonakis@uowm.gr

ABSTRACT

One of the multiple uses of the abacus in teaching mathematics is to illustrate the true nature of carrying and borrowing in addition and subtraction respectively. The understanding of carried and borrowed number requires deep knowledge of the place value concept in the decimal number system. Generally, students do not seem to understand these two concepts. In our workshop we tried through several activities to convince the participants that the Chinese abacus can help teachers to teach and students to understand and learn the aforementioned notions. The fact that the Chinese abacus enables someone to write up to fifteen in each place value and then to make exchanges between place values with his hand reinforces the understanding of carried and borrowed number. The visual and tactile perception, that the Chinese abacus provides, can support learning.

1 Introduction

Undoubtedly, one of the most important issues in the mathematics curriculum of elementary school is the understanding of place value concept in the decimal number system. However, it seems somewhat incongruous that such a system, which incorporates in its heart a small number of simple rules, would be so difficult for teachers to teach and for students to learn (Jones & Thornton, 1993; Ross, 1990; as cited in Price, 2002). In recent years, after a long period of formal and ineffective learning, mathematical education has focused more on the learning of numbers and operations (Tzekaki, 2007). The use of appropriate educational material contributes to better approach of mathematical concepts, since it is well integrated into the mathematical activity of students (Tzekaki, 2016).

One of the earliest calculating instruments, which has entered the classroom as a manipulative, is the abacus. It is an ancient tool through which one can process the place value concept in the decimal number system. According to Spitzer (1942), there are five important characteristics of arithmetic that are admirably demonstrated through this device. Firstly, the beads can represent various concrete objects. Then, the abacus can nicely illustrate that the value of each digit depends on its position to the number. Furthermore, it demonstrates the idea of a place-holder or the function of zero. Moreover, it illustrates the idea of collection. Last but not least, it reveals the true nature of carrying and borrowing.

As far as the Chinese abacus is concerned, Zhou & Peverly (2005) state that it provides a semi-concrete representation of number, children can manipulate it, and they can easily create a mental image of it. “With experience, mental calculation is enhanced by learning how to use an abacus because children learn to rely on internalized mental abacus which helps them create mental representations of numbers” (Zhou & Peverly, 2005, pp. 266-267). Stigler (1984) concluded that several pieces of evidence in his research indicate that the mental abacus incorporates important features of the physical abacus.

Bartolini Bussi states that a way to introduce the history of mathematics into classroom is through the exploration of copies of ancient instruments and artifacts (Nagaoka et al., 2000). This exploration is not limited to the visual contact only. It is extended to the tactile perception, too. Students do not just observe the instruments, they also use them. This tactile experience has the potential to motivate students and it is an important part of the cognitive foundations of mathematical activity (Nagaoka et al., 2000).

2 Background of the workshop

The workshop was based on a research study that was conducted during the school year 2016-2017 in a primary school of central Macedonia in northern Greece. The purpose of the research study was to examine if 3rd Grade students of primary school are familiar with the structure of the decimal number system and to ascertain whether the Chinese abacus, which as an artifact is in the core of the theory of instrumental genesis and semiotic mediation, can contribute to the understanding of the structure of the base 10 number system. The participants were 7 students (4 boys and 3 girls) consisting the 3rd Grade of the aforementioned primary school.

Data collection was attained through a questionnaire along with personal interviews before and after the instructional intervention, with the filling out of 35 worksheets, most of them with the aid of the Chinese abacus, as well as by observing the students' activities during the intervention. The interviews took place just before and just after the intervention, which was completed within 35 class periods.

In the questionnaire before the intervention, students were asked to read and write three-digit numbers, to perform additive analysis of numbers, to recognize place value in three-digit numbers, to make exchanges between classes of numbers, to compare and to set in order three-digit numbers, to execute addition and subtraction algorithms, as well as to explain the notion of carried and borrowed number.

During the instructional intervention, the students dealt with activities that firstly introduced them to the notion of number system in general, and then to the notion of the decimal number system. Moreover, everyone constructed his own Chinese abacus with materials provided by the teacher-researcher and they used it in several activities. The intervention's exercises and activities covered the whole range of the above aspects of the place value concept. The abacus, through its use, revealed the true meaning of the place value concept. The conceptualization of place value is the basis for number sense development.

Through a questionnaire after the intervention, students' knowledge of the above aspects was checked again.

The findings of the questionnaire before the intervention indicate that students do not have profound understanding of the notion of place value in whole numbers and in operations with them. During the intervention, the Chinese abacus was converted from an artifact into an instrument and mediated mathematical content to the students. Whereas before the intervention there were major difficulties especially in the execution of the addition and subtraction algorithm, and in the explanation of carried and borrowed number, after the intervention these difficulties were limited.

Similar research studies concerning the development of the place value concept in 6th Grade students with the aid of the Chinese abacus were conducted by Poisard (2006), and Tsiapou & Nikolantonakis (2013, 2016).

In the context of the ESU-8, we decided to organize a workshop based on a sequence of activities to introduce the Chinese abacus as a resource for the teaching and learning of number sense and place value system in primary school. According to Poisard (2017) it benefits both students' learning and teachers' classroom practices.

So, we focused on how teacher can foster the understanding of the paper and pencil algorithms for addition and subtraction work by means of the prominence of place value representation of the addends in addition and of the minuend in subtraction, and on the procedures for addition (carried number) and subtraction (borrowed number) on the abacus.

Most of the worksheets that were used in the workshop were originally designed for the aforementioned research study and our young participants filled most of them out with the aid of the Chinese abacus. In addition, the workshop's material was enriched with activities that emerged from the students' answers to the questionnaires' exercises before and after the intervention. The 11th and the 19th question (worksheet) below appertain to this category. It should also be noted that the 2nd question (worksheet) below has been modified so as to better correspond to the age of the workshop participants. The original one was asking the students to discover how the Chinese abacus works through its electronic version (http://cii.sesamath.net/lille/exos_boulier/exo1.html) as the structure and the function of the actual Chinese abacus differs a lot from the Slavonic and the spike abacus that they had already been using. The modified one asks the participants to discover the abacus' mode of use through the study of four figures depicting the successive stages of the Chinese abacus' evolution and a copy of it that was constructed by the workshop leaders and was distributed to everyone. To sum up, 21 worksheets and a copy of the Chinese abacus were available to everybody. Most of the participants were in-service teachers in different grade levels.

3 The way that the Chinese abacus works

The Chinese abacus, which is also called counting frame, known in China as "Suan Pan" (literally "counting disc"), consists of a rectangular frame. It has a height of about 20 cm and its length varies according to the magnitude of the numbers that are represented on it. It usually has more than seven rods. There are two beads in each of the top rods and five beads in each of the lower rods. The upper beads stand out from the bottom ones with a separating beam. The beads are usually round and they are made of hard wood (Korros, 2012).

The beads are activated when they are moved up from below the beam or down from above the beam. When they are moved towards the beam they are counted, while when they are moved away from the beam they are not counted.

Now, let's see how we "read" the Chinese abacus. Firstly, we will talk about the beads. There are two kinds of beads on the abacus, these of the upper and those of the lower deck. Each of the lower deck beads is worth 1. Sometimes they are called earth or water beads. The upper deck beads sometimes are called heaven beads and each of them has a value equal to 5. The rightmost rod of the abacus represents the units, the next one on the left the tens and so on. However, we may select an inner rod for the units' place, whence the rod directly to its left becomes the tens' rod and those to its right the tenths, the hundredths, etc. On the Chinese abacus there are always two heaven and five earth beads in each rod.

At the end of an action-operation on the abacus, we will never see five earth beads activated in the same rod. If this happens, they immediately return to their original position and a heaven bead is activated on the upper deck. The same thing happens if two heaven beads are activated in the same rod. They immediately return to their original position and an earth bead on the rod to its immediate left is activated.

4 Workshop activities

The workshop has the same structure as the instructional intervention of the research study. The workshop activities are divided into four parts:

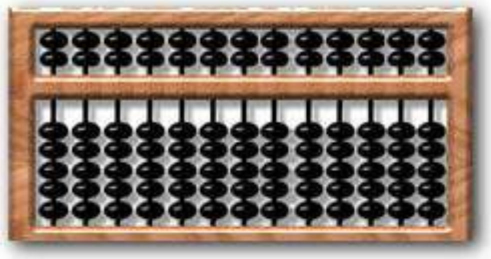
- the discovery of the structure and the mode of operation of the Chinese abacus
- the way that numbers are formed on the abacus
- the way that an addition (with or without carried number) is executed on the abacus
- the way that a subtraction (with or without borrowed number) is executed on the abacus.

The participants had the option to work either individually or in couples. Each activity was followed by plenary discussion. In general, the following pattern was applied: action-opinion formulation-validation-formalization.

Next we present the worksheets that were used in each part of the workshop with the correct answers to their questions.

4.1 First part

The first part of the workshop concerned the discovery of the tool. Initially, the participants were asked to guess what they could do with the Chinese abacus.



1. *Can you guess what you could do with the Chinese abacus?*

As most of the participants were teachers at various grade levels, it was easy for them to answer correctly that the Chinese abacus can support number representation, the execution of addition, subtraction, multiplication, division, and the finding of square and cube roots.

After studying some images representing the successive stages of the historical evolution of the Chinese abacus as a computational device, they were asked to guess how it works.

2. *After studying the images depicting the stages of the evolution of the Chinese abacus and a copy of it, which was given to you, can you guess how a number is formed on the Chinese abacus?*

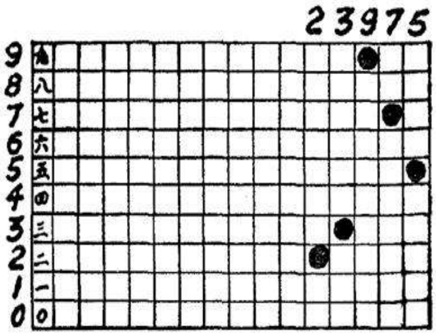


Fig. 1 Chu Pan or Bead Tray – The prototype of the Chinese abacus (600-300 BC)

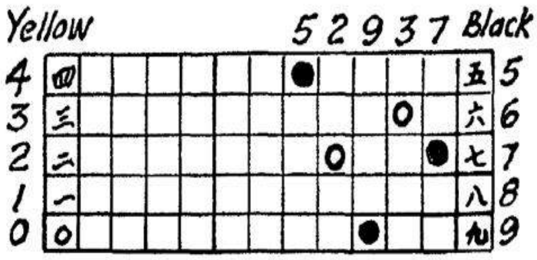


Fig. 2 Half-size tray with beads in two colors: yellow and black

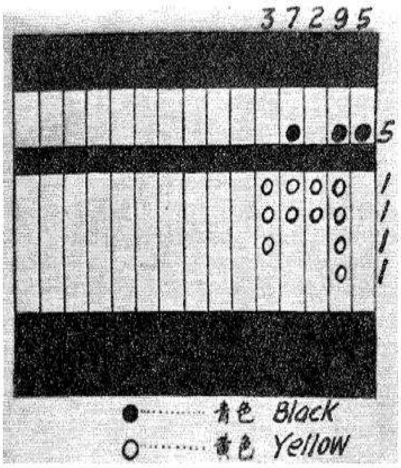


Fig. 3 The abacus with unattached beads (in use till the early 7th century)

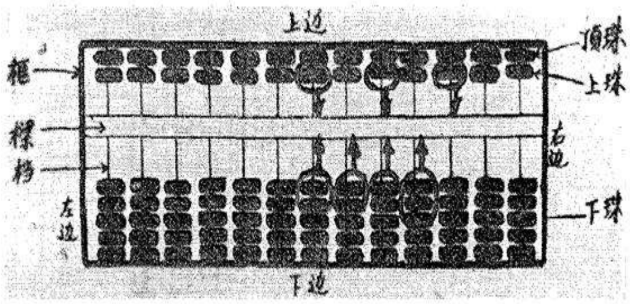


Fig. 4 The abacus having been in use for nearly 13 centuries

Chu Pan or Bead Tray (figure 1) was the initial prototype of the Chinese abacus. Its rectangular chess-board-like tray was divided into squares which had ten rows horizontally and as many columns as necessary. Each horizontal position denoted numerals 0 to 9 from bottom up. Each column represented a place value increasing from right to left. Numbers for operations were presented on the tray by placing beads in proper squares (Li, 1958).

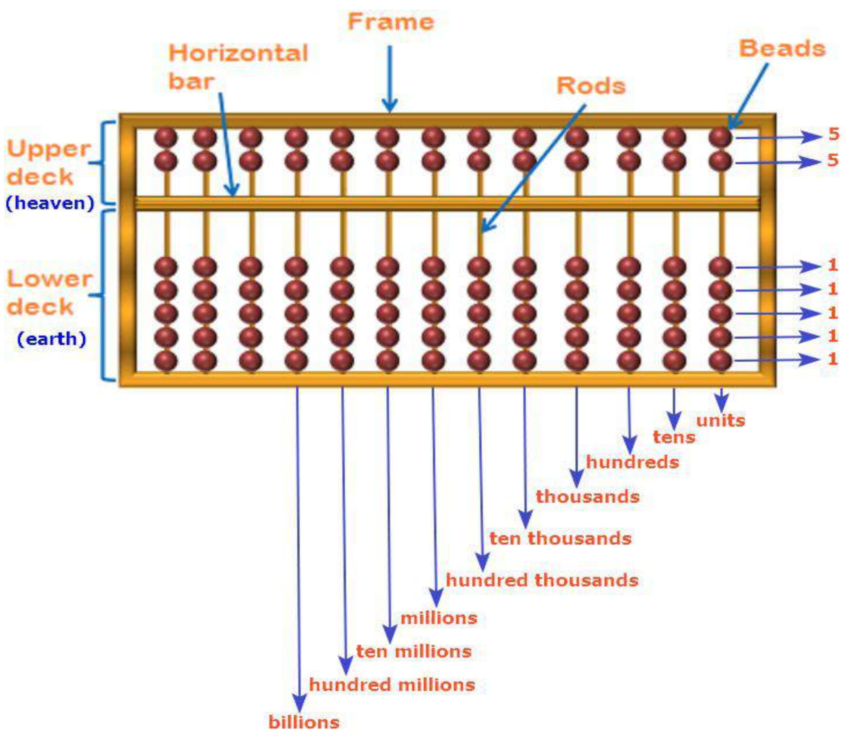
The next improvement of the Chinese abacus was the tray with beads in two different colors (figure 2). The yellow beads were denoting numerals 0 to 4, from bottom up and the black ones numerals 5 to 9, from top down. Place value was increasing from right to left, too. This improvement enabled the bead tray to be reduced in size as it was divided into five only horizontal rows of squares and as many vertical columns as before (Li, 1958).

Before the 2nd century AD Chu Pan had almost taken the form of the current Chinese abacus but with unattached beads (figure 3). The beads had not yet being attached to bamboo bar rails. Chu Pan consisted of a wooden board divided into rectangular column spaces. An upper horizontal partition separated the board in two unequal parts. On the top and the bottom of the board there were recesses for storing beads. In each column only five beads could be placed. One bead could be placed in the upper part and four beads in the lower part. The bead above the partition represented 5, while each of the four beads below the partition represented 1. Place value was increasing from right to left as in the previous figure. Though the beads that were used above and below the partition were differently colored, it was not absolutely necessary as the worth of each bead was determined by its position and not by its color. This model had existed until the early 7th century AD (Li, 1958).

The 4th figure represents the Chinese abacus that has been constantly in use for nearly 1.300 years without further improvements since the 10th century until very recently. At this stage the frame was perfected by providing a hole through the middle of each bead, by setting a round bamboo rod at each place value, and by placing an additional bead both above and below the partition at each rod. Thus the beads became attached, enabling the elimination of the top and bottom recesses for the storage of the beads. As in the earlier Chinese abacus with unattached beads, each bead above the partition is worth 5, and each below the partition denotes 1. When 5 is reached, it is exchanged with one upper bead replacing five lower beads. When 10 is reached, it is exchanged with one lower bead on the left place value replacing two upper or one upper and five lower beads on the right place value. From the point of view of simple number representation, one of the two upper beads and one of the five lower beads seem redundant. In fact both are necessary in the advanced techniques of multiplication and division operations (Li, 1958).

The workshop participants explained that the 3rd image was very helpful in discovering how the Chinese abacus works as it clearly shows the worth of each bead on both parts of the abacus. It should also be noted that some people already knew the mode of its operation, as they had already used it in the past. Then, the following picture that describes the parts of the Chinese abacus was given to them. Thus the organizers described in detail its structure.

The parts of the Chinese abacus

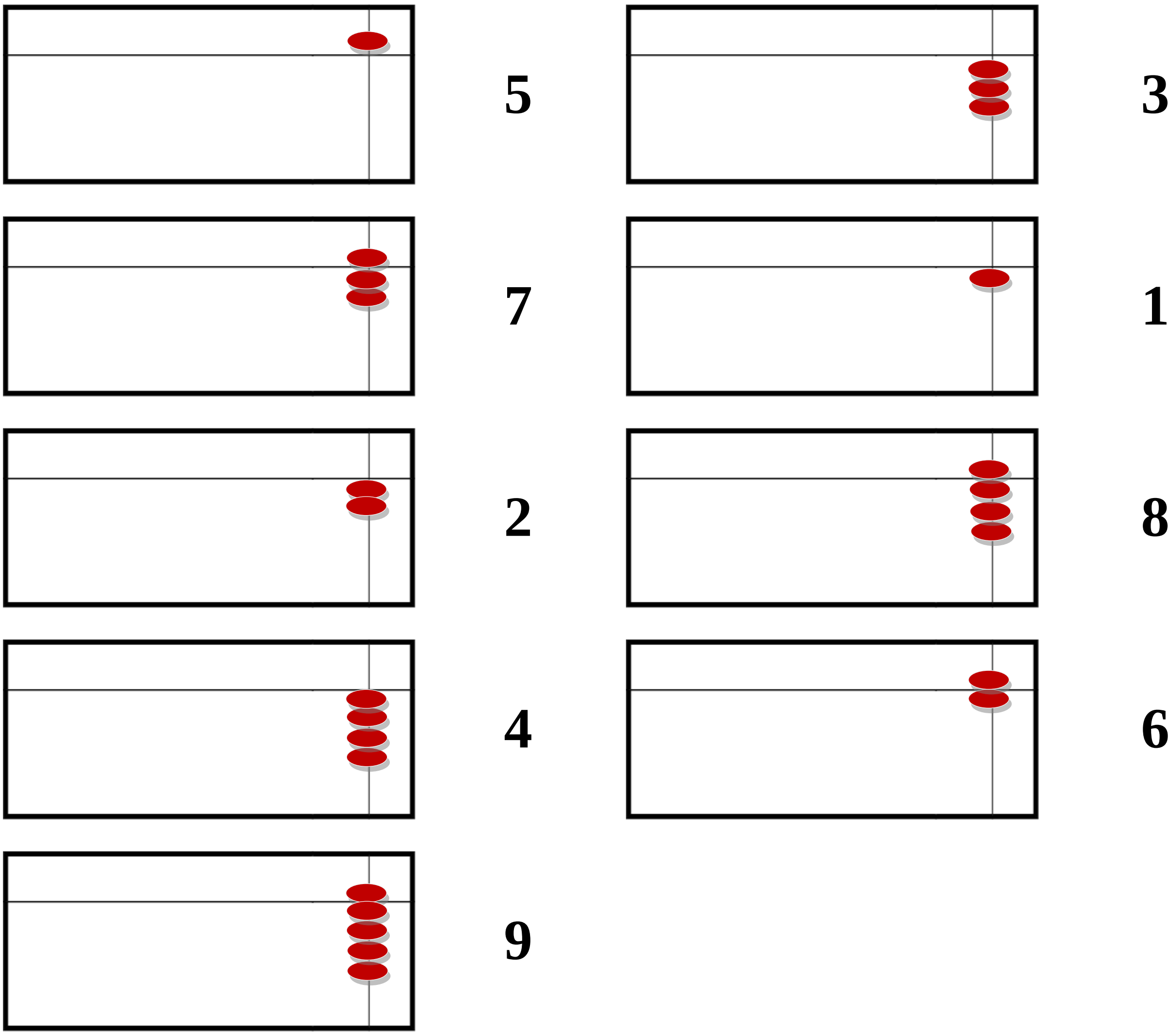


Afterwards, a video entitled "Chinese Zhusuan, Knowledge and Practices of Arithmetic Calculation through the Abacus" was presented to the participants. This video was submitted to UNESCO in the year 2012, so that the Chinese Zhusuan be inscribed on the representative list of the intangible cultural heritage of humanity. It describes the form and content of the Chinese abacus, its cultural value, its challenges and its protection measures. Zhusuan means bead computation, that is, a time-honored traditional method of performing mathematical calculations with an abacus, the Chinese abacus. Finally, in the year 2013 Chinese Zhusuan was inscribed in the list of the intangible cultural heritage of humanity.

4.2 Second part

In the second part of the workshop activities, the participants were asked to form single digit numbers both in their individual abacus and in the worksheet's sketched abaci.

3. *Can you draw beads on each abacus to represent the number written next to it?*

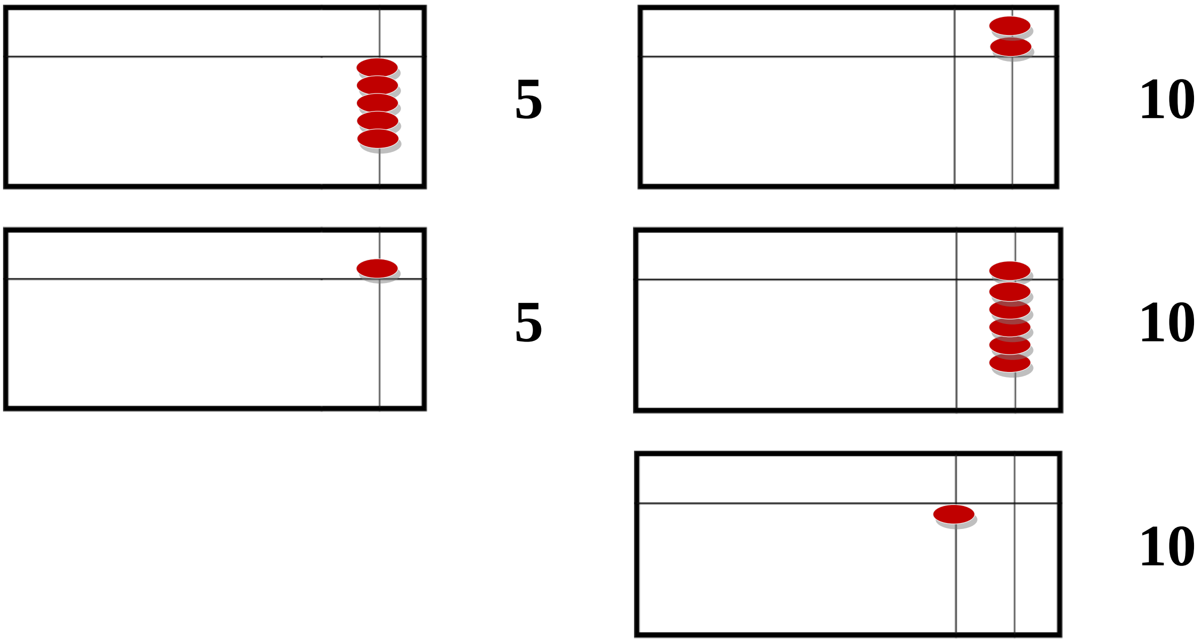


Firstly, they formed the given numbers in their individual abacus and then they drew beads in the worksheet's sketched abaci to represent the same numbers. The workshop leaders were walking around the room to check the participants' way of working. Most of them formed the given numbers correctly.

Then, they were asked to discover how the numbers 5 and 10 are represented on the Chinese abacus. They were suggested to study again the figure that shows the parts of the

Chinese abacus and the value of each bead, as it implies the different ways that the two numbers can be represented on it.

4. *Can you draw beads on each group of abaci to show the different ways the same number can be represented?*



The participants identified all the possible ways that are presented above. Afterwards, they were suggested to think which one could be the most appropriate for the representation of each number on the Chinese abacus.

5. *Which do you think is the most appropriate way to represent the numbers 5 and 10 on the Chinese abacus? Can you justify your answer?*

The representation of the number 5 with an activated bead on the upper deck of the rightmost column of the abacus and the representation of the number 10 with an activated bead at the lower deck of the left column are the most appropriate answers. Thus there are available beads, which may then be needed in the same column of the abacus in case of adding, subtracting, multiplying or dividing with another number that, in turn, must be represented on the abacus.

Time was also devoted to the identification of the role of zero in a number. The following question was given.

6. *What do you think is the role of zero in a number?*

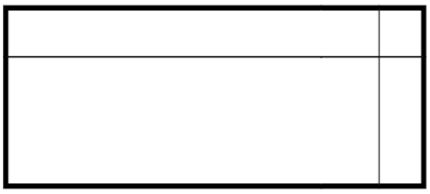
Its twofold role

- as an indication that there are no units, tens, hundreds, etc. in the number and
- as a place holder,

was highlighted by the participants.

Then, they were asked to represent zero both on their individual and the following sketched Chinese abacus.

7. *How would you represent zero in the following Chinese abacus?*

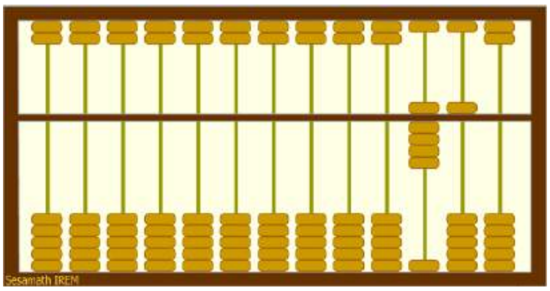


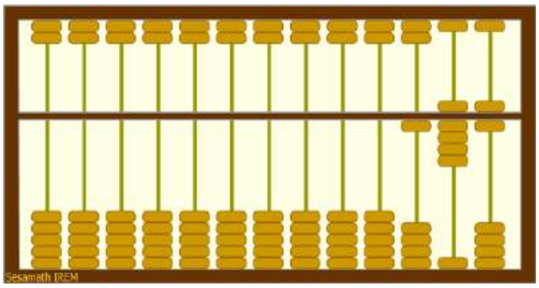
0

They easily suggested that zero be represented by activating no beads on the abacus. Most participants individually and successfully interpreted numbers that were represented in sketched abaci.

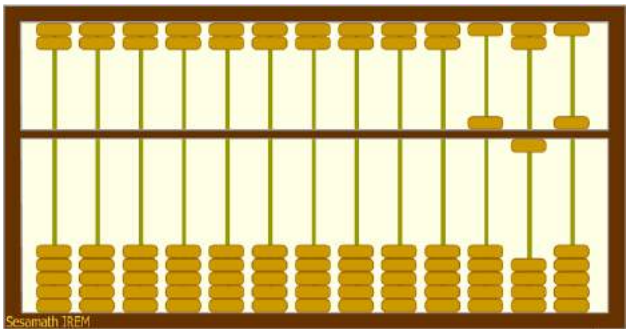
8. *Can you write above each abacus the number that it represents?*

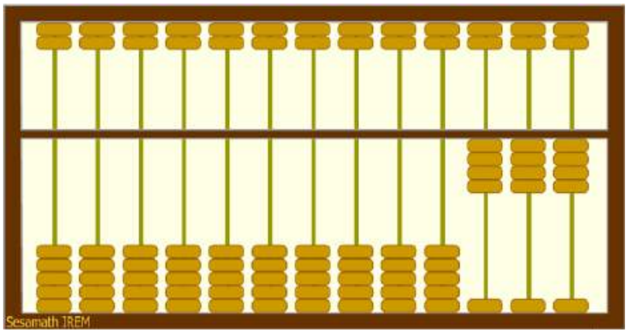
950 **196**



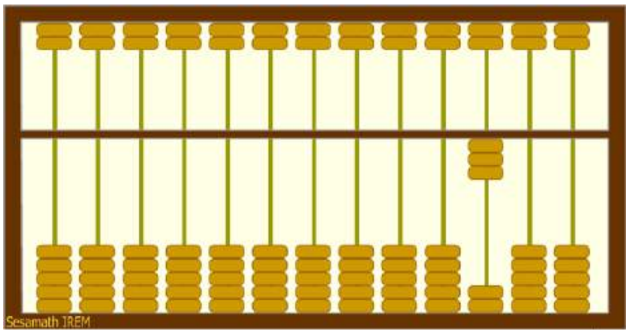


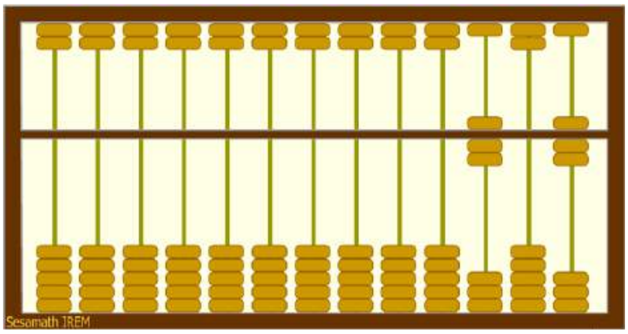
515 **444**



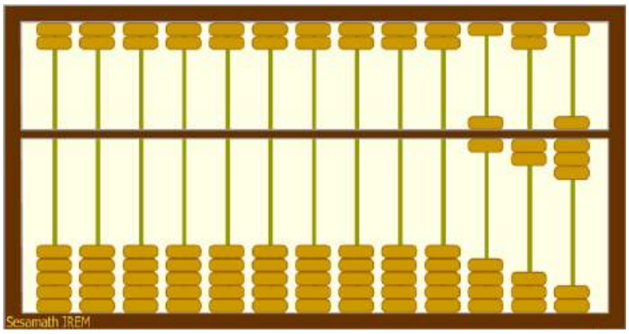


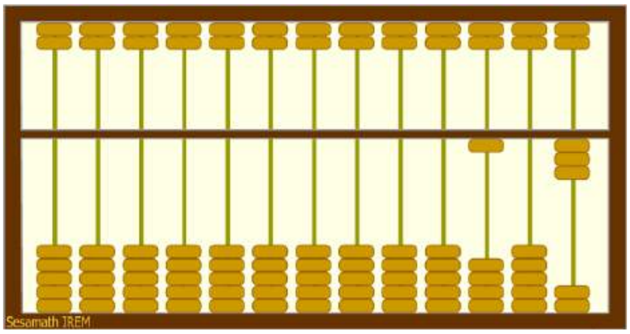
300 **707**





628 **103**

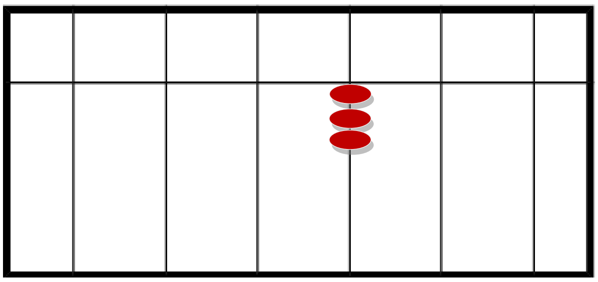




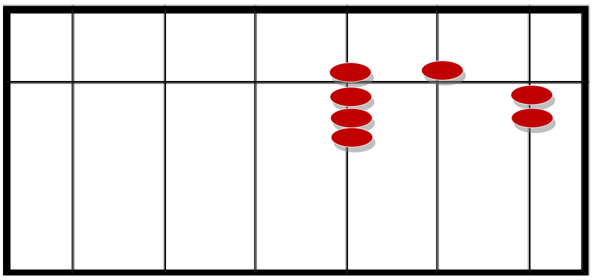
Then, they correctly formed nine three-digit numbers and one four-digit number on empty abaci by drawing beads. Previously, they verified their answers on the actual abacus.

9. *Can you draw beads on each abacus to represent the number written above it?*

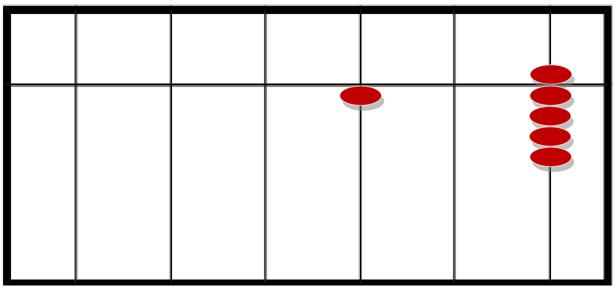
300



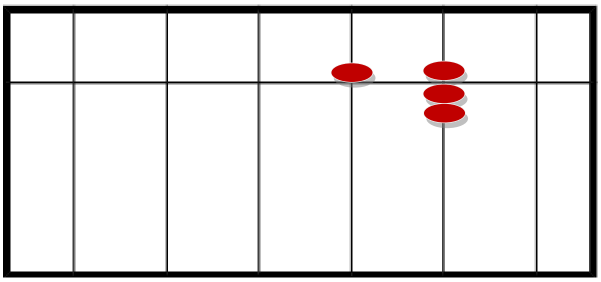
852



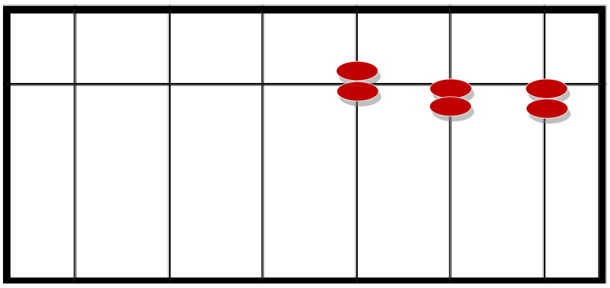
109



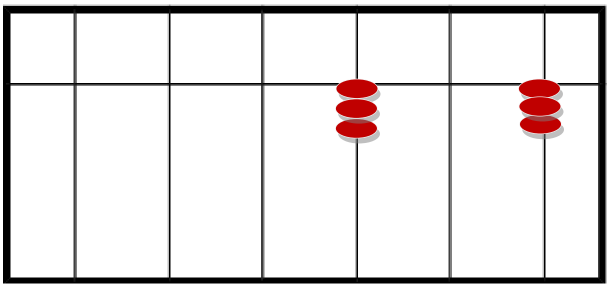
570

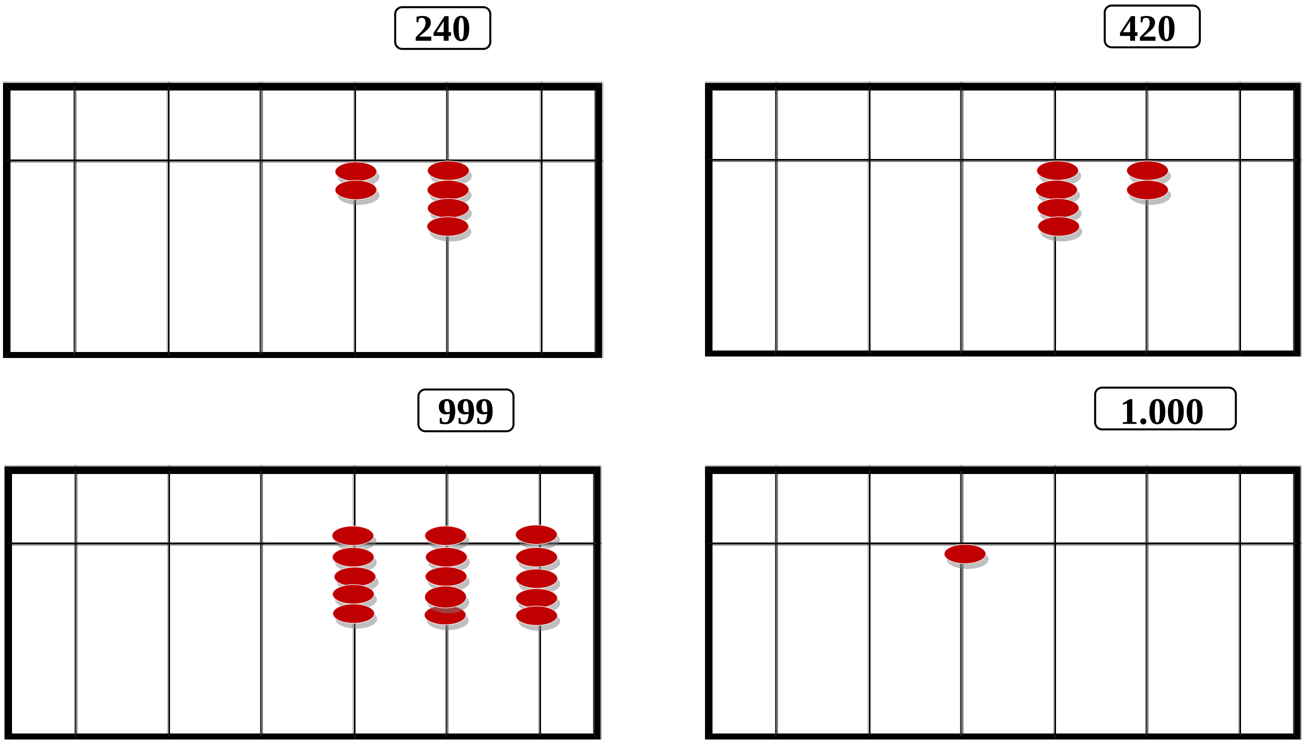


622



303





4.3 Third part

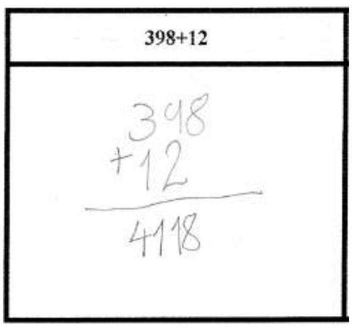
The third part of the workshop activities concerned the addition algorithm. In the plenary discussion, the participants used their experience to share common students' mistakes in the implementation of the addition algorithm.

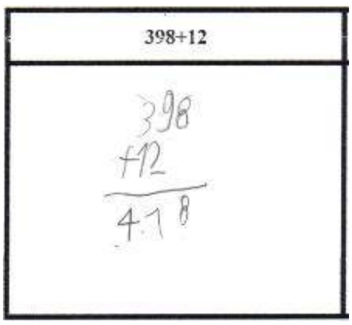
10. *What kind of mistakes do students usually make in the addition algorithm?*

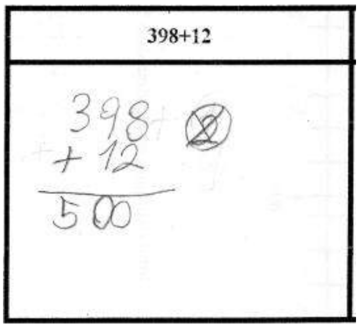
They mentioned mistakes in the placement of the addends, wrong sums at place values, non-creation of carried number, etc.

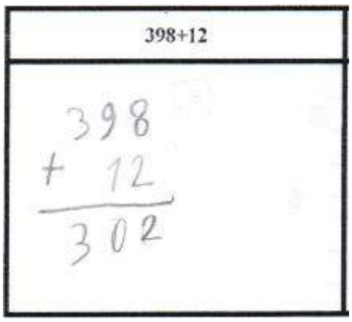
Afterwards, they worked on an activity that presented five different attempts to execute the addition 398+12. These were the answers that were given by five different students at the pre-test questionnaire of our research study. The participants were asked to identify the students' mistakes and to guess their way of thinking that led to these mistakes.

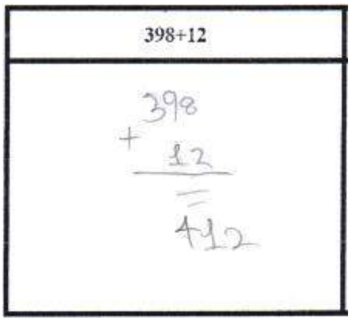
11. *Can you identify the students' mistakes while they executed vertically the addition 398+12? Can you guess their way of thinking that led to these mistakes?*











The participants were very interested in this activity. They easily identified the mistakes and they announced their remarks in the plenary discussion that followed the activity.

Then, they were asked to explain what carried number is.

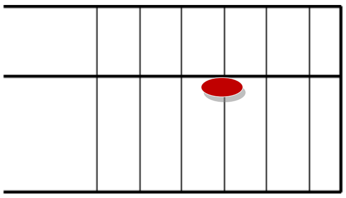
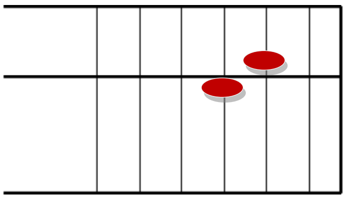
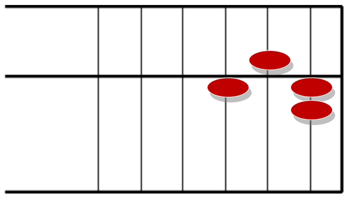
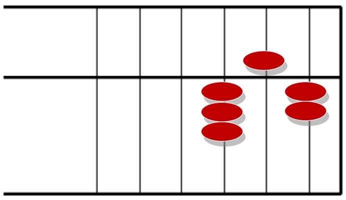
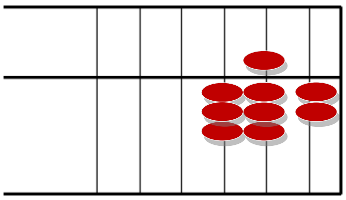
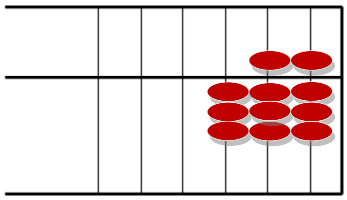
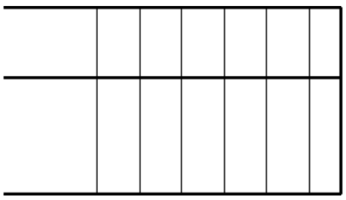
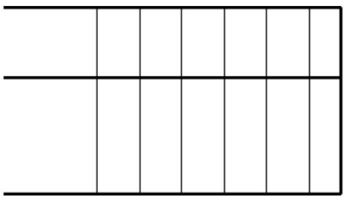
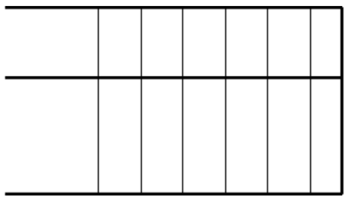
12. *Can you explain what carried number is?*

Poisard (2006), in order to explain the concept of carried number, states that in each rank of the decimal number system (units, tens, hundreds, etc.) only one digit from 0 to 9 can be written. In the addition, as soon as ten is reached in any place value, there is a transfer of numbers between ranks (always from right to left). One transfers 1 ten for every 10 units, 1hundred for every 10 tens, 1 thousand for every 10 hundreds, and so on. “The carried number enables us to manage the change of the place-value by making a transfer of numbers between ranks” (Poisard, 2006, p. 417). That is the definition she provides for carried number. Moreover, the understanding of the meaning of carried number requires a profound comprehension of the place value system.

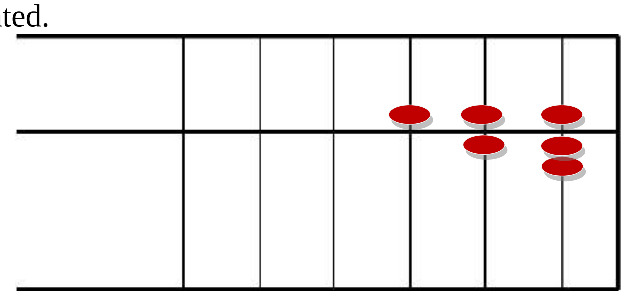
Next, the participants were asked to perform step-by-step both on sketched abaci on the worksheet and on the actual abacus an addition without carried number and an addition with carried number, following instructions. Thus the concept of carried number was visualized. At the same time, by using the concrete abacus, the visual and the tactile perception of the concept were combined (Nagaoka et al, 2000).

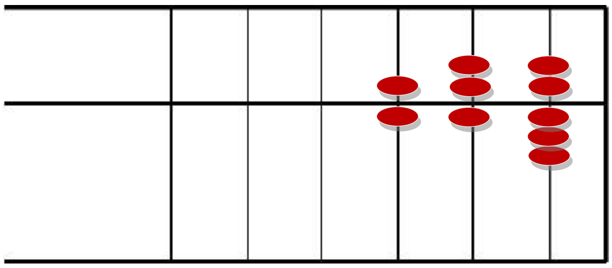
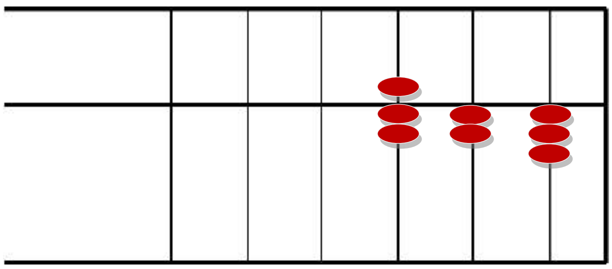
13. *Can you draw beads on the following sketched abaci to show step-by-step how you would perform the addition 152+236 on the Chinese abacus? Draw only the activated beads. On each line explain what you are doing. <u>Use only the abaci you need</u>. Finally, write the sum in the box.*

152 + 236 = **388**

1st abacus **I activate 1 hundred.** 	2nd abacus **I activate 5 tens.** 	3rd abacus **I activate 2 units.** 
4th abacus **I activate 2 more hundreds.** 	5th abacus **I activate 3 more tens.** 	6th abacus **I activate 6 more units.** 
7th abacus .... 	8th abacus .... 	9th abacus .... 

14. *Can you write vertically the following horizontal addition and find the sum? Verify the result by using your abacus. Then, follow the instructions to show, by drawing beads, the way of executing an addition on the Chinese abacus.*

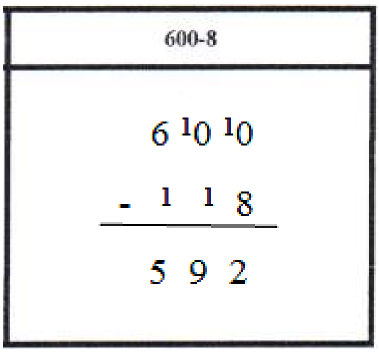
Horizontal addition	**vertical addition**	**Abacus**
		Place the number 567 on the abacus. Then, write under each place value how many Units, Tens and Hundreds are activated.  **5 6 7**

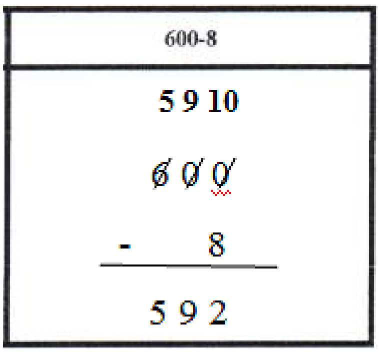
567+156= **723**	**1 1** 5 6 7 + 1 5 6 **7 2 3**	Add the number 156 by drawing beads. Then, write under each place value how many Units, Tens and Hundreds are activated. Paint the carried numbers with red color.  **6 11 13** Make the necessary exchanges between place values.  **7 2 3** Can you explain what carried number is?

4.4 Fourth part

In the fourth part of the workshop the participants dealt with subtraction. Initially, they were asked to perform the subtraction 600-8 with two different algorithms that students learn in primary school.

15. *Can you carry out the subtraction 600-8 using two different algorithms that students are taught in primary school?*





In the plenary discussion, the participants mentioned two methods: “regrouping the minuend” and “adding equal amounts”.

Then, they were asked to think about which one of the two ways could be applied to the Chinese abacus.

16. *Which algorithm do you think the Chinese abacus can support?*

They explained that the first method can be applied on the Chinese abacus, while a participant tried to explain that the “adding equal amounts” method could be applied, too.

Later, the participants were asked to mention common mistakes that students usually make in the subtraction algorithm.

17. *What kind of mistakes do students usually make in the subtraction algorithm?*

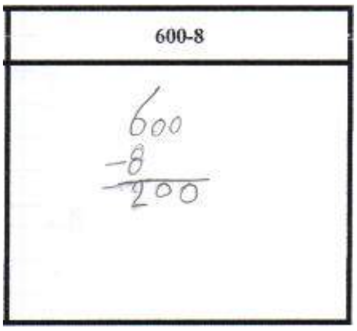
They mentioned that usually students place wrongly the subtrahend, they don’t know from which place value to borrow and they also make mistakes in the calculations.

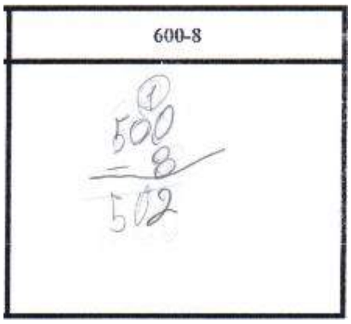
Next they were asked what borrowed number is.

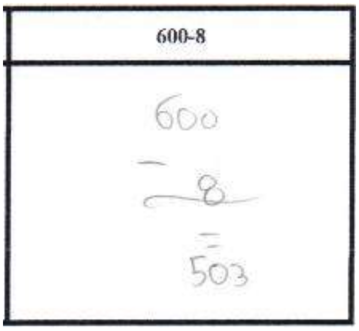
18. *Can you explain what borrowed number is?*

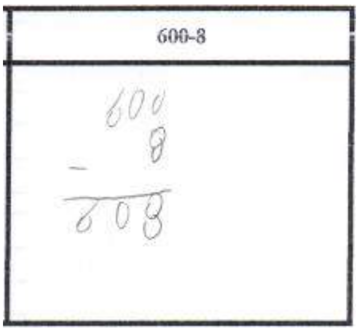
Then, as in the addition, five different attempts to perform the subtraction 600-8 were presented. Once again the participants were asked to identify the students’ mistakes and to think what led them to these mistakes.

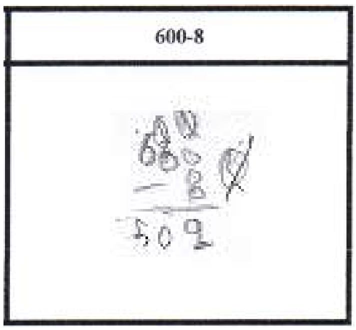
19. *Can you identify the students’ mistakes while they carried out the subtraction 600-8? Can you guess their way of thinking that led to these mistakes?*









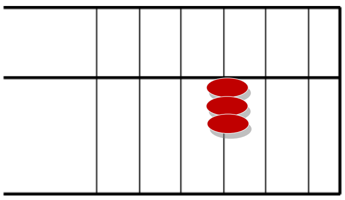
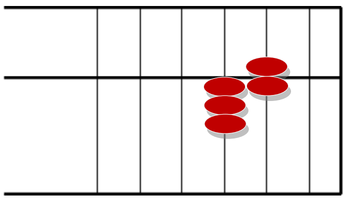
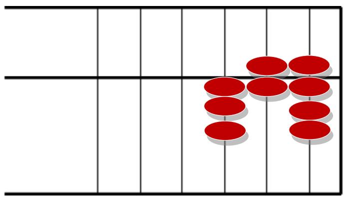


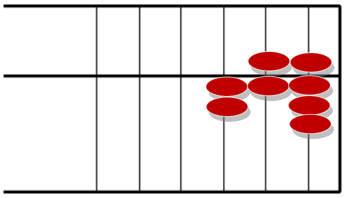
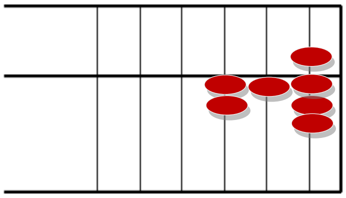
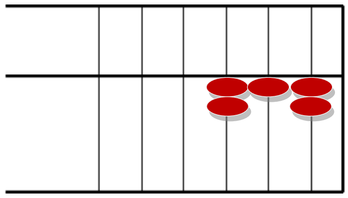
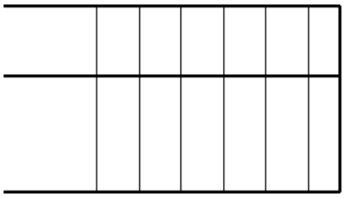
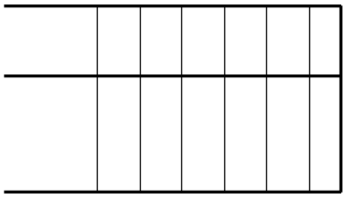
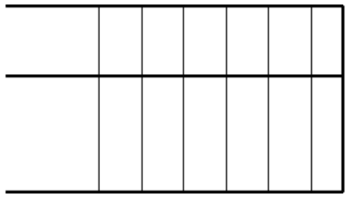
It is another activity that the participants said that it was interesting and they announced their thoughts in the plenary discussion that followed.

Finally, both in sketched and actual abaci, the participants were asked to perform a subtraction without borrowed number and a subtraction with two borrowed numbers, following the worksheet's instructions.

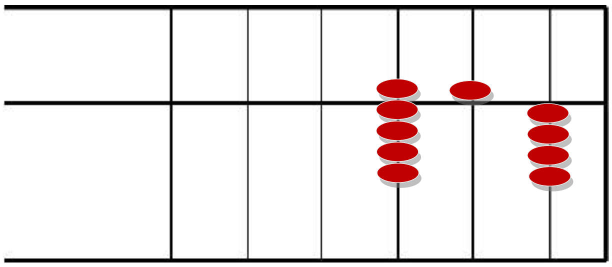
20. *Can you draw beads in the following abaci to show step-by-step how you would perform the subtraction 368-156 on the Chinese abacus? Draw only the activated beads. On each line explain what you are doing.* <u>*Use only the abaci you need*</u>*. Finally, write the difference in the box.*

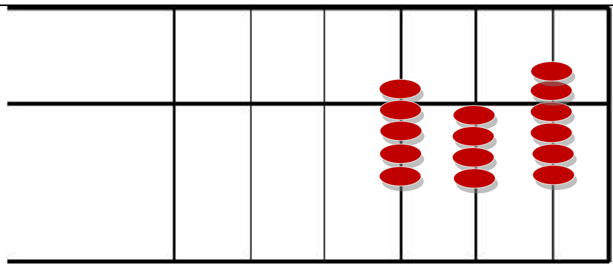
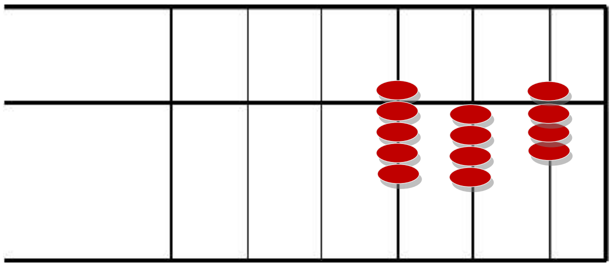
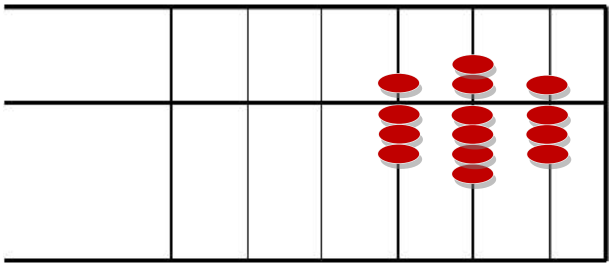
368 - 156 = **212**

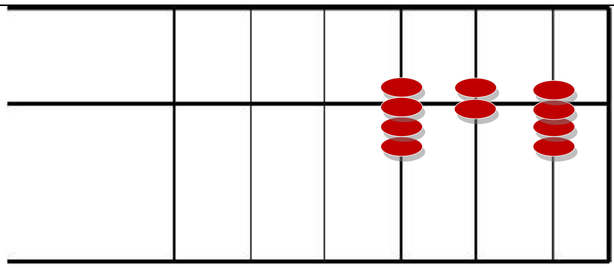
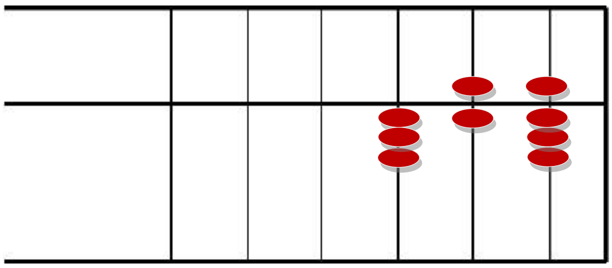
1st abacus	2nd abacus	3rd abacus
I activate 3 hundreds.	**I activate 6 tens.**	**I activate 8 units.**
		

4th abacus **I deactivate 1 hundred.** 	5th abacus **I deactivate 5 tens.** 	6th abacus **I deactivate 6 units.** 
7th abacus .... 	8th abacus .... 	9th abacus .... 

21. *Can you write vertically the following horizontal subtraction and find the difference? Verify the result by using your abacus. Afterwards, follow the instructions to show, by drawing beads, the way of executing a subtraction on the Chinese abacus.*

horizontal subtraction	**vertical subtraction**	**Abacus**
		Place with beads the number 954 on the abacus. Then, write under each place value how many Units, Tens and Hundreds are activated.  **9 5 4** Start subtracting. You cannot subtract 6 Units from 4 Units. Borrow 1 Ten (=10 Units) from the Tens’ place and make the necessary exchanges between place values. Then, draw the beads which are activated on the place values and write under each place value how many Units, Tens and Hundreds are activated. Color in red the borrowed number.

<table>
<tr>
<td>954-586= 368</td>
<td>14
8 ~~4~~ 14
~~9~~ ~~5~~ ~~4~~
- 5 8 6
3 6 8</td>
<td>
9 4 14
Subtract the Units in the Units’ place and draw what remains. Then, draw beads on the Tens’ and Hundreds’ place. Write under each place value how many Units, Tens and Hundreds are activated and go to the Tens’ place.

9 4 8
You cannot subtract 8 Tens from 4 Tens. Borrow 1 Hundred (=10 Tens) from the Hundreds’ place and make the necessary exchanges between place values. Then, draw the beads which are activated on the place values and write under each one how many Units, Tens and Hundreds are activated. Color in red the borrowed number.

8 14 8
Subtract the Tens in the Tens’ place and draw what remains. Then, draw beads on the Units’ and Hundreds’ place. Write under each place value how many Units, Tens and Hundreds are activated and go to the Hundreds’ place.</td>
</tr>
</table>

		 **8 6 8** Subtract the Hundreds in the Hundreds' place and draw what remains. Then, draw beads on the Tens' and Units' place and write under each place value how many Units, Tens and Hundreds are activated.  **3 6 8** Can you explain what borrowed number is?

At the end of the workshop we presented to the participants an outline of the theory of instrumental genesis and semiotic mediation that we used in our instructional intervention, and several examples of the implementation of the two theories.

5 Discussion

Poisard (2006) argues that for many students and teachers the notion of carried number appears not to be developed mathematically. Usually the addition is taught in the form of learning the algorithm, which means the execution of a series of steps that represent the syntactic rules of the algorithm. However, the performance of an algorithm does not require deep awareness of the meaning of the successive steps. For example, when the student sums up the units of the addends and finds a result equal to or greater than ten units, he says "one is the carried number". Then, he marks it with the digit 1 either above the tens' place that will carry it over or on the right side of the addition. However, it is not certain that the student understands that this 1 is 1 ten, which means 10 units, and not 1 unit (Fuson, 1990). This misconception is clarified by the Chinese abacus. The fact that in each column of the abacus can be represented numbers up to 15 (i.e. 15 units, 15 tens, etc.) and someone can make exchanges between the columns with his hand, enhances the understanding of the carried number concept (Poisard, 2006). Thus carried number acquires a material substance, which did not have until now, since the addition algorithm is usually taught without the use of physical manipulatives. Therefore, it is understood through the visual and tactile perception of the Chinese abacus. At the same time, the knowledge of carried number also contributes to the reduction of algorithmic errors.

This visual and tactile perception of the Chinese abacus also applies to the case of the borrowed number. As it has already been mentioned, the Chinese abacus can support the visualization of the subtraction algorithm by the type of "regrouping the minuend". This regrouping requires exchanges between place values in the opposite direction from addition. Thus, the intangible borrowing, which so far has been taught to children in the context of the subtraction algorithm, becomes tangible, and it is obvious now what is being borrowed and from where. In this way errors can be avoided when performing the subtraction algorithm, especially when the student is asked to borrow from a place value that has a zero digit.

Therefore, the Chinese abacus is a resource for the teaching and learning of the place value system in primary school. It benefits both students' learning and teachers' practice in the mathematics classroom.

REFERENCES

Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and instruction*, *7*(4), 343-403.

Korros, F. (2012). *Mathematics of flat Earth-The history of Mathematics in China until the 15th century*. Athens: Bookstars-Free Publishing. (in Greek) [Κόρρος, Φ. (2012). *Τα μαθηματικά της επίπεδης γης-Η ιστορία των Μαθηματικών στην Κίνα μέχρι τον 15ο αιώνα.* Αθήνα: Bookstars-Free Publishing.]

Li, S. T. (1958). The origin of the abacus and its development. In *ACM Annual Conference/Annual Meeting: Preprints of papers presented at the 13th national meeting of the Association for Computing Machinery* (Vol. 11, No. 13, pp.102-110). Urbana, Illinois.

Nagaoka, R., Barrow-Green, J., Bartoloni Bussi, M. G., Isoda, M., van Maanen, J., Michalowicz, K. D., Ponza, M. V., & van Brummelen, G. (2000). Non-standard media and other resources. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education: the ICMI study* (pp. 329-370). Dordrecht: Kluwer.

Poisard, C. (2006). The notion of carried-number, between the history of calculating instruments and arithmetic. In P. Groontenboer, R. Zevenbergen, & M. Chinnapan (Eds.), *Identities, cultures and learning spaces*. *Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra* (Vol 2, pp. 416-423). Adelaide: MERGA.

Poisard, C. (2017). Introducing an old calculating instrument in a new technologies environment: a praxeological analysis of students' tasks using different registers. *Re SMITECH Review of Science, Mathematics and ICT Education, 11*(2), 47-67.

Price, P. (2002). Children's difficulties with base-ten numbers: "Face-Value" and "Independent-Place" Constructs. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific*. *Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland* (pp. 592–599). Sydney: MERGA.

Spitzer, H. F. (1942). The abacus in the teaching of arithmetic. The Elementary School Journal, *42*(6), 448-451.

Stigler, J. W. (1984). "Mental abacus": The effect of abacus training on Chinese children's mental calculation. *Cognitive Psychology, 16*(2), 145-176.

Tsiapou, V., & Nikolantonakis, K. (2013). The Development of Place Value Concepts to Sixth Grade Students via the Study of the Chinese Abacus. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eight Congress of the European Society for Research in Mathematics Education* (pp. 2058-2067). Ankara: Middle East Technical University.

Tsiapou, V., & Nikolantonakis, K. (2016). The development of place value concepts and the notion of carried number among sixth grade students via the study of the Chinese abacus. *Menon: Journal of Educational Research*, 2nd thematic issue, 51-65.

Tzekaki, M. (2007). *Little children big mathematical meanings*. Athens: Gutenberg. (in Greek) [Τζεκάκη, Μ. (2007). *Μικρά παιδιά μεγάλα μαθηματικά νοήματα.* Αθήνα: Gutenberg.]

Tzekaki, M. (2016). Educational material as an element of mathematical activity. In Skoumios, M. & Skoumpourdi, C. (Eds.), *Proceedings of the 2nd Pan-Hellenic Conference "The educational material in Mathematics and the educational material in Physical Sciences: lonely trajectories or interactions?"* (pp. 380-390). Rhodes, Greece: University of the Aegean. (in Greek) [Τζεκάκη, Μ. (2016). Εκπαιδευτικό υλικό ως στοιχείο της μαθηματικής δραστηριότητας. Στο Σκουμιός, Μ. & Σκουμπουρδή, Χ. (επιμ.), *Πρακτικά 2ου Πανελληνίου Συνεδρίου "Το εκπαιδευτικό υλικό στα Μαθηματικά και το εκπαιδευτικό υλικό στις Φυσικές Επιστήμες: μοναχικές πορείες ή αλληλεπιδράσεις;"* (σ. 380-390). Ρόδος: ΤΕΠΑΕΣ, ΠΤΔΕ, Πανεπιστήμιο Αιγαίου.]

Zhou, Z., & Peverly, S. T. (2005). Teaching addition and subtraction to first graders: A Chinese perspective. *Psychology in the Schools, 42*(3), 259-272.

THE BICYLINDER OR BIRDCAGE OR MÓUHÉFĀNG GÀI

Combining a cultural approach with many other goals of mathematics education

Michel ROELENS
University Colleges Leuven-Limburg, Faculty of Teacher Education
& Maria-Boodschaplyceum Brussels, Belgium
michel.roelens@ucll.be

ABSTRACT

We want to introduce our students into some mathematical 'culture'. This noble goal, however, has many competitors in the form of other important goals of mathematics education, and time is finite. We also want to (and have to) teach calculus methods and train the students' algebraic skills. We want to show them that different methods can solve the same problem. We want to show the utility of mathematics in architecture and technology. We want to stimulate the students' spatial insight and their ability to make sketches of spatial situations. We want to show them the relationships between different parts of mathematics. We would like to make them critical of manipulation by the media or the internet.

The bicylinder is an object that allows combining all these goals. We will discuss how the bicylinder can play an interesting role in mathematics courses for students between the ages of 15 and 18, in order to combine various goals and to make the mathematics courses more cultural and versatile without spending much extra time.

1 Describing the bicylinder

1.1 Orthogonal projections and 'edges'

Consider two solid circular cylinders of equal radii, the axes of which intersect perpendicularly. While figure 1.1 shows their union, we are interested in their*intersection*.

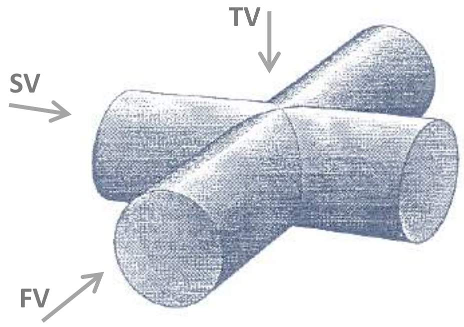


Figure 1.1: The two cylinders (Mathenjeans, 2006)

We call that intersection a bicylinder, but it has been given many names throughout history: 牟合方蓋 [Móuhéfānggài, double box-lid] (ZǔChōngzhī, 5th century),Steinmetz' solid (Charles Proteus Steinmetz, 19th and 20th century), birdcage (Stannard, 1979), equidomoid (Ferréol, 2013)…

Showing the students only the union of the cylinders (figure 1.1), we ask them to imagine whatthe bicylinder looks like and to draw its perpendicular projections: a front view (FV), top view (TV) and a side view from the left (SV) (figure 1.2). If this causes them some difficulties, they can be comforted by this quote: "It takes an unusual gift of imagination to visualize this shape clearly" (Strogatz, 2010).

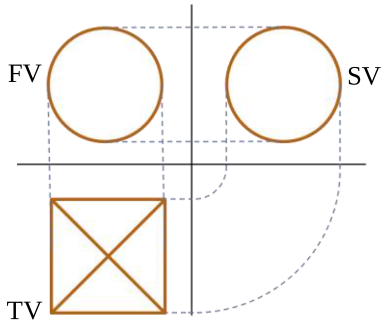


Figure 1.2: Orthogonal projections of the bicylinder

We also ask them the shape of the (curved) ‘edges’. It is a bit unusual to talk about edges when the figure is not a polyhedron, hence the quotation marks. One could say that the bicylinder has two ‘vertices’(at the top and at the bottom), four curved ‘faces’ and four curved ‘edges’ connecting the two ‘vertices’. Some students want to determine the shape of the ‘edges’ analytically. Taking the axes of the cylinders as x- and y-axes and calling the radius r, they determine the ‘edges’ with a system of equations. They solve by replacing the second equation by the difference of both and then factorizing.

$$\begin{cases} x^2+z^2=r^2 \\ y^2+z^2=r^2 \end{cases} \Leftrightarrow \begin{cases} x^2+z^2=r^2 \\ x^2-y^2=0 \end{cases} \Leftrightarrow \begin{cases} x^2+z^2=r^2 \\ (x-y)(x+y)=0 \end{cases} \Leftrightarrow \begin{cases} x^2+z^2=r^2 \\ x-y=0 \ \vee \ x+y=0 \end{cases}$$

The ‘edges’ are created by intersecting one of the cylinders with the two ‘vertical’ diagonal planes $x-y=0$ and $x+y=0$. Instead of analytically, this could just as well be discovered on the basis of symmetry. So the four ‘edges’ are halvesof two ellipses. The small axis of these ellipses is the height $2r$ of the bicylinder; the large axis is $2\sqrt{2}\,r$ (as you can deduce from the top view in figure 1.2).

Figure 1.3 shows a wooden model of the bicylinder. We deliberately don’t give it to the students from the beginning; we want to appeal to their imagination.

Figure 1.3: Bicylinder (Modellsammlung)

1.2 Making the bicylinder

If students want to make a paper model of the bicylinder, they have to know the shape of its ‘faces’ when developed in the plane. We will prove here that the plane development of a cylinder cut by an oblique plane is bound by one period of a sine graph. It can be shown with a paint roll (figure 1.4). It is alsoused by knitters: the pattern of a sleeve, which is more or less a cylinder segment, is made using (approximately) a sinusoid (figure 1.5).

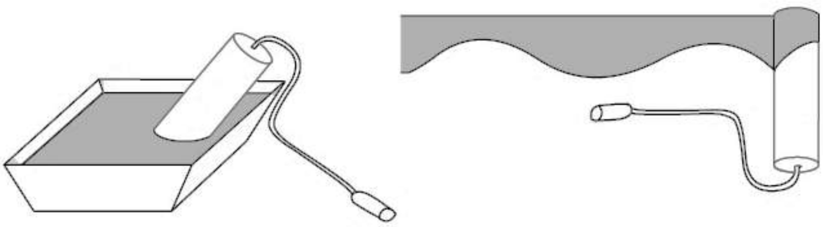
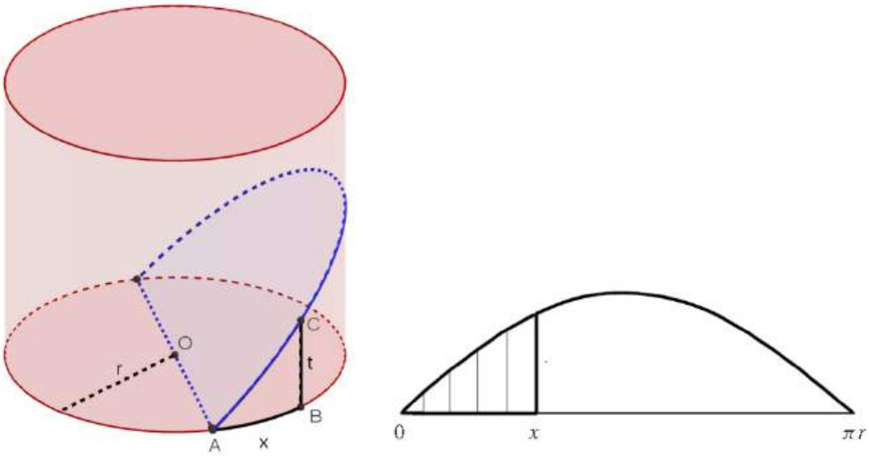


Figure 1.4: Paint roll experiment

Figure 1.5: Sleeve pattern

In order to prove that the plane development is delimited by a sinusoid, we use figure 6. By the symmetry it suffices to prove that the half ellipse (left part of figure 1.6) obtained by intersecting the volume with a plane, gives rise to half a period of a sine graph in the plane development. The solid consisting of the cylinder 'under' the half ellipse in figure 1.6a is called a *cylinder hoof*.



Figures 1.6a and 1.6b: Proof that we get a sinusoid, first part

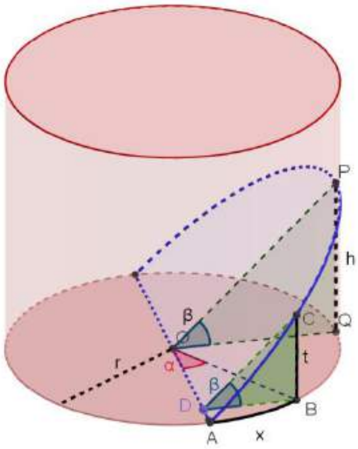


Figure 1.7: Proof that we get a sinusoid, second part

We prove that t is indeed a sine function of x (figure 1.7).

$$\begin{aligned} t &= |DB| \tan\beta && (\Delta\, DBC) \\ &= r \sin\alpha \tan\beta && (\Delta\, ODB) \\ &= r \sin\frac{x}{r} \tan\beta \\ &= h \sin\frac{x}{r} && (\Delta\, OPQ) \end{aligned}$$

Knowing this, the students can make a bicylinder from its plane development. In the case of the bicylinder, the height h of the cylinder hoof equals the radius r of the cylinders, so the development of two of the 'faces' is delimited by one period of the graphs$y = \pm r \sin\frac{x}{r}$ (figure 1.8), and in the same way the development of the other two 'faces' by $x = \pm r \sin\frac{y}{r}$ (not shown in figure 1.8).

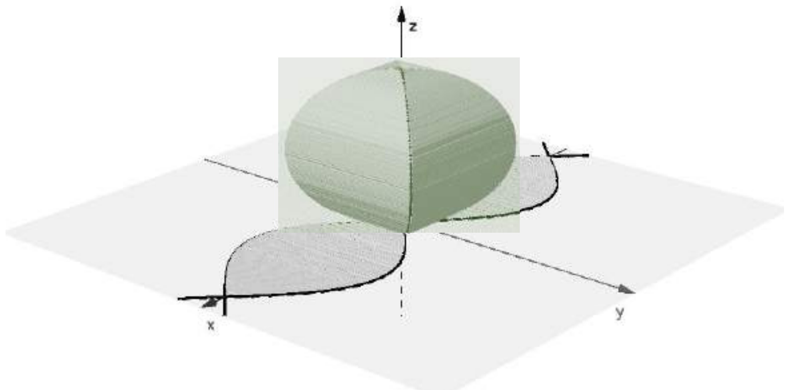


Figure 1.8: The bicylinder and the development of two of its 'faces'

2 Calculating the volume of the bicylinder

2.1 As did Archimedes?

Archimedes announces in his preface to *The Method* that the volume of a bicylinder is 2/3 of the volume of the circumscribed cube. His proof is lost but we have his proof of the volume of a cylinder hoof, which is in a non-lost part of *The Method*. With eight cylinder hoofs, we can construct a bicylinder (figure 2.2), so it is likely that Archimedes used this idea for determining the volume of the bicylinder.

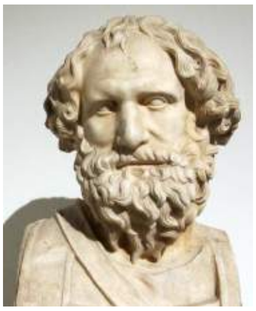


Figure 2.1: Archimedes of Syracuse (3rd century BC)

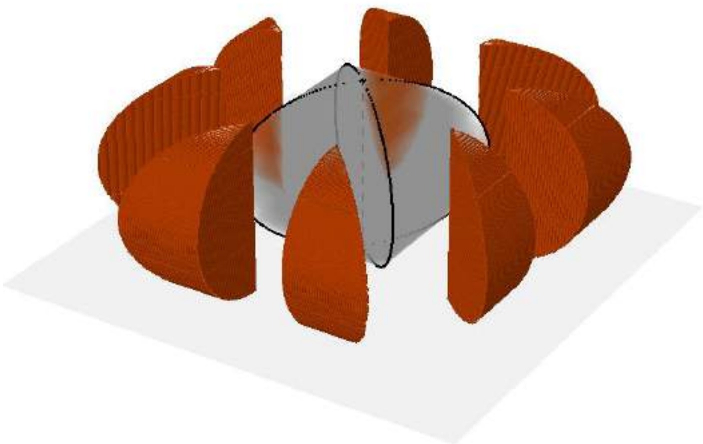


Figure 2.2: Eight cylinder hoofs form a bicylinder

Before going into the details of his proof for the volume of a cylinder hoof, let's say a few words about *The Method*.

Archimedes sent his papyrus roll with *The Method* to Eratosthenes at the famous library of Alexandria (in present-day Egypt). Later, the work was copied on parchment sheets for better preservation. Together with some other works of Archimedes, the sheets were knit together into a booklet, a *codex*. This codex disappeared until it was found in 1906 in a monastery in Jerusalem. In the 13th century, the monks had scrapped Archimedes'original texts and drawings to replace them by prayers. Archimedes' codex had become a *palimpsest*, a recycled piece of parchment. The codex was stolen in the course of the twentieth century and reappeared in 1998 in a sales hall, where it was sold by auction. A mysterious Mr. B bought it for $2,200,000. Nobody knows who Mr. B is, although there are some speculations. Fortunately, he allows scientists to study the codexusing UV- and X-rays. For more details about the story of this palimpsest, see Netz and Noel (2009).

In *The Method*, Archimedes determines areas and volumes in a revolutionary way for Greek mathematics: he determines an area by considering an infinity of line segments and a volume by considering an infinity of flat slices. Much later, this idea will become Cavalieri's principle (Bonaventura Cavalieri, 17^{th} century) and integral calculus. Sometimes he also uses the physical idea of a balance with which he 'weights' the slices, but this is not the case in the proof about the cylinder hoof.

How did Archimedes determine the volume of a cylinder hoof (i.e. of one eighth of a bicylinder)? We follow the ideas of his proof, but in a very anachronistic way, using today's algebraic notations.

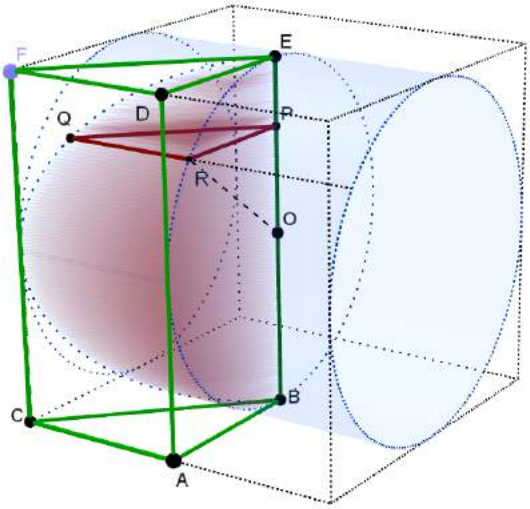


Figure 2.3: The cylinder hoof as locus of triangular slices

In a cube of edge 2, a cylinder of radius 1 and height 2 is inscribed (figure 2.3).The prism $ABC.DEF$ is one eighth of the cube. The cylinder hoof obtained as the intersection of this prism with the cylinder is one eighth of the bicylinder.Archimedes considers this hoof as the ‘locus’ of the variable horizontal triangle PQR as P varies on the segment $[BE]$ (see figure 2.3). The students calculate the area of the triangle ABC, the variable area of the triangle PQR (as a function of $x = |OP|$) and the proportion of both areas.

$$\left.\begin{array}{r} \text{area}(ABC) = \dfrac{1}{2} \\ |QR| = |PR| = \sqrt{1-x^2} \Rightarrow \text{area}(PQR) = \dfrac{1}{2}(1-x^2) \end{array}\right\} \Rightarrow \frac{\text{area}(PQR)}{\text{area}(ABC)} = 1 - x^2$$

The proportion of these areas is a quadratic function of x. The graph of a quadratic function is a parabola. This is not the way the Greeks of the time of Archimedes considered a parabola, but in a different, more geometric way Archimedes came to the same idea. Then he constructs a point S on the segment $[PR]$ such that

$$\frac{|PS|}{|BA|} = \frac{\text{area}(PQR)}{\text{area }(ABC)} = 1 - x^2.$$

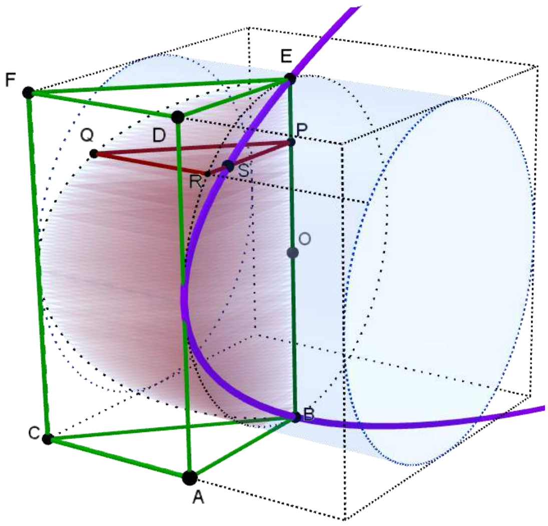


Figure 2.4: The parabola added in figure 2.3

If x varies, the point S moves on a parabola and the variable segment $[PS]$describes a parabola segment inscribed in the rectangle $ABED$ (figure 2.4).

Archimedes knows from one of his other works (*The Quadrature of the Parabola*, theorem 14) that the area of a parabola segment inscribed in a rectangle equals 2/3 of the area of the rectangle. Since each slice (line segment) of the parabola segment represents a slice (triangle) of the cylinder hoof and each slice (line segment) of the rectangle $ABED$represents a slice (triangle) of the prism$ABC.DEF$, he deduces, 'integrating' all these slices, that the volume of the cylinder hoof equals 2/3 of the volume of the prism.

So the volume of the whole bicylinder is

$$8 \cdot \left(\frac{2}{3}\,\mathrm{volume(prism)}\right) = \frac{2}{3} \cdot \big(8 \cdot \mathrm{volume(prism)}\big) = \frac{2}{3}\mathrm{volume\ (cube)} = \frac{16r^3}{3}$$

Surprisingly, the formula for calculating the volume of bicylinder does not contain a factor π. Archimedes: "Unlike spheres, cones and cylinders, this object is equal [in volume] to asolid figure bound by plane figures." (Introduction to *The Method*, cited in Hogendijk, 2002).

2.2 As did (not) Liú Huī

Figure 2.5: Liú Huī (3rd century)

Liú Huī wrote his famous Commentary on the *Jiŭzhāng Suànshù* [Nine Chapters on the Mathematics Art] (2nd century BC), wherein he added explanations and proofs to the Nine Chapters. He considers a bicylinder and its inscribed sphere (figure 2.6).

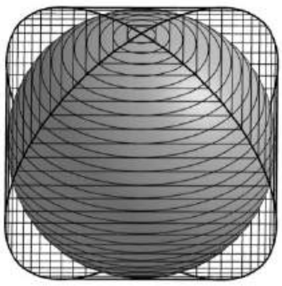


Figure 2.6: A bicylinder and its inscribed sphere (Cadav92, 2014)

He uses 'horizontal' slices. Each slice of the bicylinder is a square. The slice of the sphere at the same height is a circle inscribed in that square. So the proportion of the slices at a same height is always $\frac{4}{\pi}$. He deduces that the volume of the bicylinder is $\frac{4}{\pi}$ times the volume of the sphere. This is an early use of what we now call Cavalieri's principle (Bonaventura Cavalieri, 17th century). This principle says: *if at any height the 'horizontal' cross-sections of two solidsare in a fixed proportion, then the volumes of these solidsare in the same proportion*.

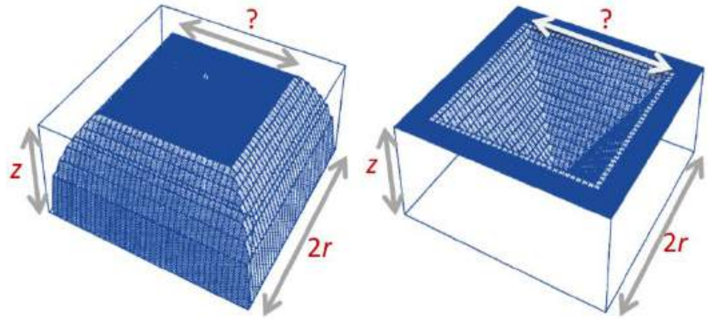
Our students can use this proportion to calculate the volume of the bicylinder from the volume of the sphere, getting the same result as in 2.1 (i.e. 2/3 of the volume of the circumscribed cube). But, unlike our students, LiúHuī did not dispose of the volume of a sphere. On the contrary, he considered this discovery of the proportion $\frac{4}{\pi}$as a step towards finding the volume of the sphere, if he would be able to find the volume of the bicylinder first.

2.3 As did ZǔChōngzhī

Figure 2.7: Zǔ Chōngzhī (5th century)

Zǔ Chōngzhī succeeded in finding the volume of the bicylinder, using a cube, a pyramid and the later principle of Cavalieri, already used by Liú Huī.

In figure 2.8a, you see the upper half of a bicylinder cut by a horizontal plane. The part above this plane has been removed. In figure 2.8b, you see a half cube hollowed by an inscribed pyramid (top down). This is also cut by a horizontal plane. The part above this plane has also been removed.



Figures 2.8a and 2.8b: Proof of the volume of a bicylinder by Zǔ Chōngzhī (De Temple, 1994)

Using a front view of figure 2.8a, the students can show that the side of the square slice at height z equals $2\sqrt{r^2 - z^2}$. So, the area of this square is $4(r^2 - z^2)$. On the other hand, the side of the section of the pyramid at height z (figure 2.8b) equals $2z$. Therefore, the area of the square 'ring', the slice at height z in figure 2.8b, equals $4r^2 - 4z^2$. Because the areas at height z are equal for each value of z, the students conclude by the principle of ZǔChōngzhī and Cavalieri that the volumes in figure 2.8a and 2.8b are equal. So, the volume of the half bicylinder is equal to 2/3 of the volume of the half cube. By symmetry, the volume of the whole bicylinder is also equal to 2/3 of the volume of the whole cube, as in 2.1.

Here we worked in half a cube; the original proof of ZǔChōngzhī divides the cube in eight parts, but this does not change much. (See Papillon, 2012, Lam, 1985 or Antony, s.d.).

2.4 As we do

The 'normal way' for our students to calculate the volume of a solid, is using an integral.Unlike the majority of the textbook exercises on volumes with integrals, the bicylinder is not a solid of revolution. The horizontal slices are squares, so the volume is the integral of the area of a square slice as a function of the height z (figure 2.9).

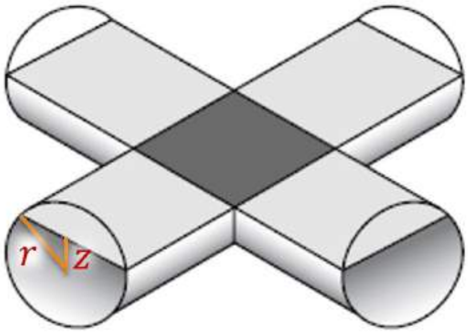


Figure 2.9: Square slice at height z

Again, the side of the square slice at height z is $2\sqrt{r^2 - z^2}$, so the volume is

$$\begin{aligned}
\text{volume (bicylinder)} &= \int_{-r}^{r} \left(2\sqrt{r^2 - z^2}\right)^2 dz \\
&= 4\int_{-r}^{r} (r^2 - z^2) dz \\
&= 4\left[r^2 z - \frac{z^3}{3}\right]_{-r}^{r} \\
&= 4\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right) \\
&= \frac{16r^3}{3}
\end{aligned}$$

3 Calculating the surface area of the bicylinder

As we saw in paragraph 1.2, the flat development of the bicylinder of radius $r$is delimited by the sine graphs $y = \pm r \sin\frac{x}{r}andx = \pm r \sin\frac{y}{r}$ (figure 1.8 in 1.2). The surface area of the bicylinder is the area of its development:

$$\begin{aligned}
\text{surface area(bicylinder)} &= 8r\int_{0}^{\pi r} \sin\frac{x}{r} \mathrm{d}x \\
&= 8r\left[-r\cos\frac{x}{r}\right]_{0}^{\pi r} \\
&= 8r(-(-r) + r) \\
&= 16\, r^2
\end{aligned}$$

Again it is striking that there is no π in the formula! The surface area is simply the area of a square with side $4r$. The fact that the area of the bicylinder is equal to the area of a square of side$4r$ is culturally interesting. It means that the quadrature is possible here, the constructionof a square with the same area as the bicylinder by means of ruler and compass, starting from the given radius r. The quadrature of the circle is one of the famous construction problems of Greek antiquity, which in the 19th century has been shown to be impossible to solve with ruler and compass.

The area of the bicylinder is $2/3$ of the surface area of the circumscribed cube $(6 \cdot (2r)^2 = 24r^2)$. For the bicylinder and its circumscribed cube the proportion of the volumes equals the proportion of the surface areas!

Hogendijk (2002) explains how the surface area can be derived from the volume, without integrals.

4 Applications of the bicylinder

Cross vaults and the joining of cylindrical pipes are an obvious application of meeting cylinders, although it is more the union than the intersection of the cylinders (figures 4.1 and 4.2).

Figure 4.1: A cross-vault (Glaeser, 2007)

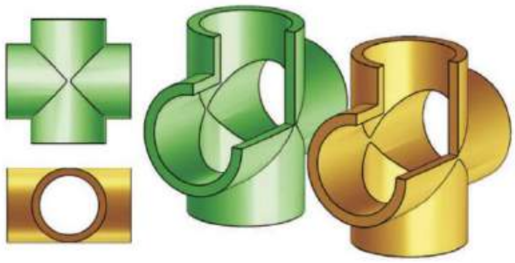


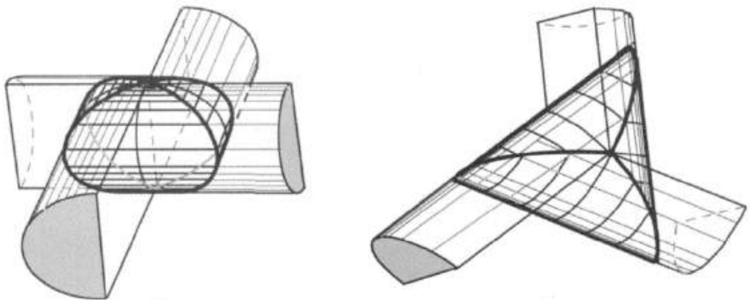
Figure 4.2: Pipes meeting at a right angle (Glaeser, 2007)

The roofs of the Château de Cheverny, one of the Châteaux of the Loire valley, have the form of half bicylinders (figure 4.3).

Figure 4.3: Chateau de Cheverny (Papillon, 2012)

5 Generalizations of the bicylinder

A first generalization consists of inventing analogue solids with a different number of curved ‘faces’ than four. It is easy to imagine making one with 6 (or $8, \ldots, 2n$) ‘faces’by intersecting 3 (or $4, \ldots, n$) cylinders, with equal radii and axes lying in one plane and intersecting in one point at equal angles of $60°$ (or$45°, \ldots, \frac{180°}{n}$). Is it also possible to obtain an odd number of ‘faces’? The number of cylinders is half of the number of ‘faces’ and one cannot use half cylinders... Why not actually? In figure 5.1a the familiar bicylinder is made in a different way and this way can be generalised to an odd number of ‘faces’.



Figures 5.1a and 5.1b: (Apostol & Mnatsakanian, 2004)

On the website Mathcurve (Ferréol, 2013) these solids are called polygonal *equidomoids*. The bicylinder is a quadrangular equidomoid. On that site it is claimed that the dome of the Cathedral of Florence is a (half) pentagonal equidomoid (that has been vertically stretched a little). In order to convince us of this, they place the following two figures next to each other (figure 5.2).

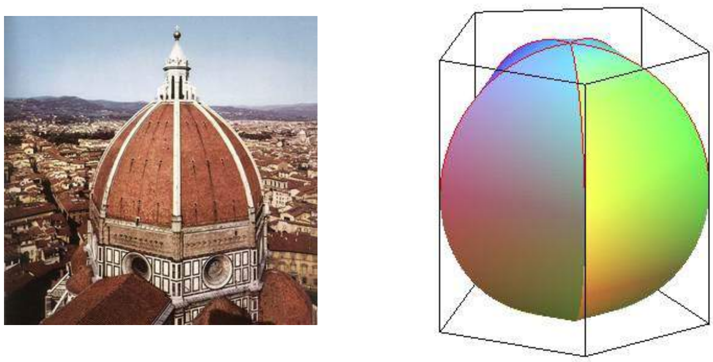


Figure 5.2: The dome in Florence and a pentagonal equidomoid (Ferréol, 2013)

However, you can clearly see in the picture that the dome has more than five ‘faces’ since four of them are visible. By entering the cathedral and looking upwards you find out that the dome is octagonal (figure 5.3).

Figure 5.3

Another generalisation is to take three cylinders with the same radius and with the axes that intersect in one point and are two to two perpendicular. The intersection of these cylinders is a*tricylinder*. This is a very interesting object, a curved rhombic dodecahedron, but we will not discuss it here (figure 5.4). Moore (1974) mentions applications in crystallography of the tricylinder and intersections of more than three cylinders, when due to increased temperature or decreased pressure a polyhedral crystal gets curved ‘faces’.

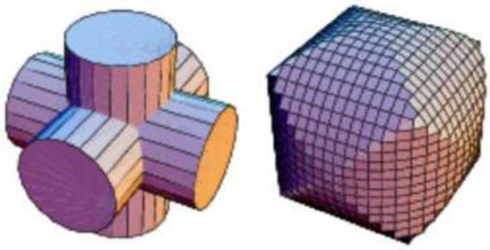


Figure 5.4: (Weisstein, 1999-2018)

6 In the classroom

In my classroom, I introduced the bicylinder as an exercise on the calculation of volume with an integral (2.4). In order to know what area they had to integrate, they had to imagine how the object is like (1.1). Only afterwards I confronted the students with other, historical, ways of finding the same result. In my esu8-workhop and in this article, I follow the chronologic order. You can find the worksheets of the workshop online: https://esu8.edc.uoc.gr/1112-2/.

What I like about the bicylinder, is that it includes a ‘normal’ textbook-like exercise (calculate its volume (2.4) or its surface area (3) with an integral), but that it goes further. Other methods than integral calculus are possible and have been discovered through history (2.1, 2.2, 2.3). The bicylinder is an challenge for the students’ spatial insight (1.1, 1.2) and it has applications in architecture and technology (4). It can even remind the students to be critical of manipulation by the media or internet (5). It is an object with a rich cultural history and it provides exemplary access to important mathematicians and important highlights in the history of mathematics.

REFERENCES

Antony or 柯志明 (s.d). *Volumes of Mouhefanggais and Spheres 牟合方蓋和球體的體積*. Hong Kong: GeoGebra Institute of Hong Kong. https://www.geogebra.org/m/kQxCtQbq (accessed 12 August 2018).

Apostol, T.A., Mnatsakanian, M.A. (2004). A fresh look at the Method of Archimedes. *American Mathematical Monthly 111*, 469-508. https://www.maa.org/programs/maa-awards/writing-awards/a-fresh-look-at-the-method-of-archimedes (accessed 12 August 2018).

Archimedes, *Περὶ μηχανικῶν θεωρημάτων πρὸς Ἐρατοσθένη ἔφοδος* [*The Method of Mechanical Theorems*]. Heath, T.H. (1912). *The Method of Archimedes, recently discovered by Heiberg*. Cambridge: University Press. http://www.wilbourhall.org/pdfs/archimedes/archimedesHeiberg.pdf (accessed 12 August 2018).

Cadav92 (2014). *A great invention: the double box-lid model, Liu Hui and his mathematic career*. https://liuhuimathmatician.wordpress.com/tag/bicylinder/ (accessed 12 August 2018).

De Temple, D.W. (1994). An Archimedian Property of the Bicylinder. *The College Mathematics Journal, 25(4)*, 312-314. https://www.maa.org/sites/default/files/0746834209146.di020763.02p0071v.pdf (accessed 12 August 2018).

Ferréol, R. (2013). *Équidomoide*. www.mathcurve.com/polyedres/equidomoide/equidomoide.shtml (accessed 12 August 2018).

Glaeser, G. (2007). *Geometrie und ihre Anwendungen in Kunst, Natur und Technik.* München: Elsevier, Spektrum Akad. Verlag.

Hogendijk, J.P. (2002). The surface area of the bicylinder and Archimedes' Method, *Historia Mathematica 29*, 199-203. http://www.sciencedirect.com/science/article/pii/S0315086002923499 (accessed 12 August 2018).

Lam, L.-Y., Shen, K. (1985). The Chinese concept of Cavalieri's principle and its applications. *Historia Mathematica 12*, 219-228. https://core.ac.uk/download/pdf/82095894.pdf (accessed 12 August 2018).

Mathenjeans (2006). http://mathenjeans.fr 2018.

Modellsammlung. https://www2.mathematik.tu-darmstadt.de/modelle/ (accessed 12 August 2018).

Moore, M. (1974). Symmetrical intersections of right circular cylinders. *Mathematical Gazette 58*, 181-185.

Netz, R., & Noel, W. (2009). *The Archimedes Codex: How a Medieval Prayer Book Is Revealing the True Genius of Antiquity's Greatest Scientist.* Boston: Da Capo Press.

Papillon, V. (2012). Mathématiques et civilisation. Les dômes du Château de Cheverny, ou Cavalieri en Chine. *Bulletin Association mathématique du Québec 52*(1), 61-64. http://archimede.mat.ulaval.ca/amq/bulletins/mar12/ChroniqueMathetciv_mars12.pdf (accessed 12 August 2018).

Roelens, M. (2015). De vogelkooi van Archimedes, Zǔ Chōngzhī, Steinmetz en anderen, *Uitwiskeling 31*(1), 8-15.

Stannard, W.A. (1979). Applying the techniques of Archimedes to the 'birdcage' problem. *Mathematics Teacher 72*, 58-60.

Strogatz, S. (2010). It slices, it dices, *The New York Times Opionator*.https://opinionator.blogs.nytimes.com/2010/04/18/it-slices-it-dices/ (accessed 12 August 2018).

Weisstein, E.W. (1999-2018). Steinmetz Solid.*MathWorld - A Wolfram Web Resource.* http://mathworld.wolfram.com/SteinmetzSolid.html (accessed 12 August 2018).

(R)EVOLUTIONS IN PROBABILITY THEORY

Students reflecting their own beliefs about mathematics by dealing with original sources from 20th century development of probability theory

Gero STOFFELS
Mathematics Education University of Siegen
stoffels@mathematik.uni-siegen.de

ABSTRACT

(i) *Materials:*

- Lime Survey for getting to know and prepare the participants of the workshop
- Excerpt from Kolmogorov (1933) (pp. 2-3)
- Original Data of pre-service teacher students' (PSTs') learning diaries, who attended the described seminar ÜberPro_WR II

(ii) The age and level of education of students to whom the workshop refers

Teachers' pre-service education

(iii*) Short Description of the Workshop*

The aim of the workshop is to share ideas of the seminar "Übergangsproblematik am Beispiel der Wahrscheinlichkeitsrechnung II" (engl. Transition problems in probability theory, short ÜberPro_WR II) – aiming to supporting students in their transition from school to university. The seminar was developed similarly to a seminar with a focus on the development of Geometry (Witzke, Struve, Clark, & Stoffels 2016) and which was refined at the University of Siegen in 2015-2017. The workshop is divided into seven phases:

1. The participants prepare the workshop by answering three questions and reading a source (7 book pages). It is important for the participants' experience as well as for the whole workshop that the chronologic order is fixed and that the given source remains unknown before answering the questions.
2. We will start discussing the online survey regarding probability theory and beliefs amongst the participants and discuss possible explanation, why it is meaningful to ask these questions. Kolmogorov (1933) claimed also, that "[t]he author set himself the task of putting in their natural place, among the general notions of modern mathematics, the basic concepts of probability theory – concepts which until recently were considered to be quite peculiar (Kolmogorov, 1933; p. v).
3. The answers given in the survey will be shared with the participants on paper cards, so that we can cluster them in an intuitive way and have the opportunity to talk about the results.
4. A short presentation (<10 min.) will introduce the outline of the seminar ÜberPro_WR II and the didactical decisions that lead to the outline. Another clustering of answers will be presented originating from the described seminar based on a categorization of mathematical world views by Grigutsch et al. (1998). This overview will motivate the next part of the workshop.
5. The participants will get an excerpt of Kolmogorov's "Foundations of Probability Theory". And shall scan this source for surprising and interesting aspects. After reading the excerpt the identified aspects will be discussed.

6. After the participants' own work with the historical sources, excerpts of the participants' diary entries, who attended the seminar ÜberPro_WRII, will be handed out to be discussed in groups of 3 to 4 participants.
7. The workshop finishes with a concluding statement, addressing the importance of students' reflections about their own beliefs and the benefits from using historical sources to engage them in their own reflections. This statement is planned to be discussed by the participants in the light of their own workshop's experiences.

For workshop participants: If you want to participate please prepare the workshop using the following link: https://umfragen.uni-siegen.de/index.php/431978/lang-en

REFERENCES

Grigutsch, S., Raatz, U., & Törner, G. (1998). Einstellung gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematik-Didaktik, 19,* 3–45.

Kolmogorov, A. N. (1933). Foundations of the Theory of Probability. Second English Edition. Translation edited by Nathan Morrison. 1956. New York: Chelsea Publishing Company.

Witzke, I., Struve H., Clark K., & Stoffels G. (2016). ÜberPro – A Seminar constructed to confront the transition problem from school to university mathematics based on epistemological and historical ideas of mathematics. *MENON* ©online *Journal of Educational Research*, 2nd Thematic Issue - The Use of History of Mathematics in Mathematics Education (pp. 66-93).

LEARNING ARITHMETIC WITH COUNTING BOARDS AND JETONS

Dominique TOURNÈS[1,3], Nathalie DAVAL[1,4], Michel MOUYSSINAT[2,5]

[1]University of La Reunion, France, [2]Institut Francophone International, France
[3]dominique.tournes@univ-reunion.fr, [4]nathalie.daval@univ-reunion.fr
[5]mmouyssinat@hotmail.com

ABSTRACT

The workshop reports on some teaching experiments which were carried out in France in elementary school classes (pupils from 7 to 9 years old) during three academic years (2014-2015, 2015-2016 and 2016-2017). The common point of student activities was the use of historical counting boards and jetons for learning the decimal numeral system and related calculation techniques.

Counting boards and jetons were one the main instruments of calculation in the West until the end of the 18th century. They were replaced progressively, not without difficulty, by the written calculation methods coming from India and used with Arabic numerals, before the latter are gradually automated by means of mechanical calculators, then electronic. This crucial place occupied by counting boards and jetons in the history of calculation led us to question the relevance of the use of these ancient material artifacts for the development or consolidation of elementary numerical apprenticeships.

In the first part of the workshop, after a short historical presentation about counting boards, participants will manipulate various models of counting boards and discover how to manage with them the simplest calculations (addition, subtraction, multiplication). In the second part, classroom experiment reports, student work, and videos will be shown as a starting point for a pedagogical discussion.

REFERENCES

Dugain, C., & Guillotin, M.-J. (2015). *Cycle 3: Remédier à une construction non-opérationnelle de la numération à l'aide d'un abaque historique*. Mémoire de master MEEF, Université de la Réunion.

Grondin, A., & Grondin S. (2016). *L'utilisation des outils de calcul et des numérations issues des civilisations anciennes à l'école élémentaire*. Mémoire de master MEEF, Université de la Réunion.

Poisard, C. et al. (2016). Les ressources virtuelles et matérielles en mathématiques: des instruments pour travailler en classe sur le nombre, la numération et le calcul. *MathémaTICE* numéro spécial 51. http://revue.sesamath.net/spip.php?rubrique135

Ries, A. (1518). *Rechnung aff der linihen gemacht durch Adam Riesen vonn Staffelsteyn*, 2nd ed. 1525, Erfurt: Mathes Maler.

Schärlig, A. (2006). *Compter du bout des doigts. Cailloux, jetons et bouliers, de Périclès à nos jours*. Lausanne: Presses polytechniques et universitaires romandes.

Trémoulu, M. (2017). *L'abaque à jetons: quel impact dans le domaine de la numération et du calcul au cycle 2?* Mémoire de master MEEF, Université de la Réunion.

Trenchant, J. (1602). *L'arithmétique de Jean Trenchant, departie en trois livres. Ensemble un petit discours des Changes, avec l'art de calculer aux Getons*. Lyon: Degabiano & Girard.

SOME ELEMENTS ON THE TRAINING IN HISTORY OF MATHEMATICS FOR TEACHERS IN FRANCE

Nathalie CHEVALARIAS
IREM de Poitiers et Lycée LP2I, Téléport 5, 86 130 Jaunay-Marigny
n.chevalarias@orange.fr

ABSTRACT

After a brief presentation of the main method of recruitment of secondary school teachers (in all disciplines, not only in mathematics), we will detail extracts of some official texts that provide the framework for the training of future teachers in mathematics: specifications for the master "MEEF", school curricula and the expectations of the national recruitment competition exam called the "CAPES". We will more particularly study the place of the history of mathematics in these texts and its theoretical impact on teachers' training.
In the second part of the article, we will specify what training (initial and in-service) is given in the history of mathematics for secondary school teachers. How many hours? During which years at university? How many training courses? On which themes? Etc. We will give examples of contents. We will also discuss the difficulties encountered by teachers, particularly in finding and working on documents that allow them to introduce the history of mathematics into their own courses.

1 Recruitment of secondary school teachers in France (some general elements)

In France, most of current teachers have been trained in two different schools. From 1990 to 2013, the initial training of French teachers was organized by the "IUFM" ("Institut Universitaire de Formation des Maîtres"), which first was independent institute and then integrated to university from 2010. Students started IUFM after the third year of university (licence degree). During the first year in this school, they prepared for the competitive exam named "CAPES"[1].When the student passed the exam, the second year was divided in two parts: part time course in a class as a trainee and part time initial training at IUFM (courses on evaluation, lesson planning, class management, …). From 2013 on, the "IUFM" was replaced by the "ESPE" ("École Supérieure du Professorat et de l'Éducation" that is to say "higher school of teaching and education") which became fully integrated in universities.

This change of schools mainly brought administrative changes with different statuses for trainers, university professors, a bigger place for research, etc. For future teachers, the concrete change had been linked with the reform of universities in 2008. From that year, the initial training of French teachers has conformed to the national LMD reform "Licence (years 1, 2, 3) – Master (years 4, 5) – Doctorate (years 6, 7, 8)". As the two years of school corresponded to years 4 and 5 of university, the creation of a Master's degree was necessary. So, now, during year 1, the ESPE prepares students for "the CAPES" exam and organizes (with the university) courses of a Master's degree in education ("Master 1 MEEF"). During year 2, the ESPE organizes courses of "Master 2 MEEF" and the students do an internship in in several classes (about 8 - 10 hours a week in total[2]). To

[1] The CAPES is a national competitive exam, consisting of written admissibility tests and, for about half of the candidates, oral admission tests. The number of candidates admitted to CAPES depends on the number of vacant positions throughout France. There is a CAPES for each discipline of the French curriculum.
[2] In mathematics, it usually corresponds to two or three classes.

become teachers, the students have to pass their Master’s degree in the university and their internship has to also be validated by the “Ministère de l’Éducation Nationale”.

This procedure is the normal recruitment of French teachers but there are other possibilities. Students can pass another Master’s degree (year 4) and prepare alone “the CAPES” exam; they can then follow only year 2 (with adjustments) of ESPE. Some students can prepare for another competitive exam (the “agrégation”) generally after a Master’s degree of research; if they before passed a Master’s degree, they only must validate their internship.

2 The Master degree “MEEF” in mathematics – some officials texts

2.1 General organization

The courses of a Master degree “MEEF” in mathematics are organized in each region by the “ESPE” and the department of mathematics of the University[3]. Different choices may be done in each region but many common points can be noticed. Indeed, the training provided by the “ESPEs” is defined by a national specification (Ministère de l’Éducation nationale, 2013-August). The following points can therefore be found in the organisation of MEEF Masters degree in mathematics:

In the first year, students go to a secondary school and shadow an experienced teacher in his or her classes. If all goes well, the student can implement some sequences with the teacher’s help. There must be four to six weeks of internship in the year, which are followed by courses where the students analyse their observations and experiments.

Most of the students' time is devoted to theoretical mathematics courses as algebra, geometry, analysis, probability, etc. These contents are evaluated in the Master’s final exams. They are also useful for the written exams of “CAPES” which are admission exams and take place in March or April. This first year ends with the oral exams of “CAPES” in June.

The second year, the students have a one-year training course (internship) as a trainee teacher. To obtain their Master’s degree, they also follow courses at “ESPE” two days a week about evaluation, lesson planning, class management, etc. They must write a Master’s dissertation on a theme linked with their experience as a teacher.

2.2 Place of history of mathematics in the contents of the Master degree

Another common point is that some courses of history of mathematics can be found in the schedules of some Master degree “MEEF” in mathematics[4]. In Poitiers, a new course was created in 2009, especially for the Master’s degree “MEEF”. What motivated this creation?

In addition to the decree of August 2013 which gives the specifications of the “ESPE”, the choices of training content are linked to the “Référentiel des compétences professionnelles des métiers du professorat et de l'éducation” (Ministère de l’Éducation nationale, 2013-July) where we can read that teachers must “have a thorough knowledge of their discipline […]. Identify the fundamental points of

[3] I only present the case of mathematics but master degree “MEEF” do exist for all disciplines of French curriculum.

[4] I will detail in next paragraph some different choices.

reference, epistemological issues and didactic problems."[5]. The contents are also linked to the definition of the CAPES oral tests. Since its creation and until 2009, one of the aims of the CAPES oral test was to "detect that candidates: […] have reflected on the aims and evolution of the discipline" (Ministère de l'Éducation nationale, 1993, p. 19, translation)[6]. In 2009, the word "history" appears in the new official text:
"This oral examination allows the candidates to show:
– their mathematical and professional culture;
– their knowledge of teaching content and curriculum;
– their reflection on history and aims of mathematics and its relationship with other disciplines." (Ministère de l'Éducation nationale, 2009, translation)[7] This text may explain the choice of my university and several others.

In practice, few oral subjects explicitly contain references to the history of mathematics. But the candidate can make it on his or her own initiative. One of my students told me that, while being asked about probabilities, he presented the problem of the Grand Duke of Tuscany. However, it is worth noting that in 2018, one of the possible subjects for the oral examination is based on an extract of Elements of Algebra (Euler) "A troop of men and women spent a thousand cents at an inn. The men paid 19 cents each, the women 13. How many men and women were there?".

2.3 Place of history of mathematics in the curriculum

Of course, the CAPES oral exams are also related to the content of the teaching curriculum. What place does the history of mathematics have in it? It may be a other reason for the introduction of history of mathematics for the future teachers. In fact, since 2009, there has been less indication of the history of mathematics in the curriculum but it can still be found in those of two last years of "lycée" (16-17 and 17-18-year-olds). Even if there haven't been explicit contents linked with history, in the introduction, it is indicated:

"Elements of epistemology and history of mathematics naturally fit into the implementation of the curriculum. The knowledge of the name of some famous mathematicians, the time when they used to live and their contribution is part of the cultural background of any student with scientific training. The presentation of historical texts helps to understand the genesis and the development of some concepts." (Ministère de l'Éducation nationale, 2010, p.1, translation)[8]

In February 2018, the Villani and Torossian's report "21 measures for mathematics teaching" reasserts the importance of the history and epistemology of mathematics:

---

[5] Original text: "Connaître de manière approfondie sa discipline […]. En situer les repères fondamentaux, les enjeux épistémologiques et les problèmes didactiques."

[6] Original text: "déceler que les candidats : [...] ont réfléchi aux finalités et à l'évolution de la discipline".

[7] Original text: "L'épreuve permet au candidat de montrer:
- sa culture mathématique et professionnelle ;
- sa connaissance des contenus d'enseignement et des programmes ;
- sa réflexion sur l'histoire et les finalités des mathématiques et leurs relations avec les autres disciplines."

[8] Original text: "Des éléments d'épistémologie et d'histoire des mathématiques s'insèrent naturellement dans la mise en œuvre du programme. Connaître le nom de quelques mathématiciens célèbres, la période à laquelle ils ont vécu et leur contribution fait partie intégrante du bagage culturel de tout élève ayant une formation scientifique. La présentation de textes historiques aide à comprendre la genèse et l'évolution de certains concepts".

“First, epistemology and history of the construction of mathematical notions, which bring a real didactic richness, are little taught in initial training. […] By taking advantage of history of mathematics, teachers place their teaching in the evolution of knowledge. In addition, students are often sensitive to the “mathematics legend”. Narrative can play a motivating role here. On the other hand, the epistemological lessons that emerge from history (the role of problems, the entanglement of concepts and techniques, the need of abstraction) are obviously made to contribute to training, in particular by overcoming short-sighted utilitarianism.” (Villani & Torossian, 2018, pp. 35, translation)[9]

Almost a year after this report, in January 2019, new curricula for the “lycée” are published and history of mathematics indeed takes a real new place. The introduction always gives a general indication: “It may be judicious to enlighten the course with historical or epistemological contextual elements. History can also be seen as a fertile source of problems that clarify the meaning of certain concepts. The “History of Mathematics” items identify some possibilities in this direction. To substantiate them, the teacher can rely on the study of historical documents.” (Ministère de l’Éducation nationale, 2019, p.5, translation)[10]. The real novelty is these items of history of mathematics! For each part of the curricula, some examples and indications are given. For example, in the part “Numbers and calculus” of the curriculum for the “seconde” (15-16 years old students), it is explained:

“The seemingly familiar notion of number is not self-evident. Two examples: the crisis caused by the discovery of irrationals by Greek mathematicians, the difference between “real numbers” and “numbers from the calculator”. It is also a matter of highlighting the gain in efficiency and generality brought by literal calculus, by explaining that a large part of mathematics could only develop once this formalism had stabilized over the centuries. It is possible to study ancient texts by authors such as Diophantine, Euclid, Al-Khwarizmi, Fibonacci, Viète, Fermat, Descartes and highlight their algorithmic aspects.” (Ministère de l’Éducation nationale, 2019, p.6, translation)[11]

In France, these new curricula may lead to an increase in training in the history of mathematics. For the moment, it is not well developed, in particular in the in-service teacher-training. Let us look at the situation in a little more detail.

[9] Original text: “Tout d’abord, l’épistémologie et l’histoire de la construction des notions mathématiques, qui apportent une réelle richesse didactique, sont peu enseignées en formation initiale. […] En tirant parti de l’histoire des mathématiques, les professeurs inscrivent leur enseignement dans l’évolution de la pensée. De plus, les élèves sont souvent sensibles à la « légende des mathématiques ». La narration peut jouer ici un rôle motivant. D’autre part, les leçons épistémologiques qui se dégagent de l’histoire (rôle des problèmes, enchevêtrement des concepts et des techniques, nécessité de l’abstraction) sont évidemment de nature à contribuer à la formation, notamment en permettant de dépasser un utilitarisme à courte vue.”

[10] Original text: “Il peut être judicieux d’éclairer le cours par des éléments de contextualisation d’ordre historique, épistémologique ou culturel. L’histoire peut aussi être envisagée comme une source féconde de problèmes clarifiant le sens de certaines notions. Les items «Histoire des mathématiques» identifient quelques possibilités en ce sens. Pour les étayer, le professeur peut s’appuyer sur l’étude de documents historiques.”

[11] Original text: “La notion apparemment familière de nombre ne va pas de soi. Deux exemples la crise provoquée par la découverte des irrationnels chez les mathématiciens grecs, la différence entre «nombres réels» et «nombres de la calculatrice». Il s’agit également de souligner le gain en efficacité et en généralité qu’apporte le calcul littéral, en expliquant qu’une grande partie des mathématiques n’a pu se développer qu’au fur et à mesure de l’élaboration, au cours des siècles, de symbolismes efficaces. Il est possible d’étudier des textes anciens d’auteurs tels que Diophante, Euclide, Al-Khwarizmi, Fibonacci, Viète, Fermat, Descartes et mettre en évidence leurs aspects algorithmiques.”

3 History of mathematics in teacher-training

To get some data on the situation in France, I sent two questionnaires. The first one was addressed to secondary school teachers. I first proposed it to all the teachers of Poitou-Charentes, the region where I teach. I got around 230 answers. To get more feedback, I asked some colleagues for help in others regions. So, the questionnaire was given to teachers both in Paris and in Franche-Comté. In those two regions, we only managed to contact teachers who are used to follow training courses (not all the teachers), but in the end, the results of the survey were similar and I therefore got 530 answers altogether. The main questions of this survey were: During your initial training did you follow a course in History of Mathematics? Do you use History of Mathematics in classroom? If yes, how? [a. in an anecdotic and occasional way? b. in a constant and integrated way? c. other option (specify)] If yes, what kind of use ? [a. some anecdotes; b. motivated introduction of a chapter, of a notion; c. reading historical texts; d. exercises at home; e. interdisciplinary works; f. other option (specify)] If no, why? [a. lack of personal interest; b. lack of formation; c. lack of knowledge; d. lack of documents; e. lack of time; f. lack of motivation; g. other option (specify)] Do you think that the use of History of Mathematics could improve your teaching? If yes, how?

Another survey was given in several universities in different regions. I send it to some members of the IREM[12], who often was secondary teachers or university teachers who teaches history of mathematics. There were two sets of questions. The first was on university courses in the history of mathematics: are there such courses in licence? If so, with what modalities (hours, semesters, contents, etc.)? Are there such courses in master degree “MEEF”? if so, with what modalities (hours, semesters, contents, etc.)? if not, why not? The second series concerned the history of mathematics courses in in-service training: number of courses offered by IREM in 2017-2018? on what subjects? courses offered by institutions other than IREM? number of courses in 2016-2017, 2015-2016 and 2014-2015?

3.1 The place of history of mathematics in studies at university

In the survey for secondary school teachers, the first question was: “During your initial training, did you follow a course in History of Mathematics?” Only 20% answered “Yes” and more than 70% of them have had less than twenty-hour courses. For example, in Poitou-Charentes before 2009, trainee teachers used to get only six hours of history of mathematics and after 2009 a new thirty-hour history course was created for them. But most teachers who have answered the survey have been recruited before 2009… In addition, the University of Poitiers does not offer courses in the history of mathematics during the three years of the “Licence de Mathématiques”. What about the other regions?

In the second survey, one of the questions was: “Do you offer courses of history of mathematics? If yes: when? How many hours? required or elective?”. Out of sixteen regions, fourteen of them offer history of mathematics in master degree “MEEF”. If they

[12] “Institut de Recherche sur l’Enseignement des Mathématiques”: The IREMs network is formed by institutes (about one per region) where primary, secondary and higher education teachers can work together to conduct research on mathematics education, and to prepare teachers’ training. Some representatives of each region can meet up in “Commissions inter-IREM” on different themes including “epistemology and history of mathematics” (there is also “didactic”, “popularization of maths”, “informatics”, etc).

don’t, history of mathematics is a required course in “Licence”. Six of them offer history of mathematics both in master degree “MEEF” and in “Licence”.
Here is a summary of the answers[13] to the survey:

	Licence		Master	
Bordeaux	S3 or S5	25h		
Caen	S6	elective 50h	S2	30h required
Clermont	S5	elective	S4	21h required
Dijon			S3 ; S4	15h + 1 seminar + 1 training day
Besançon	S3 ; S5 ; S6	18h+(24h or 29h)+ 50h		
Lille	S3 ; S6	25h elective + 72h	S3 ; S4	28h + 28h required
Limoges			S1 ; S2 ; M2	10h + 10h required + 6h (elaboration of a course)
Montpellier			M1 ; M2	M1: 16h at least M2 : 12h elective + 6h required
Paris 13 - UPC			M2	Integrated in “contents for teaching” and/or elaboration of a course (possibly linked to the master’s dissertation)
Paris 8	S1-S2-S3-S4-S5-S6	6*30h Elective (S1-S2) Required (others)	M2	
Paris - UPEM	S3-S4-S5-S6	conferences	M2	
Orléans			M2	(elaboration of a course)
Nantes	L1 ; L2	20h + 16h elective	M1	24h required
Amiens			S3	24h required
Poitiers			S1	30h required
Réunion			S3	24h required

3.2 Examples of contents of courses in master degree “MEEF”

Most common contents in the various regions are elements of epistemology and history of mathematics on some classical themes:

- History of numerations and numbers; calculus methods; algorithms
- Euclidian geometry; proofs; some elements of non-Euclidian geometry
- History of algebra; algebraic equations
- Infinitesimal calculus

Some courses are proposed to students to think about the interest and the limits of using history of mathematics in a class. The future teachers have to analyse examples of exercises for secondary students, based on history of mathematics. When these courses are given during the second year of the Master's degree, the trainee teachers are proposed to create sequences including history of mathematics. These sequences are tested in their classes and then, analysed.

[13] Table’s caption: There are three years in “License” labelled L1, L2 an L3 including six semesters labelled S1 to S6 in the columns 2 or 3. There are two years in Master labelled M1 and M2 and thus four semesters labelled S1 to S4 in the columns 4 and 5; When I don’t know the precise semester or when the course is a year-long one, I indicate L… or M…, otherwise I write the semester.

In the Master degree “MEEF” of Poitiers, as the students haven’t had any history of mathematics courses in “Licence”, the sequences are divided into different times with different aims. There are times of lecture course to give them bases and points of reference about mathematicians and history of mathematics. Students also have to read historical texts. That allows them to do mathematics through historical documents and confronts them with unfamiliar notations and vocabulary. The reading of historical texts can be followed by different types of questions as detailed by Evelyne Barbin in *De grands défis mathématiques d'Euclide à Condorcet,* (Barbin, 2010, p. 66): interpret in the historical context, compare texts, write in the manner of a mathematician, interpret in modern terms, etc. For example, in a course about infinitesimal calculus, I present some parts as problems of tangent or quadrature in Antiquity, as the method of indivisibles of Cavalieri or as the method of tangent circles of Descartes. However, the students have to read themselves some texts in different aims. About the tangent to a circle by Euclid (book 3, prop 16) they have to understand the type of reasoning; from a text on the quadrature of cycloid by Roberval, they have to write in modern notations the areas of the figure; from an extract of *Analyse des infiniment petits pour l’intelligence des lignes courbes* by the Marquis de l’Hospital, they have to calculate other differences in the same manner as $d(xy)=xd(y)+yd(x)$. The students also prepared brief oral presentations, from a historical text. Last years, the main texts they studied were extracts of *Euclids' Elements* as book I, proposition 47 (Pythagoras’theorem) or proposition 32 (sum of angles in a triangle), *La Disme* of Stevin, texts on the resolution of second-degree equations by Descartes or by Al Khwarizmi, etc. They had to present the author, explain the mathematical content of the text and analyse the link with contents of the current curriculum as can be seen in the example in annex 2.

3.3 History of mathematics in the in-service training: Difficulties and paradox

It is more difficult to find history of mathematics in the in-service training. Fifteen regions completed the survey about in-service training and only six of them answered they could offer a training in history of mathematics in 2018! For the past four years[14], only five regions have succeeded in maintaining their training in history of mathematics at the same quantity each year (“Académies” of Besançon, Paris, Lille, Montpellier, Limoges); several regions stopped providing such courses. In the “Académie” of Grenoble, the number of them has decreased each year but there is still one proposed in 2017-2018. Many others haven't given any for several years.

The different topics offered within in-service training in 2018 were “Focus on mathematics in Germany” and “Mathematics and philosophy: Model and reality” (IREM of Franche-Comté), “Ancient instruments” (IREM of Grenoble), “Mathematics and other disciplines throughout history” (IREM of Lille), “History of mathematics and algorithms” (IREM of Limoges), “History of astronomy in mathematics class” (IREM of Paris 7) and “Voting systems and general will” (IREM of Montpellier).

In the “Académie” of Poitiers, for years we have had fewer and fewer elective training (including those in history of mathematics) because there are more institutional trainings for example on algorithms and programming. So, we tried to include a touch of history of mathematics while studying other themes of training like “probability and models in

[14] In the survey, I asked them for 2014-2015; 2015-2016; 2016-2017 and 2017-2018 in order to see the evolution. Of course, many of them had proposed in-service trainings before 2014.

lycée" in 2015, or "teaching with magnitudes in collège" last year. We hope that the introduction of the history of mathematics into the new curricula will make it possible to offer more training.

Though there is little training related to history, paradoxically it seems to be appreciated. I do not have enough statistical data on this issue but in my own experience as a trainer, I have noticed some very positive reactions. For example, in the "Académie" of Poitiers, in 2014 and 2015, elements of history of mathematics were integrated in the training about probabilities whereas they weren't announced in the title of the course. And, at the end, when the colleagues had made an assessment of their daytime, it was one of the most appreciated aspects of the training! In Franche-Comté, the courses in history of mathematics have become a tradition as it can be seen in one of the answers to the survey: "The math history training at the IREM of Franche-Comté has been offered and accepted by the rectorate without interruption since 1986. It brings together between 25 and 35 participants every year. Its duration is traditionally 3 days (with sometimes a negotiation to 2 days for budgetary reasons). The mathematics and philosophy workshop has been taking place every spring since 2015 and is a continuation of a workshop from 2013 on Aristotle's syllogisms. It brings together 20 to 30 participants. Its duration has increased from one day to two days since 2017."

Secondly, there are few trainings but an actual use in class! In the survey for secondary school teachers, while 70% of teachers said they had no initial training in the history of mathematics, 70% said they use it in class. Of course, as colleagues responded voluntarily to the survey, it is likely that they did so willingly when they used the history of math, hence this high percentage. However, this result remains paradoxical in relation to their training.

3.4 Documentary resources on history of mathematics for teachers

The answers to this survey show that, even if teachers don't use history of mathematics, it isn't a lack of personal interest (only 3%) or lack of motivation (3% too). Some of them are afraid of wasting time in class (around 9%). The main difficulties indicated by the teachers are the lack of training, the lack of knowledge and the lack of references or sources. Let us look at the documentary resources that French mathematics teachers can access.

Many documents exist but it is not necessarily easy to find them and especially to study them alone. The in-service trainings do help colleagues to use these materials. Let us quote some examples of resources on history of mathematics in France.

For more than 40 years, the IREM have been linking university research and teachers, often through the organization of training courses. The IREM are quoted in the Villani-Torossian report as an important resource for the teachers. In history of mathematics, many of their members (professors of university or secondary teachers) take part to the initial training, previously described. The epistemology and history of mathematics inter-IREM commission has already produced numerous books for teachers. The first one was *Mathématiques au fil des âges* (Dhombres, Dahan-Dalmedico & Bkouche, 1987) and the last one *Passerelles : enseigner les mathématiques par leur histoire au cycle 3* (Moyon &

Tournès, 2018). Most of book references can be found in the Website of the IREM institutes[15].

One of the current issues is the way to access to these resources. As there are fewer and fewer in-service training sessions, how to make these resources known to teachers? Internet use must be taken into account. In the end, it becomes a way for the teachers to learn by themselves. In France, “Culture Maths”[16] and “CNRS - Images des maths”[17] are two institutional websites where you can find, among other things, articles on the history of mathematics. In the website of APMEP (National association of mathematics teachers), a working group on history of mathematics[18] lists documents, books, sites, videos, etc. that can help colleagues to learn about the history of mathematics and to find ideas for classroom activities.

And obviously, IREM-related websites can be found. Websites of certain IREM-groups provide documents on the history of mathematics prepared for training, such as the “IREM de Caen-Normandie”[19]. Publimath (managed by an inter-IREM commission) is a bibliographic database on mathematics education[20], including many references in the history of mathematics. Many of these documents are available online. Recently, a new use of website has been experimented on the website of the epistemology and history of mathematics inter-IREM commission. The new pages[21] created are linked to the chapters of the book *Passerelles : enseigner les mathématiques par leur histoire au cycle 3*. They allow readers to download documents for the class and they provide references and additional links.

4 Conclusion

Through the answers to the two surveys and some official texts, we have seen that history of mathematics has been maintained last years in the initial training, or even developed in some regions including mine. We can hope that the recent projects of the curriculum can contribute to the recovery of the in-service training in this field. For now, some teachers wanted to use history of mathematics, if the projects of curriculum is carried out, teachers will therefore need knowledge and ideas to construct activities for students. Particularly, the members of the IREM will be able to propose training sessions to help them.

However, even if there is more training, it will take time for as many teachers as possible to be trained. The issue of the diffusion of the resources is at stake. Of course, books will be written but a lot of teachers will keep on finding resources online. But how can we work on our own from a old text found online? How can we find additional information among the multitude of websites? How can we promote reliable and well-documented websites? These questions will be part of the challenges of the future training sessions.

[15] http://www.univ-irem.fr/spip.php?rubrique163&debut_article_numerotes=5#pagination_article_numerotes
[16] http://culturemath.ens.fr/
[17] https://images.math.cnrs.fr/-Histoire-des-mathematiques-.html
[18] https://www.apmep.fr/-Histoire-des-maths-
[19] https://irem.unicaen.fr/spip.php?article104
[20] see the article of Hombeline Languereau and Annie Michel-Pajus, in this volume (ch. 2.6).
[21] http://www.univ-irem.fr/spip.php?rubrique505

REFERENCES

Barbin, É. (2010). Nombres et grandeurs: des pythagoriciens aux algébristes de la Renaissance. In É. Barbin, (Ed.) *De grands défis mathématiques d'Euclide à Condorcet* (pp. 65–82). Paris: Vuibert.

Dhombres. J., Dahan-Dalmedico, A., & Bkouche, R. (Eds.). (1987). *Mathématiques au fil des âges*, IREM groupe Épistémologie et histoire. Paris: Gauthier-Villars.

Moyon, M., & Tournès, D. (Eds.) (2018), *Passerelles: enseigner les mathématiques par leur histoire en Cycle 3*. Bouc-Bel-Air: ARPEME.

Villani, C., & Torossian, C. (2018). 21 mesures pour l'enseignement des mathématiques. http://cache.media.education.gouv.fr/file/Fevrier/19/0/Rapport_Villani_Torossian_21_mesures_pour_enseignement_des_mathematiques_896190.pdf (accessed 22 November 2018).

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (1993). Note relative à l'épreuve sur dossier des concours externes du CAPES, du CAPET, du CAPEPS et du CAPLP2, *Bulletin officiel spécial* n° 5 du 21 octobre 1993 (pp. 19-20).

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2009). Arrêté du 28 décembre 2009 fixant les sections et les modalités d'organisation des concours du certificat d'aptitude au professorat du second degré. https://www.legifrance.gouv.fr/affichTexte.do?cidTexte=JORFTEXT000021625818 (accessed 13 March 2019).

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2010). *Bulletin officiel spécial* n° 9 du 30 septembre 2010. http://cache.media.education.gouv.fr/file/special_9/21/1/mathsS_155211.pdf (accessed 22 November 2018).

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2013 - July). Arrêté du 1er juillet 2013 Référentiel des compétences professionnelles des métiers du professorat et de l'éducation. https://www.education.gouv.fr/pid25535/bulletin_officiel.html?cid_bo=73066 (accessed 13 March 2019)

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2013 - August). Arrêté du 27 août 2013 fixant le cadre national des formations dispensées au sein des masters «métiers de l'enseignement, de l'éducation et de la formation» https://www.legifrance.gouv.fr/jo_pdf.do?id=JORFTEXT000027905257 (accessed 13 March 2019).

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2019). Bulletin officiel spécial n°1 du 22 janvier 2019. https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/95/7/spe631_annexe_1062957.pdf (accessed 10 March 2019).

Annex: Example of oral exam in Master degree "MEEF" (Poitiers)

Etude d'une méthode de résolution d'équation du second degré par Al Khwarizmi

(traduction Ahmed DJEBBAR dans *L'algèbre arabe, genèse d'un art*, Vuibert Adapt, 2005)
Quant à la justification de *un bien et dix racines égal trente-neuf dirhams*, sa figure est une surface carrée de côtés inconnus, et c'est le bien que tu veux connaître et dont tu veux connaître la racine. C'est la surface (AB), et chacun de ses côtés est sa racine. Chacun de des côtés, si tu le multiplies par un nombre parmi les nombres, quels que soient les nombres, sera des nombres de racines, chaque racine étant comme la racine de cette surface. Comme on a dit qu'avec le bien il y a dix de ses racines, nous prenons le quart de dix – et c'est deux et un demi – et nous transformons chacun de ses quarts < en segment> avec l'un des côtés de la surface. Il y aura ainsi, avec la première surface, qui est la surface (AB), quatre surfaces égales, la longueur de chacune étant comme la racine de la surface (AB) et sa largeur deux et un demi, et ce sont les surfaces (H), (T), (K), (J). Il <en> résulte une surface à côtés égaux, inconnue aussi, et déficiente dans ces quatre coins, chaque coin étant déficient de deux et un demi par deux et un demi. Alors ce dont on a besoin comme ajout afin que la surface soit carrée, sera deux et un demi par lui-même, quatre fois ; et la valeur de tout cela est vingt-cinq.
Or nous avons appris que la première surface, qui est la surface du bien, et les quatre surfaces qui sont autour de lui et qui sont dix racines, sont <égales à> trente-neuf en nombre. Si on leur rajoute les vingt-cinq qui sont les quatre carrés qui sont dans les coins de la surface (AB), la quadrature de la surface la plus grande, et qui est (DE), sera alors achevée. Or nous savons que tout cela est soixante-quatre, et que l'un de ses côtés est sa racine, et c'est huit. Si on retranche de huit l'équivalent de deux fois le quart de dix – et c'est cinq – , aux extrémités du côté de la surface la plus grande qui est la surface (DE), il reste son côté trois, et c'est la racine de ce bien.

Questions :
1) Présenter l'auteur.
2) Le problème posé est, en écriture moderne, la résolution de l'équation : $x^2 + 10x = 39$
Comment est désignée la quantité x^2 ? Comment est-elle représentée géométriquement ?
Comment est désignée la quantité x ?
3) Exposer la résolution proposée par Al Khwarizmi en langage actuel.
4) On trouve dans l'ouvrage d'Al Khwarizmi une deuxième méthode pour résoudre la même équation :

(traduction de J. Høyrup, reproduit dans KOUTEYNIKOFF Odile, « Regard historique sur la résolution des équations du second degré », Repère IREM n°28, juillet 1997)
Il y a une autre figure qui conduit à la même chose. C'est la surface (AB) qui représente le trésor. Nous voulons donc lui ajouter ses dix racines. Pour ce faire, nous divisons les dix en deux, ce qui devient cinq, et nous construisons deux surfaces sur deux côtés d'AB, à savoir les surfaces G et D, dont les longueurs égalent cinq, ce qui est la moitié des dix racines, tandis que la largeur de chacun d'eux égale le côté du carré AB. Alors cinq sur cinq nous manque opposé au coin de AB : ce cinq étant la moitié des dix racines que nous avons ajoutées à deux des côtés de la première surface. Nous savons donc que la première surface, qui est le trésor, et les deux surfaces sur ses côtés, qui sont les dix racines, font ensemble trente-neuf. Pour compléter la grande surface en carré, seul cinq sur cinq, ou vingt-cinq, fait défaut. Nous ajoutons à ceci trentre-neuf, pour compléter la grande surface SH. La somme est soixante-quatre. Nous extrayons sa racine, huit, qui est une des côtés de la grande surface. En lui enlevant la même quantité que nous lui avons ajoutée antérieurement, à savoir cinq, nous obtenons trois comme reste. Ceci est le côté de la surface AB, qui représente le trésor ; c'est la racine de ce trésor et ce trésor lui-même est neuf. Ceci est la figure

a) Faire la figure correspondant à cette deuxième méthode.
b) Comment peut-on utiliser cette figure en classe de $1^{\text{ère}}$ S ?

DEVELOPING GEOMETRIC PROPORTIONAL THINKING TO 6TH GRADE STUDENTS WITH THE USE OF A HISTORICAL INSTRUMENT OF ERRARD DE BAR LE DUC

Triantafillos DELIGIANIDIS[1], Kostas NIKOLANTONAKIS[2]
University of Western Macedonia, Florina School of Education,
Department of Primary Education, P.O. Box 21, Florina, 53 100, Greece
[1]soflak@yahoo.gr, [2]knikolantonakis@uowm.gr

ABSTRACT

This paper describes a teaching experiment on the notion of Proportion to a Sixth Grade class in Greece. The teaching was based on the incorporation of History into Mathematics teaching through the study of a primary historical source and the use of a reconstructed 16th-century historical instrument of Errard de Bar-le-Duc (Errard, J., 1594). Prior to the intervention the pupils' personal geometric knowledge was investigated with the help of a pre-test, consisting of the ability to solve verbal and numerical proportions, to solve analogue tasks and to distinguish proportional and non-proportional situations. This was followed by a didactic intervention, which included a series of eight lessons. The collection of data during the series was done with field observations, recordings and worksheets. Upon completion, the pupils' personal geometric knowledge was rechecked through a post-test. Also, with the help of a questionnaire, the students' view of the teaching series and the integration of the History of Mathematics were explored. The results of our post-test showed that there was a clear improvement in the ability to formulate the students' mathematical proportional thinking. They improved their performance, used more strategies to solve the problems, gave examples of recognition and problem-solving, developed the capacity to analyze quantities in a given situation, and applied the multiplication relationship to their data. They have also been able to work together and harness the benefits of teamwork.

Keywords: Ratio and Proportion, History of Mathematics, historical tool, Primary Education

1 Introduction

Ratios and proportions are widely used in mathematics, science, and everyday life (Karplus, Pulos & Stage, 1983).

However, out of the fifteen chapters of the section dealing with Ratios and Proportions in the School Manual of the sixth grade, two activities only approach the subject geometrically.

Researchers such as Lawson & Chinnappan (2000) argue that Geometry offers the framework for resolving various problems and developing learning strategies to solve problems in other thematic areas.

On the other hand, the use of the History of Mathematics and its integration into teaching is almost non-existent in Greek reality. It is limited only to some historical notes at the beginning of the chapters, which are usually left unused in teaching.

At the same time, the new Curriculum (Pedagogical Institute, 2011) provides for the inclusion of handwritten and digital tools in the teaching of mathematics.

We therefore consider a geometric teaching approach of ratios and proportions useful and constructive, from the perspective of History and the use of a historical tool.

2 Research Methodology

The design of the interventions was based on the hypothesis that students can learn mathematical notions by following a history-based teaching approach that uses History as a "tool" (Jankvist, 2009). Our work follows the principles of the "modules approach". They may consist of material focused on a particular subject of the curriculum, suitably adapted for short-term classroom intervention. However, they may have a larger extent, requiring more teaching hours, and are intended to study a subject that is not included in the curriculum. The use of History of Mathematics through teaching courses can be done in a variety of ways, such as studying texts from original sources or working with students with worksheets. The worksheets conduct students to work on the mathematical notions and they can contain also historical information, biographies and other material for the didactic situations. Finally, on a larger scale, this approach may consist of courses or a study of books, or extensive research work by students. The historical material is the object of a project, in our case the study of a primary source and reconstruction of the historical tool is the entry, which will ultimately lead to the discovery of its properties by the students.

Bussi (2000) argues that History of Mathematics can be applied to school activities by studying and constructing copies of ancient tools and other artifacts that are reconstructed with the help of historical sources. She believes that almost all the tools of ancient and modern technology incorporate a lot of mathematical knowledge that is accessible through a careful and targeted study. They can be used as a major mobilization for both young and old students, but also adults. Particularly students, who do not love Mathematics, enjoy working with physical objects that are closest to their everyday experiences. Cerquetti – Aberkane (1997) has already proposed some activities with a tool from Errard's book.

The main research question was: *How effective in shaping the students' Proportional thinking, can be an experimental geometric approach that uses the study of a primary historical text and a reconstructed historical instrument?*

Eight didactic interventions were designed aiming at studying translated primary sources and constructing the tool based on the historical text to discover the properties of Proportions incorporated in the use of the instrument and ultimately to implement these properties for measurements on the real space. During these interventions' students worked alternately in small groups on outdoor activities and individually in classroom.

3 The implementation

The implementation started in the school yard, where groups of students were assigned a measuring distance assessment. The diversity of the results demonstrated the need to use an instrument to measure them with greater precision. Then, we presented a translated excerpt of the historical text, entitled *La géométrie et pratique générale d'icelle*, (Errard, J., 1594) which describes an instrument suitable for measuring inaccessible distances. The pupils constructed a similar instrument, according to the instructions of the text and used it during their experimental work to discover the properties of Proportions of the similar triangles formed via the use of the tool.

Finally, they used these properties to resolve problems of measuring inaccessible distances and moreover to design a scale plan of the school surroundings.

Our main path was to use and evaluate the idea of passing from the experiment in the real space into the paper-pencil classroom environment and from the real space drawing into geometrical figure. Via these interventions we have tried to see how our students developed their geometrical – proportional thinking by passing from experimental geometry (Natural geometry) into a deductive way of thinking (Natural axiomatic geometry-Euclidean geometry).

3.1 First Teaching Intervention

Students were divided into groups of 4-5 people. The teams went to the school yard, where they were given a worksheet. They were asked to work in whatever way they wanted and make estimations for the height of different objects from their own environment, such as the flagpole in the yard of the school or a tree in the courtyard. After initially locating objects in real space, students tried to estimate the heights visually, by using various objects of known length as a measure. Then they tried to describe how they worked to get the most accurate estimation. After completing the process and the worksheet, they returned to the classroom where they had to present their assessments and discuss their documentation. The diversity of the results demonstrated the need to use an instrument to measure them with greater precision:

Harris: Yes, but if we had something to measure it (a tree), we wouldn't have had so many variations.
Teacher: What do you mean?
Harris: Yeah, an instrument ...
George: Yes, but how? Would we go over the tree to measure?
Teacher: Do you think it is difficult to measure the tree because it is tall?
George: Yes.
Teacher: And if there was a tool that we could measure from a distance, would we have variations in our measurements again?
George: Yes, but fewer.
Teacher: Why is this?
Harris: Someone might hold it wrong, another slightly above ...
Teacher: Does it depend on the use of the tool
Harris: Yeah.

3.2 Second Teaching Intervention

In this phase we attempted to integrate the History of Mathematics into the teaching. For this purpose, an authentic 16th-century text by Jean Errard de Bar-le-Duc had been selected. This was the first and second chapter of his work “Géométrie et pratique générale d'icelle”. The text was in French and it was accompanied by a translation in Greek as children did not know the language. For the sake of clarity, those parts that were considered to be particularly difficult for children were simplified. Using the projector and photocopies, Errard's historical text and tool was presented to students. The children showed great interest in Errard's book and began to ask questions about its authenticity, its origin and if there were any similar texts printed at that time in Greece:

Thomas: What does it write in the title, sir?

Teacher: “Geometry and General Practice”

Raphael: Is it printed in Paris? Is it authentic?

Teacher: Yes. In Paris and it is the authentic one.

Thomas: Did we (the Greeks) have such books back then?

Teacher: It would be very difficult, since we were in a period of Ottoman rule. But since you mention it, I want you to look at some of the references later in the text and discuss them.

LA GEOMETRIE
ET
PRACTIQVE GENERALLE
D'ICELLE.
Au Tres-Chrestien Roy de France & de Nauarre,
Par I. Errard de Bar-le-duc, Ingenieur ordinaire de sa Maiesté.
SECONDE EDITION.
A PARIS,
Chez GVILLAVME AVVRAY, au haut de la ruë S. Iean de Beauuais, au Bellerophon couronné.
Auec priuilege du Roy.

Figure 3.1: The cover page of the Book's 2nd edition

Figure 3.2: One image of tool's use

The presentation aimed at students to find out the purpose for which the text was written, and that similar concerns about measuring inaccessible distances have occupied man from very old age. It also helped pupils to realize that the use of such a tool could solve the problem they faced in the previous phase and stimulated their interest in the construction of this tool:

Spyros: The tool in the first image looks different than that in the second image.

Teacher: It's the same tool. Just the first one shows how to use it.

Panagiotis: It measures from the one side of the river to the other.

Teacher: Right. It targets a particular point, the tree.

Eleni: Some triangles are formed ...

Teacher: Yes, not really. They are conceivable. But for these we will talk below.

3.3 Third Teaching Intervention

The aim here was to carry out the construction of the tool by the students themselves. For this purpose the teacher ensured that each group had the materials for the construction of the instrument: three numbered bars, two by one meter and one by hundred and twenty centimeters, a movable connector for the vertical bar and a foldable joint with a viewfinder for the moving rod. We also used a photocopy of the historical text for each child, with translated instructions for constructing the instrument:

"So I would like the construction to be as follows: two brass bars completely straight be linked to diabetes as AB, AC, and each one and a half each foot approximately, and one inch wide, moving and rotating in center A. Next, on the AB bar (which we will call a

position), there is an engraving on which will apply a semicircle labeled here as D, on which one stands another EF bar (called base) which will be of similar length but equal width with the others, or a little narrower, and applied with the semicircle against length of AB, and in a way that in the center of E (which will be right on the straight line AB), we can close and make the angle we want with rod AB, along and joining the surfaces of the rods of the seat and its movable AC bar, and which, however, can be stabilized through a screw applied to the so-called semi-ring. Being so made, and the binoculars are focused on the ABC points (as we usually do in all instruments) each of the so-called three bars will be divided into 300 equal parts or in any other number we would like".

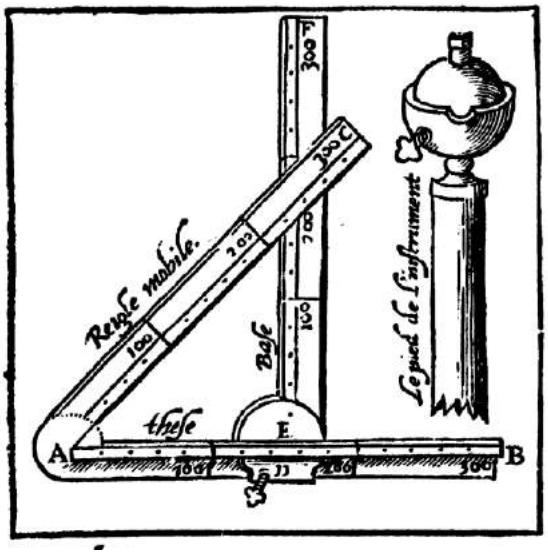


Figure 3.3: The picture for the description of the tool

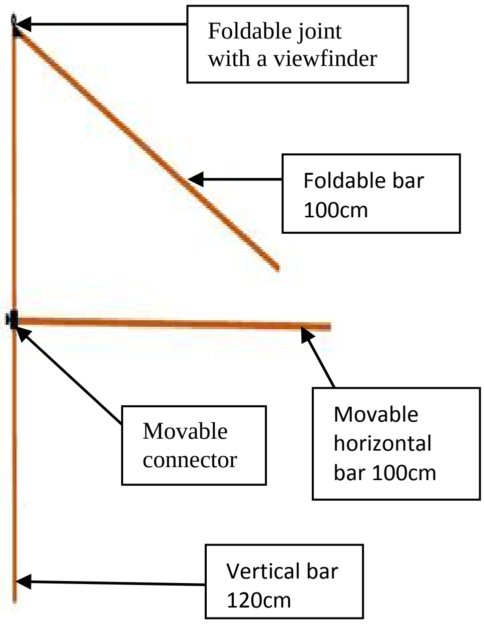


Figure 3.4: The parts of Errard's tool in modern scheme

The children's attention was greatly gained by the images of the historical text, which in addition to their artistic value, were able to provide a wealth of information on the construction material and how it was assembled:

Konstantinos: It's made of metal and it has a screw on it... it looks like a machine.

Helen: This ball, here, what is it?

Teacher: It's the joint. He presents it in detail in the text below.

The whole construction was generally considered simple and it was completed in four simple steps. The assembly of the tool caused great satisfaction to students and stimulated their creative mood:

Thomas: This is the same as in the picture!

Teacher: But of course, and you made it yourself!

Marianna: When will we use it?

Teacher: From the next lesson.

At the same time, teamwork and interactions between team members created the appropriate conditions for exploring the attributes of the tool provided in the next interventions.

Figure 3.5: Picture of the tool assembled by the students

3.4 Fourth Teaching Intervention

At this point we tried to train students to use the Errard's tool, by measuring accessible distances in the schoolyard, transfer their measurements onto the paper, find that two rectangular triangles are formed and that the distance requested is the basis of the big triangle.

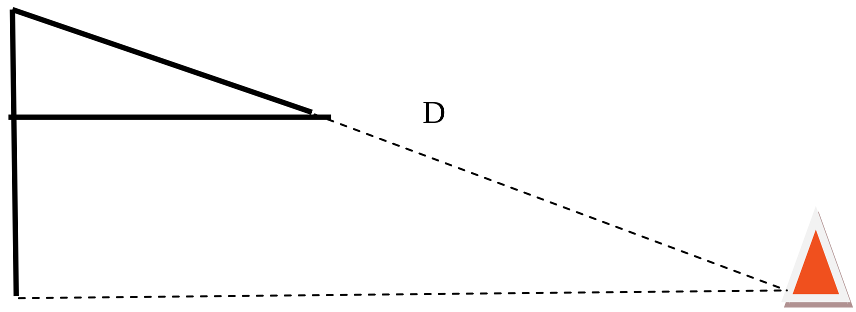


Figure e 3.6: Using the tool to measure accessible distances in the schoolyard

Students worked with the Errard's tool using a worksheet, confirmed their measurements with a tape measure and transferred the triangles formed on a sheet of paper. They kept a record of all the measurements done in the yard. In order to simplify the process, we took AB=20cm. in all measurements:

	segment	1st measurement in cm	2nd measurement in cm	3rd measurement in cm
Group A	AB	20	20	20
	AC	120	120	120
	AD	50	65	70
	CE	300	400	450
Group	AB	20	20	20
	AC	120	120	120

B	AD	60	77	62
	CE	420	520	450
Group C	AB	20	20	20
	AC	120	120	120
	AD	85	75	100
	CE	330	540	480
Group D	AB	20	20	20
	AC	120	120	120
	AD	50	35	65
	CE	300	250	370

Table 3.1: The results of the measurements in the yard

Although students used the tool for their measurements, its properties were yet completely revealed. They will be discovered in the next lessons.

3.5 Fifth Teaching Intervention

This time pupils worked in the classroom, using their notes and schedules from their previous session. They presented their findings in class and discussed the results. The comparison led to the conclusion that the sides of the triangles are associated with a proportional relationship.

At this point, we looked back at the historical text to find references about the property of Proportion that we had just discovered. It was a real surprise for students to realize that some other people, centuries ago relied on this piece of knowledge of mathematics.

3.6 Sixth Teaching Intervention

The knowledge just gained from the previous activity had to be verified in practice and applied to a multitude of cases so as to be generalized and formalized in the consciousness of students.

So, we asked the children to work in groups, alternating roles to get the proper familiarity with using the tool. They placed a variety of objects (cones) at accessible distances in space and then targeted them with the tool and tried to measure their distance. Then they measured the distances by a tape measure and confirmed their calculations**:**

Group	Distance measured with Errard tool	Distance measured with tape measure
A	4,60m.	4,80m.
B	2,40m.	2,40m.
C	3m.	3m.
D	3,90m.	3,80m.

Table 3.2: The divergence in pupils' measurements

When recording the results, it was found that there was little variation in the measurement results between the instrument and the confirmation with the tape measure. The students were able to give some good explanations for this:

Marianna: Maybe we did not just target correctly the cones (objects).

Panagiotis: The horizontal bar is divided by 5cm. Our measurement can be between those spaces…

Angelina: The instrument has to be upright, otherwise you get a false measurement!

3.7 Seventh Teaching Intervention

The students at this lesson worked in groups in the yard of the school. They placed various objects, as indicators, on the four corners of the building, as well as other selected parts of the courtyard. Then they made the measurements with Errard's the tool and recorded their results. The teams took different points and worked alternately.

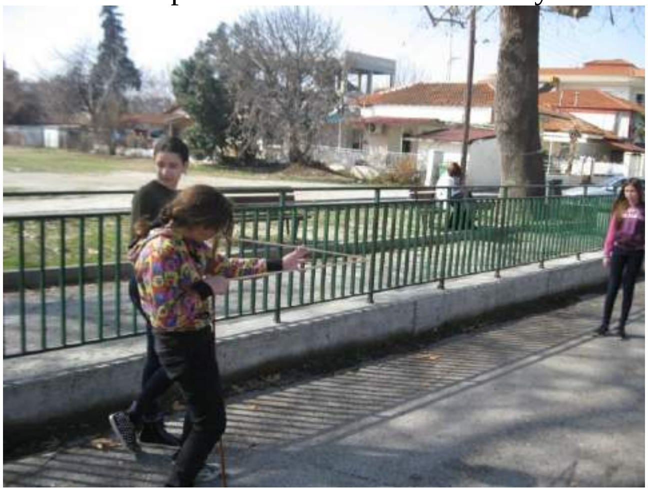


Figure 3.7: The students working with the tool in the yard

The results of the measurements were not any longer confirmed by metric measurements, as the distances were considered to be inaccessible and the functionality of the tool to measure such distances had also been confirmed:

Group	Type of measurement	Measurement result
A	Width of the building	28m.
B	Width of the schoolyard	85m.
C	Length of the building	54m.
D	Length of the schoolyard	130m.

Table 3.3: The measurement results of the building

3.8 Eighth Teaching Intervention

In this last session the students worked individually in the classroom. They had a draft of the school from the previous activity with its dimensions and now they had to transfer their measurements to the paper and to scale the ground plan of the building and the yard of the school using their geometric instruments.

4 The evaluation

The results of the post-test gave us an insight into the configuration of the pupils' personal space as it was formed after the didactic intervention. Overall, the results showed a significant variation in pupil's performance compared to the pre-test results.

If we look at the results of the post-test in general and compare it to the pre-test, we can say that there is clearly an improvement in both quantitative and qualitative results.

The tests were based on four main axes: a) the distinction between proportional and non-proportional situations; b) the ability to solve analogue projects; c) the ability to solve numerical and verbal proportions and d) the ability to compare ratios.

The post-test gave us useful information. In the first axis, students had to distinguish proportional from non-proportional situations such as, for example, the price and the quantity of a product. Here we had no significant improvement since the correct answers increased from the pre-test from 110 to 114.

In the second axis where we examined the ability to resolve proportional projects, we had a significant improvement. Here, the right answers grew in the post-test from 21 to 38, the wrong answers decreased from 32 to 24 and the students who did not respond at all, also dropped from 16 to 7. Apart from the quantitative data, we also had a significant improvement in the qualitative characteristics of the pupils' abilities. Thus, in most cases, students used multiple heuristic methods to solve the exercises, adopted the multiplicative reasoning to solve the problems, motivated their choices and tried to solve even those problems that could not control. Characteristic is the case of Konstantinos, who, after seeing a tree with a height of 1.5 m makes a shade of 3m and the height of another tree shading 9m is being sought, says: "*The height of the tree is half, i.e. 4.5m.*"

We also had the same improvement in the third group of exercises where we looked at the ability to solve numerical and verbal proportions, where for example pupils had to complete the fourth term of a proportion with a number or a word. Here the correct answers to the post-test increased from 45 to 58, the errors dropped from 33 to 30, while students who did not respond at all dropped from 16 to 4.

The last group of exercises that examined students' ability to compare ratios gave us the following comparative results. The correct answers to the post-test increased from 19 to 27, the errors decreased from 18 to 13 and the students who did not respond at all dropped from 9 to 6.

Improving performance can be seen in the diagram below, which compares student performance in both pre-test and post-test:

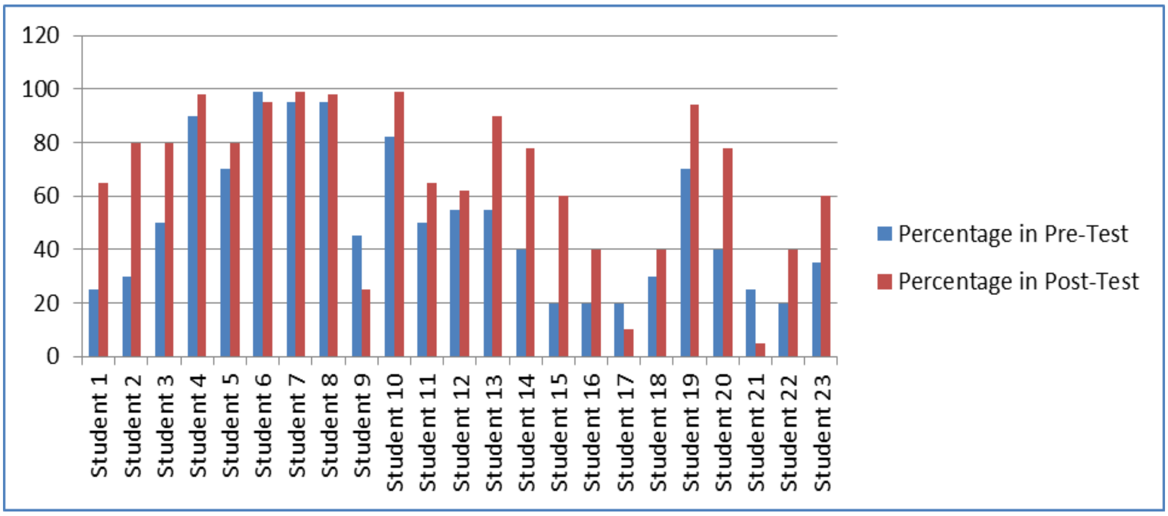


Diagram 4.1: Students' performance in pre- and post-test

Thus, we see that most students after the intervention have made significant progress, with about one third of them approaching excellent performance.

Summing up the above and attempting to answer our research question, we would say that our experimental geometric approach to Ratio and Proportion has succeeded in bringing positive results to the development of the proportional thinking of our students.

5 The evaluation of the teaching series by the students

The evaluation of the teaching series was based on a questionnaire and aimed at the following axes: the general evaluation, the historical text, the tool and the usefulness of the series.

Summarizing the children's answers to the general assessment of the teaching series, all students responded positively. The answers they gave to justify their point of view were quite interesting and seem to converge to the following:

Eleni: They (the lessons) were useful, educational and entertaining.

Harris: It gave me an incentive.

Angelina: I like to spend my time with Mathematics

Chronis: I liked that we all worked together

Konstantinos: It helped us in Mathematics.

We see, therefore, that all students have positively evaluated the possibility of other similar practical teaching interventions because they believe that they have a different benefit in their learning work.

In the second part of the questions, where the text was evaluated, students were given the opportunity to give multiple answers. The first question asked students to characterize the text. According to their answers, all 23 students in our sample, found the historical text interesting, 21 students found it pleasant, 18 found it comprehensible, while one added that he found it detailed. In the second question, the children had to answer if the text was related to Mathematics. All students responded positively and completed what concepts were included in the text. The most typical answers were: “rectangular and equal triangles, angles of the triangles, ratios and proportions of triangle sides, distance measurement, straight lines and Euclidean Geometry”.

We see that most students have responded positively to the text, since they considered it interesting, enjoyable and comprehensible, while everyone was able to understand the relationship of the text with mathematics and to distinguish the mathematical concepts included in it.

The next category of questions involved the evaluation of the tool itself. Generally, their experience of building and using the tool seems to be positively evaluated by the students themselves, since most of them found the construction and use of the tool easy, they feel it helped them and they would like to construct similar tools in the future:

Stefanos: I like to see things I've construct with my classmates.

Angelina: Nice experience and helps you learn.

Panagiotis: Tools are interesting and you learn a lot from their use

Harris: I want to discover new things and go deep into the magic of mathematics.

Vasiliki: It was fun.

Nikos: I want to learn about other tools.

Great impression also made to the students that a 16th century tool was used to teach mathematics. Thus, 7 students were surprised that with a historical tool they were able to measure inaccessible distances, 5 others that they had the experience to see in practice a 16th century idea and other 2 that they did not know the existence of such a tool.

The last question in the questionnaire attempted to assess the pupils' view of the usefulness of the teaching series and its practical use. Here, 19 students estimated that they will use what they have learned in their daily lives.

The following responses of the children, who responded positively, stand out:

Spyros: I will need them if I become a mathematician, engineer or topographer.

Sophia: I can appreciate distances better.

Harris: I can measure distances without a measure.

Marianna: I can calculate the actual distance of two areas on a map.

Raphael: To calculate inaccessible distances.

6 Conclusion

By making a general assessment of the eight didactic interventions, we would say that the goals were largely achieved. The pupils worked together, discovered the benefits of incorporating the History of Mathematics into their work, construct a historical tool with their own hands, used it, discovered the properties of similar triangles, and led to the discovery and use of scales. They therefore met, in an experiential way, the ratios and the proportions our research sought.

It is true that the use of History in this work was relatively limited. This happened because we had chosen to approach the subject in a way that History is mainly used as a "tool". Also, the extent of the teaching intervention was limited and there was always the risk of getting away from our original goal which was, developing geometric proportional thinking to 6th Grade Students.

Nevertheless, a descent attempt was made to rebuild the critical steps of the historical development of the tool, by leading a sequence of mathematical problems of scalar difficulty in order for the student to build his/her knowledge on the previous experience (Tzanakis et. al 2000).

There is, of course, much space for a deeper and more detailed discussion on the key points of the historical evolution of the subject we are examining and the way they have affected mathematical knowledge. We hope, however, that we will have the opportunity to develop them further into some other project.

REFERENCES

Bartolini Bussi, M. G. (2000). Ancient instruments in the mathematics classroom. In J. Fauvel, & J. van Maanen, (Eds.), *History in mathematics education: The ICMI Study*, New ICMI Study Series, vol. 6 (pp.343-351). Dordrecht: Kluwer.

Cerquetti-Aberkane, F., Rodriguez, A., & Johan, P., (1997). *Les maths ont une histoire activités pour le cycle 3*. Paris: Hachette éducation.

Errard, J. (1594). *La géométrie et practiqve generalle d'icelle*. Auvray.

Jankvist U. T., (2009), A categorization of the "whys" and "hows" of using history in mathematics education, *Educational Studies in Mathematics*, *71*(3), 235 – 261.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Early adolescents' proportional reasoning on ‘rate’problems. *Educational studies in Mathematics, 14*(3), 219-233.

Lawson, M. J., & Chinnappan, M. (2000). Knowledge connectedness in geometry problem solving. *Journal for Research in Mathematics Education, 31*(1), 26-43.

Pedagogical Institute (2011). Curriculum for Mathematics in Compulsory Education. Retrieved June 11, 2017 from http://ebooks.edu.gr/2013/newps.php

Tzanakis, C., Arcavi, A., de Sá, C. C., Isoda, M., Lit, C.-K., Niss, M., de Carvalho, J. P., Rodriguez, M., & Siu, M. K. (2000). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel& J. van Maanen (Eds.), *History in mathematics education: The ICMI study*, New ICMI Study Series, vol. 6 (pp. 201–240). Dordrecht: Kluwer.

MATHEMATICS AND SCIENCE HISTORY CONTEXTS OF MATHEMATICS TEXTBOOKS IN TURKEY

Burcu DURMAZ
Süleyman Demirel University, Turkey
drburcudurmaz@gmail.com

ABSTRACT

In last few years in Turkey, it has been suggested and emphasized that giving details of history of mathematics in the curriculum may help increasing students' attitudes towards mathematics and success (Ministry of Education, 2013). In addition to this it is a known fact that science lessons, which are integrated with not only scientists' successes but also experiences and challenges, have a potential to increase the students' science success (Lin-Siegler, Ahn, Chen, Fang, and Luna-Lucero, 2016). In this context, the lives and study of mathematicians and scientists may make mathematics lessons more meaningful for the students. But when we examine mathematics curriculum from primary school to high school grades, we cannot see such an emphasis on primary school mathematics program as well as with those in other grades. For these reasons, this study aims to investigate the density and effectiveness of the context and content of the history of mathematics and science in Turkish mathematics textbooks across all grades. For this purpose, two independent researchers analyzed 21 mathematics textbooks for 1st grade to 12th grade. All of the textbooks are still used in mathematics classes even if they were published after 2013 (the revision year of the mathematics curriculum). The data in the textbooks were classified by grade level; name of the mathematician/scientist; the delivery of information (non-historical, anecdotes, notes with daily use of/application the historical knowledge, notes without daily use of/application the historical knowledge, etc.), and learning area (geometry, numbers, algebra, etc.). By clarifying the criteria for the textbooks, the researchers discussed the conflicts of their work and opinions after working on the data separately. At the end of the study it was determined that neither the content about mathematicians and scientists nor information about the origins of the mathematical concepts or symbols are sufficiently given in the textbooks. It was also seen that the limited context of history of mathematics and science was primarily given at the beginning of the new subjects and meant only for reading and not to use along with the lesson. Scholars have noted that the weakest way of using historical content in mathematics lessons is giving them as short historical anecdotes at the beginning of the new chapters (Baki & Bütüner, 2013; Erdoğan, Eşmen & Fındık, 2015). Lastly, it was found that the textbooks written collaboratively by scholars, teachers, and experts were better than those by means of history of mathematics and science. According to the findings of this research, it can be said that history of mathematics and science contents are not enough to nurture the students' mathematical knowledge and increase attitudes towards mathematics (Ministry of Education, 2013). Thus, it is suggested that the authors of the mathematics textbooks need to be trained on how to connect the history of mathematics and science with mathematics curriculum in effective and useful ways.

REFERENCES (10pt fonts)

Baki, A., & Bütüner, S. Ö. (2013). The ways of using the history of mathematics in 6th, 7th and 8th grade mathematics textbooks. *Elementary Education Online*, *12*(3), 849-872.

Erdoğan, A., Eşmen, E., & Fındık, S. (2016). Ortaokul matematik ders kitaplarında matematik tarihinin yeri: ekolojik bir analiz, *Marmara Üniversitesi Atatürk Eğitim Fakültesi Eğitim Bilimleri Dergisi*, *42*(42), 239-259.

Lin-Siegler, X., Ahn, J. N., Chen, J., Fang, F. F. A., & Luna-Lucero, M. (2016). Even Einstein struggled: Effects of learning about great scientists' struggles on high school students' motivation to learn science. *Journal of Educational Psychology*, *108*(3), 314.

Ministry of Education (2013). Ortaokul matematik dersi (5-8. sınıflar öğretim programı). Ankara, MEB Talim Terbiye Başkanlığı Yayınları.

USING HISTORY TO TEACH COMPLEX NUMBERS

Hui GU[1,3], Bin HOU[2,4]
[1]Wuxi Qingshan High School, China, [2]Wuxi Taihu High School, China
[3]guhuisky@163.com, [4]1021000396@qq.com

ABSTRACT

As usual we used textbook approach to introduce the imaginary unit *i*, whose square equals to -1. When class ended, a student asked if negative numbers could have logarithms. Does log(-1) exist? His question reminded us to reconsider how to make more sense of the imaginary unit *i*. There must be some more natural and intuitive alternative to define it. The historical development of mathematical concepts and the way that they develop in an individual mind are observed to be very much alike (Jankvist 2009). With an exploration of a history-based classroom practices and students' performances in non-routine problems solving, our research aims to examine when and how, and in what context to introduce the imaginary unit and what geometric intuition can serve to enhance students' understanding.

The case study took place with a class of 11th grades students (16-17 year old) in our high school. The theoretical framework used to study the teachers' classroom practices is the double approach (Vandebrouck 2012). We articulate both the analysis of students' and teachers' activities in order to identify, understand and interpret the link between the teaching of complex numbers using original sources and the corresponding students' activities. In the highlight of theoretical framework, we attempt to answer the following two research questions:

1. Can students' own solutions to a cubic equation together with Cardan's attempt, Leibniz's doubt and Bombelli's discovery pave the way for the introduction of the imaginary unit *i*?

2. Can geometrizing the imaginary unit by Wessell, Argand, and Gauss enable students to visualize the concept and acquire an intuitive understanding on dealing with non-routine, open-ended "real-life" challenges thereafter? The data that we collected consists of videos of classroom sessions and interviews with students.

Our research indicates that teachers themselves first need to be well equipped with the history of mathematics so as to better judge how students should acquire such knowledge. The concern about the logarithms of negative numbers raised by the student mentioned above can be encouraged to be an interesting after-class historical project. Meanwhile, we confirm that the utilization of original sources in mathematics classroom can activate students' engagement to make mathematical discoveries, when different strategies flash into their minds and a repertoire of diverse representations are compared and the best is then chosen. Also, we address some obstacles to overcome. In the presentation, a detailed teaching scenario with the historical package, teachers' implementations and students' activities as well as implications for teaching and research will be presented and discussed.

Questions discussed in our class included the following:

Question 1: Find the solution to the equation $x^3 = 15x + 4$.

Question 2: Calculate the above cubic equation with Cardano's formula; share what you discover.

Question 3: Leibniz considered the following situation: Let *x* and *y* be positive values, and $x^2 + y^2 = 4$ as well as $xy = 5\sqrt{5}$. Can you find the value of $x + y$ and *x*, *y* respectively? Explain why.

Question 4: With Bombelli's equation $\sqrt[3]{a+b\sqrt{-1}} + \sqrt[3]{a-b\sqrt{-1}} = c + d\sqrt{-1} + c - d\sqrt{-1} = 2c$, can you shed new light into $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$ (this is what we got when we applied Cardano's formula to solve the cubic)?

Question 5: Comparing with the number -1, what conclusion can you draw about the algebraic and geometric representation of the imaginary unit *i*?

Question 6: Share your ideas about Wessel's work with your classmates and infer the geometric representation for $\sqrt{-1}$ (Ever since Wessel, then, multiplying two directed line segments together has meant the two-step operation of multiplying the two lengths, with length always taken to be a positive value, and adding the two direction angles. These two operations determine the length and direction angle of the product, and it is this definition of a product that gives us the explanation for what $\sqrt{-1}$ means geometrically).

Question 7: A treasure hunt with imaginary numbers (quoted from one two three… infinity by George Gamow.)

REFERENCES

Jankvist, U. T. (2009). A categorization of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics*, *71*(3), 235-261.

Vandebrouck, F. (Ed.). (2012).*Mathematics classrooms: Students' activities and teachers' practices.* Rotterdam: Sense Publishers.

TSCHIRNHAUS' TRANSFORMATION MATHEMATICAL PROOF, HISTORY AND CAS

Uffe Thomas JANKVIST[1,4], Morten MISFELDT[2,5], Mario Sánchez AGUILAR[3,6]

[1]Danish School of Education, Aarhus University, Denmark
[2]Aalborg University and Høgskolen i Oslo og Akershus, Denmark-Norway
[3]CICATA, Legaria, InstitutoPolitécnico Nacional, Mexico
[1]utj@edu.au.dk, [2]misfeldt@learning.aau.dk, [3]mosanchez@ipn.mx

ABSTRACT

The paper addresses the potential of using history of mathematics in combination with ICT to illustrate the notion of proof and proving, including also the role of examples and counterexamples, to undergraduate students and relates such use to research findings regarding students' difficulties. The historical case used to illustrate this is Tschirnhaus' transformation from 1683, suggesting a presumed method for solving *n*-degree algebraic equations, and we draw on a small empirical example from a Roskilde University undergraduate mathematics student project report. Based on this example we further discuss the potential use of history in combination with CAS in a classroom setting drawing on the notions of justificational mediations, sociomathematical norms and scientific debates.

Keywords: History of mathematics, original sources, CAS, mathematical proof and proving, counterexamples, "the mathematical underworld", justificational mediations.

1 Introduction

It has been pointed out previously that HPM research often do not make use of general mathematics education research frameworks (Jankvist et al., 2015). This is a pity for at least two reasons. Firstly, mathematics education research frameworks have a variety of lenses to offer HPM. Secondly, it may make it easier to communicate HPM research to the rest of the mathematics education community. In this paper, we shall address the topic of mathematical proof, examples and counterexamples under the assumption that HPM has much to offer in this regard – and that digital tools may also have a significant role to play.

Before we begin, a warrant should be provided to the reader, which is that this is a theoretical paper. Taking our departure in what the literature states regarding students' difficulties regarding mathematical proof and counterexamples, we outline a selection of theoretical constructs to potentially be used in relation to using elements of the history of mathematics in this endeavour. We also address elements of potentially using digital technologies in this respect, this in particular in relation to an illustrative and empirical example which serves as an invitation to think further. The example draws on the history of the so-called "Tschirnhaus' transformation" from 1683.

2 Students' difficulties with mathematical proof

As phrased by Duval (2007, p. 137), "Proof constitutes a crucial threshold in the learning of mathematics. Why do so many students not succeed in truly crossing it?" One reason is given by the Education Committee of the EMS (2011, p. 51) in their series of "Solid Findings" articles: "Mathematical thought concerning proof is different from thought in all other domains of knowledge, including the sciences as well as everyday experience; the concept of formal proof is completely outside mainstream thinking." Dreyfus (1999, p. 94) has observed that “[i]ndeed, research results on students’ conceptions of proof are amazingly uniform; they show that most high school and college students don’t know what a proof is nor what it is supposed to achieve. Even by the time they graduate from high school, most students have not been enculturated into the practice of proving or even justifying the mathematical processes they use”. Besides the differences to everyday reasoning, students’ difficulties with the notion of proof also stem from that they may have never learned what counts as a mathematical argument (Dreyfus, 1999). This is backed by Duval (2007, p. 159), who distinguishes two kinds of failures on students’ behalf: (1) “Dysfunctions in valid reasoning, such as status confusion, non-distinction between a statement and its converse, etc.” (2) “Gaps of deficiencies in the progress of a proof”. (For Duval, “status confusion” not only refers to different forms of reasoning in mathematics, but also to the different status of statements within a proof, e.g. hypothesis, property, and conclusion).

One type of status confusion is that of proof vs. example. Harel and Sowder (2007) have coined the notion of *empirical proof schemes*. The empirical proof schemes come into play when using examples to justify the truth of general (universal) statements, which is what Bell (1976) refers to as empirical justifications. Balacheff (1987) distinguishes between two subcategories: the naïve empiricism, which consists in the checking of assumed special cases, e.g. the first two or three instances; and the crucial experiment, which consists in checking for assumed general cases, e.g. if it is true for the numbers 1, 17, and a billion, it is true for all natural numbers.

Counterexample is yet another difficulty for students related to mathematical proof and reasoning, and one which is also subject to status confusion. Iannone and Nardi (2005) distinguish three, at times conflicting, roles that counterexamples play in learning and doing mathematics: (1) In the affective role the “counterexample has to be emotionally convincing for the students (strengthen their certainty). If it is based on what the student perceives as some minor technicality [...] the counterexample may not convince the student that the proposition is false.” (Nardi, 2008, p. 89) (2) The cognitive role “consists in conveying to the students that all counterexamples are the same, as far as mathematical logic is concerned; that a single counterexample can refute a proposition; that a proposition does not need to be always false in order to be false; and, that one occasion of falsity suffices.” (p. 89) (3) The epistemological-cum-pedagogical role “has to do with what can be learned from a good counterexample – for example in mathematics we use counterexamples to identify which elements of a false statement would need to be amended in order to transform this statement into a theorem.” (p. 89) (see also Peled & Zaslavsky, 1997).

If, for example, the textbooks do a poor job in enabling students to distinguish between different forms of reasoning – and their status – in mathematics, the job is left entirely to the teachers. Often, they might do this by asking students to “explain” and “justify” their reasoning. But this is a task which is related also to the students’ perception (or beliefs) of mathematics and what it means to do mathematics. Dreyfus (1999, p. 106) says: “...the requirement to explain and justify their reasoning requires students to make the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures. This implies acquiring new attitudes and conceiving of new tasks: The central question changes from ‘What is the result?’ to ‘Is it true that...?’”.

3 Question and hypothesis

Indeed, it is a “solid finding” that empirical proof schemes are widespread among students. Still, this also builds on a rather rough picture of what mathematics is – e.g. abstract vs. concrete – and often underplays the role of examples. Teaching students a sharp distinction between examples (that serve the purpose of generating ideas for propositions) and proofs (that serve the purpose of validating these propositions) might lead to a somewhat sterile image of what mathematics is – e.g. a collection of proved theorems as well as experience in finding theorems by investigating examples and proving them deductively. The history of mathematics has something to offer here: Firstly, in terms of original sources on methods and examples. Secondly, in terms of providing students with an idea about how mathematics has evolved over time (Jankvist, 2015).

Hence, the aim of this paper is to *explore the relationships between proof, examples and counterexamples through a use of history and historical sources in the classroom*. The hypothesis we build upon is that we firmly believe that digital technology may have a role to play in such an exploration (and we shall return to this later). We shall provide one example of an original source, or excerpt of one such, that may to some degree illustrate this. The source is that of Tschirnhaus’ transformation from 1683, where he proposed a method which he apparently believed could be generalized to solve algebraic *n*th degree equations.

4 Tschirnhaus’ transformation

Ehrenfried Walther von Tschirnhaus (1651-1708) was a German philosopher and mathematician, who among other things in the summer of 1675 established a very close relationship with Leibniz (1646-1716) while in Paris. Tschirnhaus is acknowledged for four major contributions:

- On catacaustics (1682)
- On quadrature or integration (1683)
- On the Tschirnhaus transformation (1683)
- His (philosophical) logic, *Medicina mentis* (1687)

We are concerned with the third, which is entitled *Methodus auferendi omnes terminos intermedios ex data æqvatione*, which in English translates to *Method for eliminating all intermediate terms in a given equation*.

With reference to Descartes’ *Geométrie*, Tschirnhaus builds on the fact that it is always possible to eliminate the second degree term of any third degree equation. He considers a third degree equation in *y*:

$$y^3 - qy - r = 0 \text{ and } z = y^2 - by - a,$$

for appropriate a and b. By eliminating y between these two, Tschirnhaus finds “the resulting equation” which is what is displayed in Figure 4.1.

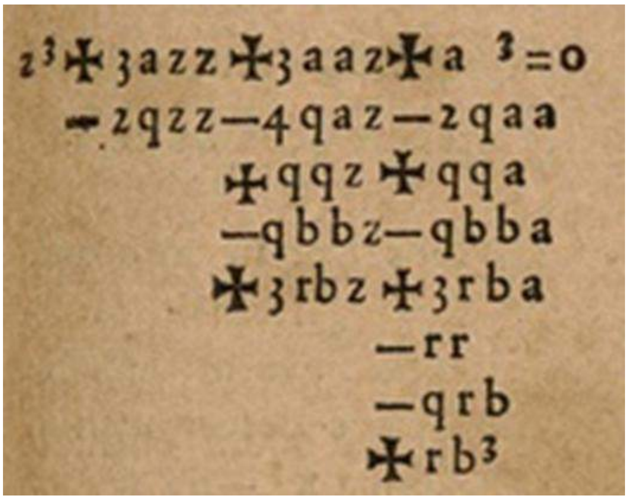

$z^3 + 3azz + 3aaz + a^3 = 0$
$- 2qzz - 4qaz - 2qaa$
$+ qqz + qqa$
$- qbbz - qbba$
$+ 3rbz + 3rba$
$- rr$
$- qrb$
$+ rb^3$

Figure 4.1: Excerpt from *Methodusauferendi terminus intermedios ex data aequatione*

The columns in Figure 4.1, arranged according to the degree of the *z*-term, are the key to Tschirnhaus' observations. Thez^2 terms in column 2 are eliminated if

$$3a - 2q = 0.$$

The *z*-terms in column 3 are eliminated, if

$$3a^2 - 4qa + q^2 - qb^2 + 3rb = 0,$$

a second degree equation in *b* with *a* = 2*q*/3. Tschirnhaus thus arrives at an equation of the form

$$z^3 - t = 0,$$

which has roots$\sqrt[3]{t}$, $\omega\sqrt[3]{t}$, and $\omega^2\sqrt[3]{t}$.

As remarked by Kracht and Kreyszig (1990, pp. 17-18), in "the remaining portion of his paper, Tschirnhaus discussed expressions *a* and *b* suitable for eliminating the third term in an equation of degree 4, 5, or 6 from which, just as in the cubical case, the second term is already absent" and that the "paper was supposed to explain a general method, and one wonders to what extent the author himself believed his claim: '*Et sic idem processus observaturadtres, qvatuor, qvinqve & terminus auferendos*'. In the continuous correspondences between Tschirnhaus and his old friend, Liebniz, the latter certainly expressed his doubts in relation to the generality of Tschirnhaus' method. Already as early as in 1678 or 1679, Leibniz challenged Tschirnhaus' idea:

> Concerning your ... method for finding the roots of an equation, which for solving
>
> $$x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$$
>
> consists in assuming
>
> $$x^4 + bx^3 + cx^2 + dx + e = q \qquad \text{[}q\text{ should read }y\text{]}$$
>
> and then eliminating x by y and ... eliminating the middle terms in the resulting equation. ... I do not believe that it will be successful for equations of higher degree, except in special cases. I believe that I have a proof for this. (Leibniz, 1678-79, quoted from Kracht & Kreyszig, 1990, p. 27)

As noted by Kracht and Kreyszig, Leibniz' last sentence of course refers to the transformation under consideration, not to a forerunner of Abel's famous proof for the insolvability of the quintic by radicals.

5 An empirical case from a mathematics undergraduate program

In a Roskilde University mathematics bachelor's thesis on algebraic equation solving from Cardan (1501-1576) to Cauchy (1789-1857), three students describe and discuss Tschirnhaus' transformation in the light of other and related events in the history of algebraic equation solving (Backchi, Jankvist & Sağlanmak, 2002). Besides the original source from 1683, they relied on the research by Kracht and Kreyszig (1990) as well as Tignol's (2001) book about Galois theory. The students work through both the historical presentation, i.e. that of Tschirnhaus, of the transformation, and the modern by Tignol, which relies on calculating determinants and using matrices.

As pointed out by the students, the reason that Tschirnhaus' method works for n=3 is that "the system of the n-1=3-1=2 equations, in this case only leads to a second degree equation in b, since (n-1)!=(3-1)!=2" (Backchi et al., 2002, p. 43). For n=4 and n=5, the intermediate equation "will in worst case become an equation of degree (4-1)!=3!=6 or (5-1)!=4!=24" (p. 43).

From our perspective the interesting aspect of this thesis is the way that the students "test" Tschirnhaus' transformation. The students use CAS (*Maple*) to empirically "test" Tschirnhaus' transformation for n=4 and n=5 in order to further their understanding of what "goes wrong". For n=4 they obtain a third of page long expression for the intermediate equation, visually illustrating the inefficiency of the method (pp. 152-153). For n=5, *Maple* crashes before completing the calculations (p. 165).

Surely, the students "test" can also be done in CAS without the modern day notation and use of matrices introduced by Tignol (2001), although it undoubtedly will be a more cumbersome task. Our purpose in this particular paper, however, is not to discuss and compare modern day mathematical notation to that of previous times. Rather we seek, as previously mentioned, to illustrate that digital technologies can be used to empirically "test" mathematical conjectures, as a kind of "*technological un-likelihood test*". This is to say, the technology does not provide a traditional counterexample in the usual sense, but it can play the role of illustrating to the students why it is unlikely that a given mathematical conjecture holds - in the example above, because the intermediate equations simply increase so much in complexity.And unlike modern day mathematics, the history of mathematics is full of conjectures which has already been proven wrong, and which for that reason can act as illustrative examples in the teaching and learning of mathematics.

6 The "mathematical underworld"

With outset in this example, it makes sense to look at the distinction between deductive and inductive reasoning in mathematics. The classical picture that we as educators try to convey is that of mathematics as a pure deductive discipline, where inductive or experience-based reasoning is "wrong", "problematic" or "superfluous". However, the example of Tschirnhaus' transformation – as it is described above – can help us unpack this notion of mathematics as "pure", or phrased in another way; introducing Tschirnhaus' transformation and using CAS to explore his conjectures allows us to open up another layer of mathematical practice in relation to education. The students in the example above do not follow the classical image of mathematics. Rather they explore through computer work the likelihood of Tschirnhaus' hypothesis in an informal way.

If we distinguish examples from counterexamples, and again from proofs and conjectures, the mainstream story of mathematics is that examples allow for inductive reasoning, but only in a heuristic manner. Nothing is proved by example! Hence, examples are thought of as ways to generate ideas for further exploration and formulation as theorems and proofs (Johansen & Sørensen, 2014, p. 140). On the other hand, counterexamples are part of deductive reasoning as ways of rejecting propositions or proving converse statements (by contradiction or counterexamples). This means that the only "allowed" inductive reasoning in a mainstream view of mathematics is in relation to heuristic treatments of examples. But this mainstream story leaved out two forms of reasoning that are relevant, and that we believe it is worthwhile focusing on; namely inductive reasoning in relation to proofs and in relation to counter-examples. Our interest here is howfailed attempts to proof, or difficulties with conducting calculations, build mathematical intuition, both within the individual mathematician and mathematics student and in the mathematical society at large.We know from the literature on mathematical practice that inductive reasoning in mathematics is not limited to the heuristic work with generating conjecture through examples. Several autobiographical accounts (e.g. Thurston, 1994) as well as work in the philosophy of mathematical practice (Misfeldt & Johansen, 2015) point to the importance of development of strategic ideas and intuitions about routes to be pursued in mathematical research as well as dried out areas and problems in which one can be stuck for so long that it challenges one's career.

This phenomenon, i.e. that mathematicians and the mathematical society are accumulating inductive knowledge about proofs and counterexamples, is on the one hand a critically important part of mathematical practice, and on the other hand not a part of the official mainstream story about mathematics. We shall refer to such use of inductive reasoning and knowledge generation in relation to mathematics in ways that are not solely related to heuristic treatment of examples as "*the mathematical underworld*". This term is close to what Reuben Hersh (building on Goffman 1978) describe as the backside of mathematics (Hersh, 1997). The backside of mathematics is defined in relation to the front side, and where the front side is described as formal and precise ordered and abstract, the backside is fragmentary, informal, intuitive and tentative (p. 36). Hersh describes how the philosophy of mathematics tends to not see the backside of mathematics (of course new approaches in the philosophy of mathematical practice tries to address this) and we can add to this that mainstream teaching of mathematics, has the same problem. How do we address the backside of mathematics in teaching? The informal and intuitive nature makes the backside hard to point to in a precise manner. We can define the "mathematical underworld" as the object of teaching when trying to address the backside of mathematics in teaching. The underworld is where the backside lives and can be studied. In that sense, the mathematical underworld represents knowledge generation in the mathematical field that transcends the clear distinction between modes of discovery and of justifications.

This leads us to the question of how to understand and teach the mathematical reasoning process in a way that does not neglect this mathematical underworld. In order to do that we base ourselves in a model developed with an outset in Lakatos and Polya and published in the proceedings of the previous ESU (Misfeldt, Danielsen & Sørensen, 2015). This model describes an ideal reasoning process, focusing on the relation between inductive reasoning (from example to conjecture) and deductive elements such as counterexamples and refinements of proofs.

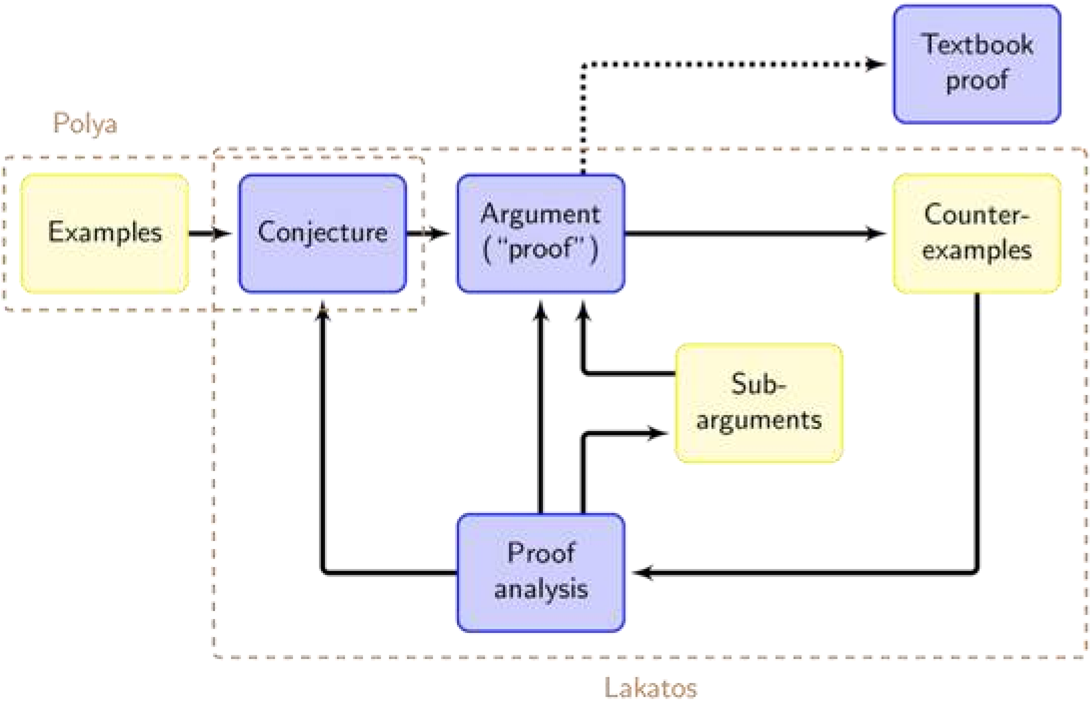


Figure 6.1: This figure taken from (Misfeldt et al., 2015, p. 437), describes the reasoning process in mathematics from ideation to proof. The model shows the interplay between proofs and counterexamples, based in Lakatos, but the model does not capture inductive reasoning that is not involved in working with examples. The yellow labels designate inductive reasoning, and are intimately connected to examples.

The model in Figure 6.1 formalizes the mainstream image of the role of examples in mathematical proof work (Misfeldt et al., 2015). But the role of inductive and instrumented reasoning in the mathematical underworld is not clearly described here. If we apply the model to the way the students use CAS to work with Tschirnhaus' transformation, we see the phenomenon these students experience is not completely captured by the model, and in particular by the wording applied in the model. The attempt to create examples arguing for the potential conjecture fails and this sparks interest in understanding the limitations of the methods rather than proving the general claim.

We now take a look at some of the circumstances related to how the considerations discussed above may find their application in a classroom setting. We do so by outlining a selection of theoretical constructs that we deem potentially relevant for such an endeavour. These are Misfeldt's and Jankvist's (2018) notion of justificational mediations, Yackel's and Cobb's (1996) notion of sociomathematical norms, and finally Alibert's(1988) notion of scientific debate.

7 Justificational mediations

The use of CAS for learning mathematics is often understood with the instrumental approach to mathematics learning (Trouche, 2005), which focuses on how students transform CAS tools to personal instruments. One critical concept in the instrumental approach is that of mediation that designates the way in which tools support goal directed activities and hence mediate between a student and his/her goal. The literature highlights a critical distinction between epistemic and pragmatic mediations referring to whether the students aim at understanding certain phenomena or at solving specific tasks (Artigue, 2002).

When working with the use of CAS in the context of proofs and proving activities, we have described four core questions/aspects about how CAS mediate proving (Misfeldt & Jankvist, 2018). Aligning with the instrumental approach, this can be conceptualized as using CAS for justificational mediations (Misfeldt & Jankvist 2018). These four aspects of justificational mediations are:

1. Does the CAS use establish truth? This is the core function of a justificational mediation. To what extent does the CAS output act as warrant in an argument?
2. Does the CAS use allow interaction and experimentation? This highlights the degree to which students can change parameters, explore phenomena etc., and therefore to what extent the students still have agency when working with CAS in relation to proofs.
3. Is the argumentation inductive, deductive or authoritarian? What type of proof scheme is in play and what type of warrant do CAS provide.
4. Does the argument highlight important aspect of the proof or the mathematical relationships?

Both the second and the fourth aspects are to some extent related to epistemic mediations. Still, we suggest that these four aspects of justificational mediations more or less capture the important aspects of using CAS tools in proving activities.

Looking at the case of the students' work with the Tschirnhaus transformation, we can make the following observations related to the four questions/aspects:

1. The CAS use does not establish truth in the usual sense – what we do learn from the use of CAS, however, is that the computations of the Tschirnhaus transformation in the case of fourth and fifth degree polynomials become very lengthy and complicated and do not seem to work in the sense of providing a mathematical result. This is not the same as establishing truth in a classical sense.
2. The use of CAS is a case of experimentation, in the sense that the reported activity, more or less is an experiment where the students could – and did – play with attempts to calculate the Tschirnhaus transformation for specific polynomials of higher degree than 3. In the case, there is a direct continuity between calculating the general case (as attempted in the student project) and experimenting with specific examples.
3. The argumentation here is both deductive and inductive. In a sense the attempt is to calculate the general Tschirnhaus transformation, for polynomials of degree 2, 3, 4, …. In cases where this strategy works, the argument is straightforward deductive and fulfils all mathematical standards in the classical sense. Yet, when this plan breaks down (from polynomials of degree 4 and onwards) the argument is different. The CAS breakdown is not by any means a valid mathematical argument

for the impossibility of Tschirnhaus' hypothesis. Still, it does support the idea that this hypothesis might not hold water, in an inductive fashion. This type of non-valid/non-kosher mathematical knowledge is what we have designated "the mathematical underworld".

4. The relation to the main ideas in the proof of the Tschirnhaus transformation is slightly awkward here; since we are looking at a negative result (the general Tschirnhaus transformation cannot be calculated). However, CAS provide a good idea about the reason why the result is unobtainable.

8 Sociomathematical norms

Sociomathematical norms were observed and named by Yackel and Cobb (1996), who in a teaching situation noticed that aspects which could neither be described as purely mathematical norms nor purely as classroom social norms were in play. Yackel and Cobb (1996, p. 461) define sociomathematical norms as "normative aspects of mathematics discussions specific to students' mathematical activity" and describe the difference to social norms as "The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm. Likewise, the understanding that when discussing a problem students should offer solutions different from those already contributed is a social norm, whereas the understanding of what constitutes mathematical difference is a sociomathematical norm".

Sociomathematical norms are negotiated between the students and the teacher, and may thus vary from classroom to classroom. This negotiation builds on already "taken-as-shared" perceptions within the classroom, and as such they are "… intrinsic aspects of the classroom's mathematical microculture. Nevertheless, although they are specific to mathematics, they cut across areas of mathematical content by dealing with mathematical qualities of solutions, such as their similarities and differences, sophistication, and efficiency. Additionally, they encompass ways of judging what counts as an acceptable mathematical explanation." (Yackel & Cobb, 1996, p. 474). Hence, explanations and justifications are themselves made the objects of reflection.

The case highlights a type of inference in mathematics, different from the established and accepted reasoning. This "mathematical underworld" poses an educational problem, since exploring the Tschirnhaus transformation and realizing that there seems to be a breakdown when attempting to calculate the transformation for the general polynomial of degree 4 and 5, is not a proof or even an established mathematical result. However, it does constitute a type of mathematical knowledge that should be object for teaching, namely establishing sociomathematical norms that both allow for discussing and working with the type of phenomena described as the "mathematical underworld", while avoiding the type of student mistakes and misconceptions that easily result from working with inductive reasoning and non-formal mathematical work. This dilemma calls for specific teaching strategies allowing students and teachers to discuss the epistemological status of various pieces of knowledge and intuitions about mathematics. We suggest that the combination of historical sources and CAS is a good context for developing such discussion, and in the following section we will describe one pedagogical strategy that develops this discussion.

9 Scientific debates

Under the heading of generating *scientific debate* Alibert (1988, p. 32) provides three steps for making this happen in the mathematics classroom:

1. The teacher initiates and organizes the production of scientific statements by the students. These are written on the blackboard without any immediate evaluation of their validity.
2. The statements are put to the students for consideration and discussion. They must come to decisions about their validity by taking a vote; each opinion must be supported in some way, by scientific argument, by proof, by refutation, by counterexample.
3. The statements that can be validated by a full demonstration become theorems; those found to be incorrect are preserved as "false-statements" associated with appropriate counter-examples.

In an empirical example of a classroom scientific debate described by Alibert (1988), students came up with counterexamples to a proposed statement, which then resulted in new hypotheses, empirical observations, derived conjectures, and eventually an argument which required a formal proof.

Although our small empirical example with the undergraduate student report is not one of classroom interactions, we do see a resemblance to the activities. Here, however, it was neither the students nor the teacher who as part of step 1 provided the scientific statements, but the history of mathematics! This approach could easily be transformed into a classroom activity, where the teacher posed a mathematical conjecture from the history of mathematics on the blackboard. In the case of Tschirnhaus' transformation, the voice of Leibniz could play the role in framing the scientific debate.

In our empirical example, CAS came to act as a justificational mediator in terms of judging the potential validity of Tschirnhaus' transformation. The same could certainly be the case in a classroom setting. Also, here CAS would have a significant role to play in order to guide the students' discussions in step 2. One sociomathematical norm to be established here would be that a "technological un-likelihood test" certainly is not the same as an actual counterexample. This is to say, step 2 of the scientific debate provides, as mentioned above, the opportunity to discuss the epistemological status of mathematical statements and arguments, i.e. the combination of both history and technology here comes to serve as a way of counteracting the "status confusion" as described by Duval.

Step 3 in Alibert's scientific debates would in our example correspond to the students' reasoning of the degree of the intermediate equations in Tschirnhaus' transformation. Depending on what historical conjecture, false or true, was posed by the teacher in step 1, step 3 will result in a theorem with a proof or an actual counterexample.

Such scientific debates may be used to establish "healthy" sociomathematical norms in a classroom by: developing students' deductive proof schemes; assisting in discarding their empirical proof schemes and/or external conviction proof schemes; illustrating the difference between proofs that prove and proofs that explain; illustrating the difference between different types of mathematical arguments (cf. Dreyfus) and the status confusion of which Duval talks.

10 Final remarks

The example described above as well as the way that it can be viewed through the lenses of three different mathematics education frameworks allows us to suggest that the relation between history of mathematics and the use of technology can be mutually fruitful.

As mentioned, the history of mathematics is rich on examples of conjectures that turned out not to be true. By having students look at and work with such examples they may come to grasp not only some of the differences in mathematical statements, but also the very need for formal proof to begin with. In that sense, the history of mathematics has a role to play in assisting students in overcoming the "status confusion" of which Duval's speaks. But apart from motivating the need for formal proving, historical examples can also open up new perspectives on what types of mathematical knowledge is relevant to address from an educational perspectives.

In this paper, we have brought light on the "mathematical underworld" of inductive reasoning in relation to hypothesis testing and theorem proving. Hereby we have come closer to understand types of reasoning and knowledge that have played (and still are playing) a role in developing mathematics, but are not considered as part of mainstream mathematics. We have been able to activate "computer evidence" to open up and discuss historical examples that would be too laborious to address using classical algebraic methods. We have labelled this computer evidence as justificational mediations, aligned with the instrumentational approach of mathematics education (Trouche, 2005). Furthermore, we have used the concepts of sociomathematical norms as well as the didactical idea of scientific debates to unfold the pedagogical challenges that the mathematical underworld poses to teaching. By developing sociomathematical norms that value working with examples and counterexamples, and in a complex way includes inductive reasoning, we argue that it is important to fully grasp the nature of mathematics as a discipline (e.g. Jankvist, 2015) and that historical sources in combination with digital tools are useful for that.

Yet, the problems of status confusion and development of empirical proof schemes persist, and hence we end the paper by suggesting deliberate teaching strategies based in scientific debates to support students' understanding of all the different natures of mathematical knowledge that has been and still are important for mathematical practice.

REFERENCES

Alibert, D. (1988). Towards new customs in the classroom. *For the Learning of Mathematics, 8*(2), 31–35.

Artigue, M. (2002). Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245–274.

Backchi, D. H., Jankvist, U. T. V., & Sağlanmak, N. (2002). *Algebraisk ligningsløsning fra Cardano til Cauchy: Et studie af kombinationers, permutationers samt invariansbegrebets betydning for den algebraiske ligningsløsning før Gauss, Abel og Galois*. IMFUFA-tekst nr. 409/2002. Roskilde: Roskilde University

Balachef, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics*, *18*(2), 147–176.

Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, *7*(1-2), 23–40.

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics*, *38*(1-3), 85–109.

Duval, R. (2007). Cognitive functioning and the understanding of mathematical processes of proof. In P. Boero (Ed.), *Theorems in school: From history, epistemology and cognition to classroom practice* (pp. 137–161). Rotterdam: Sense Publishers.

Education Committee of the EMS (2011). Do theorems admit exceptions? Solid findings in mathematics education on empirical proof schemes. *EMS Newsletter, 82,* 50–53.

Goffman, E. (1978). *The presentation of self in everyday life*. London: Harmondsworth.

Harel, G., & Sowder, L (2007). Toward a comprehensive perspective on proof. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Charlotte, NC: Information Age Publishing.

Hersh, R. (1997). *What is mathematics, really?* (Vol. 18). New York: Oxford University Press.

Iannone, P., & Nardi, E. (2005). Counterexamples: Is one as good as many? In *Proceedings of the 4th Mediterranean Conference in Mathematics Education* (Vol. 2, pp. 379–388). Palermo, Italy.

Jankvist, U. T. (2015). Changing students' images of "mathematics as a discipline". *The Journal of Mathematical Behavior*, *38*, 41–56.

Jankvist, U. T., Mosvold, R., Fauskanger, J., & Jakobsen, A. (2015). Analysing the use of history of mathematics through MKT. *International Journal of Mathematical Education in Science and Technology*, *46*(4), 495–507.

Johansen, M. W., & Sørensen, H. K. (2014). *Invitation til matematikkens videnskabsteori*. Copenhagen, Samfunds Litteratur.

Kracht, M., & Kreyszig, E. (1990). E. W. von Tschirnhaus: His role in early calculus and his work and impact on Algebra. *Historia Mathematica, 17*(1), 16–35.

Misfeldt, M., Danielsen, K., & Sørensen, H. K. (2015). Formal proof and exploratory experimentation: a Lakatosian view on the interplay between examples and deductive proof practices in upper-secondary school. In E. Barbin, U.T. Jankvist & T.H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education. Proceedings of the Seventh European Summer University* (pp. 433–446). Copenhagen: Aarhus University.

Misfeldt, M., & Jankvist, U. T. (2018). Instrumental genesis and proof: Understanding the use of computer algebra systems in proofs in textbook. In L. Ball, P. Drijvers, S. Ladel, H.-S. Siller, M. Tabach & C. Vale (Eds.), *Uses of technology in primary and secondary mathematics education. Tools, topics and trends* (pp. 375–385). Cham: Springer.

Misfeldt, M., & Johansen, M. W. (2015). Research mathematicians' practices in selecting mathematical problems. *Educational Studies in Mathematics*, *89*(3), 357–373.

Nardi, E. (2008). *Amongst mathematicians. Teaching and learning mathematics at university level*. Boston: Springer.

Peled, I., & Zaslavsky, O. (1997). Counter-examples that (only) prove and counter-examples that (also) explain.*Focus on Learning Problems in Mathematics*, *19*(3), 49–61.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, *30*(2), 161–177.

Tignol, J.-P. (2001). *Galois' theory of algebraic equations*. Singapore: World Scientific Publishing.

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 197–230). New York: Springer.

von Tschirnhaus, E. W. (1683). Methodus auferendi omnes terminos intermedios ex data æqvatione. *Acta Eruditorum*, 204–207.

Yackel, E. & Cobb, P. (1996).Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458–477.

THE TEACHING OF LOGARITHMS IN UPPER SECONDARY SCHOOL FROM A HISTORICAL PERSPECTIVE

Eleni LAPPA[1], Kostas NIKOLANTONAKIS[2]

University of Western Macedonia, Florina School of Education,
Department of Primary Education, P.O. Box 21, Florina, 53 100, Greece
[1]elelappa@gmail.com, [2]knikolantonakis @uowm.gr

ABSTRACT

Secondary school students (16 – 17 years old) face difficulties concerning the creation of a structured mental scheme about the concepts of logarithm and logarithmic function. This paper focuses on a didactic intervention following the Jankvist's modules approach and utilizing the History of Mathematics (HM) as a tool for the conceptual understanding of mathematical notions and more broadly as a goal for the perception of the human evolving nature of Mathematics. The use of HM, through studying historical texts and the awareness of the concept historical evolution, combined with the collaborative teaching method and the Geogebra software. This case study aimed to address the following research questions: a) Does the use of HM as a tool affect students' conceptual understanding of the logarithmic concepts? b) Does the HM also be promoted as a goal for students even merely to approach Mathematics as an evolving cultural product?

Keywords: History of Mathematics, Semiotic Instrumental Discursive dimensions, Geogebra, Collaborative teaching model, logarithm, logarithmic function.

1 Introduction

The abstract nature of mathematical concepts is considered as a central source of difficulties for students in the process of mathematics' conceptual understanding. Mathematical concepts are the main subjects of instructional approaches mainly through their representations, which cannot often be perceived by the senses (Sfard, 1991). What is required in mathematical education is the kind of comprehensive understanding, which focuses on the knowledge not only of 'what I have to do' but also of 'why should I do it' (Skemp, 1976). In every attempt to observe the learning path of a mathematical concept, the separation of mathematical knowledge into procedural and conceptual divisions lies in the forefront. The execution of routine algorithms, as well as the usual manipulations performed on mathematical objects, refers to the procedural knowledge which is necessary for the accomplishment of a mathematical work but is not capable of acquiring a fully integrated cognitive structure relating to the particular mathematical concept. At a higher level, conceptual knowledge is required to fill in the answer to 'why'. Conceptual knowledge is not an isolated part of knowledge but includes the necessary links to create a structured and flexible cognitive scheme (Gray & Tall, 1994). The adequately structured communication between the epistemological and cognitive content of the mathematical concept through the development of semiotic, instrumental and discursive dimensions, is achieved by developing two qualitatively different and at the same time complementary approaches of this concept: the operational and the structural ones (Sfard, 1991). Operational approach perceives the concept as a process, giving a dynamic character and is expressed by performing actions, whereas structural considers the concept as an object and it is the product of self reflection processes.

The concepts of logarithm and logarithmic function are basic parts of the curriculum content of the educational system. Logarithm is a notion with a long evolutionary course. Being at first an auxiliary tool for the calculation of difficult and time-consuming operations, has become, in its functional form, an instrument for explaining various kinds of phenomena applicable in many and varied scientific fields. However, students usually face difficulties in constructing a concept image that incorporates the exponential process into the mathematical object - the numbers called logarithms (Thomaidis, 1986; Vagliardo, 2006; Panagiotou, 2014).

In parallel, students are also called to deal with the concept of the logarithmic function. Function is considered to be a fundamental concept of mathematics while at the same time the effort of its conceptual approach faces a variety of challenges (Sfard, 1991; Sierpinska, 1992; Hitt, 1998; Artigue, 1999; Gagatsis & al., 2006). Difficulties are located in points such as the idea of functional dependence as a co-variation, the understanding of the role of the independent and dependent variables, the manipulation of functional symbolism, the use of different semiotic registers and various representations, the transition from algebraic to functional thinking, as well as the dual nature of the concept as a process and as an object (Artigue, 1992; Gagatsis & al., 2006; Kieran, 2007; Lagrange & Psycharis, 2014; Minh & Lagrange, 2016). Therefore, the teaching of logarithmic function faces two major obstacles: the first concerns the difficulty of understanding the logarithmic concept and the second is the one that emerges from the concept of function itself.

The reasons justifying the utilization of History of Mathematics (HM) in instructional approaches are varied and widely discussed (Fauvel, 1991; Furinghetti, 1997; Tzanakis & al., 2002; Fried, 2008; Jankvist, 2009; Tzanakis & Thomaidis, 2012). Characteristically, it could be said that mathematical discipline is not exhausted in its final results, as its development path involves years of efforts, doubts and revisions (Tzanakis & Thomaidis, 2000). The knowledge of the evolution trajectory of a mathematical concept can contribute positively to overcoming epistemological and didactic obstacles so as to achieve conceptual understanding. Teaching the concept in its final refined form, according to the current Curriculum, is an example of what Freudental calls ‘anti – didactical inversion’ (Barbin, 2015).

2 Methodology

This intervention was designed in the context of Jankvist’s modules approach, using a PowerPoint file, an introductory worksheet and 3 main worksheets. It was applied to a 16-student class (17 years old) in the upper secondary school. The role of the HM in this intervention is twofold. On the one hand, from a macroscopic point of view, through the negotiation of the historical evolution of logarithms, history was used as a goal to highlight as far as possible the dynamic and evolving nature of the content of mathematical discipline, its links with the social and cultural environments in which it develops, as well as its connection to other fields of science. On the other hand, history was used as a tool to achieve a more wholistic approach to mathematical concepts, focusing on key components of the historical evolution that are necessary in constructing an integrated image of these concepts. The most important stages in the historical evolution of the logarithms were presented emphasizing the necessity for estimating difficult numerical operations in the historical context of 17^{th} century, as though the reasons that led to the invention of the concept, the importance of logarithms in the modern reality, as well as the individuals who determined with their contribution the development of this historical route. The method of Prosthaphaeresis in the arithmetic-geometric progressions’ correspondence was also studied in the Napier kinematic model. The logarithm was defined as the 'number of ratios' in a continuous ratio within the bounds of a historical context in which the notion of power had not yet been clarified. The way that Napier constructed his logarithmic tables and the potential of estimating the logarithm of any positive number by the hyperbolic areas were also discussed. The historical review was completed by presenting and discussing logarithmic applications in various fields. The students then worked on translated historical texts by Leonard Euler from his book “Introductio in analysin infinitorum” (1748) which contained the definition of the logarithm and the logarithmic rules as well as the definition of the function as analytical expression with reference to the reverse function.

A combination of various teaching tools was used in this instructional approach for the creation of a structural mental scheme by the students concerning the specific concepts. The collaborative teaching method favored the exchange of views, the development of arguments and the active participation of students. At the same time, the digital environment of the educational software Geogebra, in combination with the historical texts provided a proper environment for experimentation and exploration, driving the development of the instrumental dimension of work for conceiving the logarithmic function as a co-variation reverses to the exponential. Through the observation of the development of the groups’ work along the axes of semiotic, instrumental and discursive dimensions and the analysis of students’ individual answers in two questionnaires completed after the end of the intervention, the following questions were addressed: a) Does the use of HM as a tool affect students' conceptual understanding of the logarithmic concepts? b) Does the HM could be also promoted as a goal for students to even merely approach mathematics as an evolving cultural product?

2.1 Worksheets

2.1.1. Introductory, first and second worksheets - logarithms

The introductory worksheet contained difficult and time-consuming numerical calculations (multiplications, divisions, cubic root extractions) which students were asked to estimate. After their initial failed attempts and in the course of the intervention they managed to make the same calculations using logarithms.

The first worksheet contained Leonard Euler's text (1748), concerning the symbolism of logarithms and the definition as a power exponent and as a function value, approaching the logarithmic concept as a process and at the same time as a result:

> **“***Just as, given a number a, for any value of z, we can find the value of y [α^z] so in turn, given a positive value for y, we would like to give a value for z, such that α^z = y. This value of z, insofar as it is viewed as a function of y, it is called the LOGARITHM of y”* (Down & al., 2004).

The knowledge of the text was intended to enrich the theoretical frame of reference in the workspace of the groups. The following tasks in this worksheet aimed at the acquirement of a comprehensive understanding of the definition and the symbolism through the clarification of the exponential and logarithmic relation of the variables involved and the application of the definition in order to deal with relative activities. On the other hand, the first worksheet aimed at the investigation of students' knowledge about basic components concerning the concept of function such as the set of values and the domain of the function. The responses of the groups were used to design the third worksheet, which focused on the concept of the logarithmic function in order to examine in detail the nature of function as a co-variation, the notion of reversibility and the alternating roles of dependent - independent variables. The groups were asked to apply the exponential process, included in the definition, in order to activate the operational nature of the logarithmic concept. Through these actions, it was expected that students would make a first step towards the internalization of the process and the identification of the loga symbol as an autonomous mathematical object.

On the same wavelength as the first worksheet, the second one contains Euler's text, focusing on the logarithmic rules which proof is indirectly given to the text as a result of the definition:

"In like manner if log $y = z$, then log $y^2 = 2z$, log $y^3 = 3z$, etc., and in general log $y^n = nz$ or log $y^n = n$ log y, since $z =$ log y. If follows that the logarithm of any power of y is equal to the product of the exponent and the logarithm of y… If we already know the logarithms of two numbers, for example log $y = z$ and log $v = x$, since $y = \alpha^z$ and $v = \alpha^x$, it follows that log $vy = x + z =$ log $v +$ log y. Hence, the logarithm of the product of two numbers is equal to the sum of the logarithms of the factors" (Down & al., 2004).

The usefulness of rules for estimating logs of many numbers using logarithms of few ones is also emphasized without providing clear reference on the ease of numerical operations that is achieved through the rules. Students were expected to delve deeper into the internalization of the logarithm concept by using the definition in constructing proof of logarithmic rules and developing the required justification on answering the questions were posed.

1.1.2. Third worksheet – logarithmic function

The third worksheet contained two phases. In the first one, a historical text including the Euler's definition of the function as analytical expression constituted the core of the work. There was also reference to the uniqueness of the dependent variable for each value of the independent one and the existence of the inverse function, switching in fact the roles of dependent and independent variables.

"*a function of a variable quantity ...[as] an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities … a single-valued function is one for which, no matter what value is assigned to the variable z , a single value of the function is determined … If y is any kind of function of z, then likewise z will be a function of y"* (Down & al., 2004) .

The tasks following the text were structured so as to lead the focus of the students' work on the concepts of co-variation, dependent and independent variables and inverse function. Through a real problem in the seismology field, students were also expected to deal with the notions mentioned above, focusing on the exponential and logarithmic functions.

The second phase of the worksheet was characterized by the involvement of the Geogebra mathematical software for studying the logarithmic function. Students had to swift independent with dependent variables finding out the mathematical formulas of inverse functions, to get to the proper formula from the graph of the logarithmic function, to create graphical and algebraic resolutions of the logarithmic equations and to justify the symmetry of the inverse functions' graphs in terms of the roles of variables. The questions aimed at the construction of a transparent mathematical meaning by the students concerning the concept of the logarithmic function and its relationship with the reverse exponential one.

2.2 Questionnaires

Two questionnaires were distributed to the students without advance warning, one week after the end of the intervention, that means about one and a half month since the beginning of the course. The aim was to examine at what extend students could use their knowledge, without preparation and repetition, to solve logarithmic equations and inequalities and to gain an understanding on the logarithmic function with proper application of logarithmic rules. At each step justification was required.

The first questionnaire was designed to enable students to express their personal views about: 1) the use of history in the didactic intervention, 2) the initial necessity leaded to the invention of logarithms as well as their use in modern reality and 3) the dynamically evolving character of mathematical concepts. The second questionnaire focused on finding out current students' knowledge of the logarithm and logarithmic function as well as its applicability to resolve related activities as already been mentioned.

3 Analysis of results

3.1 First and second worksheets - logarithms

Students took advantage of the texts' dynamics, through the manipulations they used. More specifically, the symbols were decoded and the relative information was constructed, the exponential process was used in experimentation and researching frameworks to resolve the activities, and at the same time, based on the contents of the texts, formal and non-formal arguments were developed during the justification of the manipulation of the symbols and of the procedures proposed by the students in the groups. Characteristically, concerning the question about the definition of logarithm as a functional expression, groups used initially the exact words of the text to answer the question and indeed the sentence in which the word logarithm had been written in capital letters, without mentioning the exponential relation that links the variables, which means that at this initial stage they were unable to clarify the definition. As it may be seen from the discussions in the groups, students probably carried away by a norm that 'requires' the formulation of a short and formal definition. As students continued working on the remainder activities the inadequacy of the initial, automated approach to the definition was highlighted. Groups were forced to revert to the definition of the text and to examine relations of variables and constants as well as the range of acceptable values that variables could get. They used specific numerical values and processed algebraic notation using generalized variables, even by changing labels of the variables of the text, thus the semiotic dimension of the work was developed. Groups produced a verbal expression of the definition:

- If the number rises to an exponent, then, the exponent is the logarithm of the outcome.

The typical exponential procedure and its rules, well-known to the students, were used as artifacts for constructing the definition in the developing of the instrumental dimension of work. A representative dialogue in a group concerning the phrase of the text *"Just as, given a number α, for any value of z, we can find the value of y [= a^z], so, in turn, given a positive value for y, we would like to give a value for z, such that $\alpha^z = y$"* follows:

- If I know z and also the α, then I can find y. If I know y then how can I find z? What does it mean?
- Then say that z is the logarithm of y. Is it a result of estimation or a correspondence?
- The logarithm counts the ratios. It is a number. Is it both?

Although the last question was not answered by the members of the group at this stage, the previous dialogue demonstrates a creative progress of the work towards both the view of logarithm as an object and the dual nature of the function as a process and result. The etymology of the word 'logarithm', which was a key element in the presentation of the logarithm historical evolution, seems to have contributed positively to the attempt of constructing, according to Tall & Vinner (1981), the appropriate 'concept image'. In parallel, the correspondence of arithmetic and geometric progressions was used by the groups to justify and verify the solution of logarithmic equations such as $\log_2 4 = x$, which initially were solved by using Euler's definition.

However, although the definition had already been clarified and was applied successfully, students were struggling to accept $\log_5 112$ as a number that was a solution to a given equation ($5^x = 112$). Similarly, the question of "what is equal to $\alpha^{\log_a u}$?" proved to be of high difficulty for all groups. This is a fact which demonstrates the kind of obstacles students need to overcome when dealing with the logarithmic concept. The groups were experimented by applying specific 'convenient' numerical values to the variables and by using the Euler's definition both as a tool and as a component of the reference framework for validating the results. In this way the discursive dimension of the work was developed. It is noteworthy however, the formation of reflective abstraction in students' learning as it characteristically came from a student's point of view about the outcome of $\alpha^{\log_a u}$:

- The outcome is u because the logarithm is an exponent ... That is, here is the power of base α which is raised to the exponent that equals u ... so the outcome is u.

The evolving development of the work from the first to the second worksheet indicated a dynamic course of internalization of the definition and integration of the exponential process into the object - logarithm. In the second worksheet students analyzed the equalities of the text and led to the construction of proof of the logarithmic rules by flexible manipulations of the definition:

- Since $\log y = z$, α is raised to z, so $\alpha^z = y$ ... $y^2 = (\alpha^z)^2$, $y^2 = a^{2z}$ ... $2z$ is the exponent to the base α and thus the log of y^2. That is, $\log y^2$ equals to $2\log y$.

The semiotic dimension of the work developed without any problems in the manipulation of the symbols and the decoding of the relations of the text. The use of the definition activated the instrumental dimension of the work and the validation of the construction of proof contributed to the development of the discursive dimension. Characteristically, a student stated:

- If I know the logarithm of two numbers then I know the logarithm of their quotient ... because if I know that the logarithm of y is equal to z then α^z is y and in the same way α^x

is v. If I divide them, the result is α^{z-x} ... that is, z-x is the exponent of α, hence the logarithm of the quotient.

It should be noted that groups distinguished the simplification of estimations in Euler's rules due to the reduction of multiplication to addition and division to subtraction, which was the basic reason leaded to the invention of logarithms and a significant element in the presentation of the historical evolution of logarithmic concepts. Hence, the correspondence of arithmetic and geometric progressions was brought to the forefront of the groups' work one more time. This finding was used by students for justifying the usefulness of logarithmic rules.

3.2 Third worksheet – logarithmic function

In the third worksheet, except of the Euler's text with the definition of function as analytical expression and a reference to the concept of the inverse function, a function of exponential formula ($I = I_0 \cdot 10^R$, $I_0 > 0$) involving the earthquake magnitude (R) and intensity (I), in the context of a real problem in the field of seismology was also included. Groups were asked to decide if this expression was a function according to Euler's definition and to define the dependent and independent variables. They created functional expressions that were either the result of a specific application of their knowledge in context of a discipline like physics, e.g. the displacement in a uniform linear motion, or were given in the typical form of a functional symbolism, e.g. $y = z + 1$. They also focused on the uniqueness of the y value (value of function) obtained for each z value. Groups used elements of the history of logarithms, trying to relate the meaning of function with the Euler's and Napier's logarithmic approximations. They found out that there is a co-variation of variables in both approaches. Groups analyzed the functional symbolism of the text and used it in their experimental efforts. The discussion focused on the role of independent and dependent variables and constants in the expressions that students presented. They had no problem to recognize the relationship between magnitude and intensity of earthquakes as a functional expression and to identify dependent and independent variables. They stated:

- Changing values of magnitude results in change of corresponding values of intensity ... the intensity is depended on the magnitude that changes. The intensity is y and the magnitude is z.

They compared and contrasted the Euler's definition with the one which they had already been taught regarding function as correspondence of values. Specifically, they said:

- Each numerical value of x is assigned to only one numerical value of y. In the text x is z. Saying that for any value of z is defined a single value for the function, it means that every x is assigned to a single y that is the value of the function.

Afterwards the preceding process, the members of the groups concluded that the given relation of earthquake magnitude and intensity was consistent with Euler's definition of function:

- It is an expression formed by the variable R and constants ... is the way in which the intensity changes when the magnitude changes.

The given relation was also examined as co-variation of variables, in which R is arithmetic and I geometric progression. It was found that the arithmetic change in values of magnitude results in a geometric change of intensity. The result was the expected appearance of logarithms while shifting variables as it was requested in the worksheet:

- *R* is arithmetically altered and *I* geometrically … *R* is the logarithm of intensity.

Groups agreed that shifting the variables leads to a new expression that retains the properties of function definition mentioned in the text:

- The switch of the variables leads to the inverse function.

At the same time they criticized the phrase of the text: *"if y is any function of z, then z will be a function of y"*. They argued that this may not be the case and justified their claim using contradictory examples like $y = z^2$.

Subsequently, the work was transferred to the digital environment of the Geogebra mathematical software. Students constructed graphical representations by matching pairs of independent and dependent variables to points using the Cartesian axes. Conversely, they managed to match the graph appeared on the screen as an illustration of exponential function points, where the coordinates were reversed, to the appropriate logarithmic formula despite the initial difficulties faced at this point. Groups had to return to the text and the definition:

- The expression solved with respect to the dependent variable is the formula of function ... now the dependent is magnitude, *R*.

At each step of the work students combined the elements of the historical text concerning the concept of function and the reverse function, with the supervision of multiple representations of function (formula, graph, value table) in the digital environment of Geogebra in order to accomplish graphical and algebraic resolution of the worksheet's activities:

- In the spreadsheet (value table) there is just a single value of the dependent variable for each value of the independent one, either when the dependent is *I* or when is *R*. The same is apparent in graphs too, as each number on the horizontal axis (horizontal coordinate) corresponds to a single point of the graph.

Groups concluded that logarithm is a monotonic increasing function when base α is greater than1 and a decreasing one when $\alpha < 1$ both in a graphical way, using the software, and in an algebraic way using the definition and suitable numerical values. They also studied the domain and the set of values of the logarithmic function as a reverse co-variation of the exponential and explained the symmetry of the graphs about the line $y = x$:

- The horizontal coordinates of the points on the logarithmic graph are the vertical ones of the points of exponential function. This is because reversion of functions means reversion of the variables.

The development of the discursive genesis of the work, as a result of attempting to produce adequate logical explanation, was either based to the exponential process and the operation of its rules

- If $c < k$ then $\log c < \log k$ because if $\log c = x$ and $\log k = y$, since $c < k$, it is $\alpha^x < \alpha^y$ so $x < y$,

or to direct references to logarithm as a number - exponent:

- If $b > c$ then $\log b > \log c$ because greater power means greater exponent when there is the same base higher than one.

Groups analyzed relationships and dependencies between elements of graphs, assigned the values of dependent and independent variables to point coordinates and used the elements of the enriched referential framework, i.e. the definition of the logarithm and the logarithmic function set of values, in developing lines of argumentation to support their conjectures. The use of the software promoted the development of the instrumental and discursive dimensions of work.

4 Conclusion

The knowledge of historical elements concerning the logarithmic concept prompted the use of the definition as a tool for the working on the activities as well as for the decoding of the symbolism, developing the instrumental and semiotic dimensions of the work and helping, in that way, the integration of the exponential process into the subject - logarithm. The correspondence of arithmetic and geometric progressions and the etymology of the word ‘logarithm’ were complementarily used along with the latest Euler’s definition when attempting to deal with the activities required in the worksheets. Although this process was not followed by all groups and in every phase of the work, the knowledge of the historical evolution of the concept, since the initial role of logarithm as a tool and the necessity leaded to its invention until the latest definition and its functional form, provided a more wholistic approach to mathematical concepts, which favored the development of a conceptual understanding. According to students' responses to the questionnaires, the familiarization with the historical evolution activated their interest and triggered their motives, as through the presentation of historical data, frequently asked questions by the students about the necessity of existence of the mathematical concepts and their usefulness in modern reality were answered.

The second pole of HM utilization in this didactic approach concerned the study of historical texts. It is true that students’ responses to the questionnaires have shown that they faced difficulties processing the texts. This is a fact that could be expected as, through the texts, they dealt with concepts at their source, studying in a different ‘language’, more authentic and less ‘refined’ than these they are used to in school textbooks. They developed a kind of dialectical relationship with texts. As it appeared in the transcripts, the ‘rhetorical’ formulation of mathematical relations in a significant part of the texts was a challenge for students, depriving the "facilitation" of an automatic and purely procedural perception of concepts. As shown in the course of the work during the intervention and in the transcripts, the collaborative teaching method functioned supportively to overcome the difficulties that arose. The cognitive abilities of the text were exploited by the groups, not only through study and analysis, but even through criticism of the provided information such as the prerequisites for the inverse function. Therefore, historical texts functioned as cognitive tools in the course of students' conceptual understanding of logarithm and logarithmic function as they were used as levers of work development along the axes of the semiotic, instrumental and discursive dimensions. In parallel with using historical texts, the digital software functioned in an auxiliary way for studying the logarithmic function and its properties, thus its contribution to this instructional approach was considered positive by the students. The transcribed conversations in the groups and the successful respond to the activities of the questionnaires have demonstrated a satisfactory degree of integration of the exponential process into the mathematical object - logarithm as well as the approximation of the logarithmic function as inverse co variation.

At the same time, students seemed to realize that mathematical discipline is an evolving product that is linked to the general social context in which it develops. Therefore, from a macroscopic point of view it can be said that HM contributed, in this case, in students' approaching to mathematics not as a mockery of definitions, theorems and exercises of unjustifiable existence, but as a human cultural product. However, it is natural that the use of HM specifically in a particular subject, in this case the logarithm, is not sufficient for an overall and conscious review of students' opinion about mathematics discipline. This was partly reflected in some of the students' responses who, on the one hand, agreed about the evolving nature of Mathematics but, on the other hand, they still doubted whether the didactic intervention influenced significantly their views. Despite the fact that this research is a case study which may have limitations in generalization, we could say that the utilization of HM facilitates the teaching of logarithms in secondary education and also provides an alternative way to approach mathematics from a different perspective.

REFERENCES

Artigue, M. (1992). Functions from an algebraic and graphic point of view: cognitive difficulties and teaching practices. *The concept of function: Aspects of epistemology and pedagogy*, *25*, 109–132.

Artigue, M. (1999). The teaching and learning of mathematics at the university level. *Notices of the AMS*, *46*(11), 1377–1385.

Barbin, É. (2015). Philosophies and theories behind history and education: thirty years after Hans Freudenthal. *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU*, p. 21–40, Barbin, É., Jankvist, U., & Kjeldsen, T. H. (Eds.), Copenhagen: Danish School of Education, Aarhus University.

Dawn Leigh Anderson, D., L., Berg, B., Ann Willis Sebrell, A., W., Smith, D., W., (2004). Exponentials and Logarithms, *Historical Modules Project***,** Mathematical Association of America.

Fauvel, J. (1991). Using history in mathematics education. *For the learning of mathematics*, *11*(2), 3–6.

Fried, M. N. (2008). History of Mathematics in Mathematics Education: a Saussurean Perspective. *The Mathematics Enthusiast*, *5*(2), 669–703.

Furinghetti, F. (1997). History of mathematics, mathematics education, school practice: Case studies in linking different domains. *For the learning of mathematics*, *17*(1), 55–61.

Gagatsis, A., Elia, I., & Mousoulides, N. (2006). Are registers of representations and problem solving processes on functions compartmentalized in students' thinking? *Revista Latinoamericana de Investigación en Matemática Educativa*, *9*(1), 197–224.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for research in Mathematics Education*, *25*(2), 116–140.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior*, *17*(1), 123–134.

Jankvist, U. T. (2009a). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational studies in Mathematics*, *71*(3), 235–261.

Jankvist, U. T. (2009b). History of modern applied mathematics in mathematics education. *For the Learning of Mathematics*, *29*(1), 8–13.

Kieran, C. (2007). Learning and Teaching Algebra at the Middle School through College Levels: Building Meaning for Symbols and their Manipulation. In F. K. Lester (Ed.). *Second Handbook of Research on Mathematics Teaching,* (pp. 707–762). Charlotte NC: Information Age Publishing Inc.

Lagrange, J. B., & Psycharis, G. (2014). Investigating the potential of computer environments for the teaching and learning of functions: A double analysis from two research traditions. *Technology, Knowledge and Learning*, *19*(3), 255–286.

Minh, T. K., & Lagrange, J. B. (2016). Connected functional working spaces: a framework for the teaching and learning of functions at upper secondary level. *ZDM*, *48*(6), 793–807.

Panagiotou, E. (2014). History of Logarithms and its Pedagogical Dimensions. *Epistimes Agogis*, 92–113, (in Greek).

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Elements of Pedagogy and Epistemology*, (pp. 23–58). Washington, DC: Notes and Reports Series of the Mathematical Association of America.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics teaching*, *77*(1), 20–26.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Thomaidis, Y. (1986). Origin and Applications of Mathematics Teaching Theory - The Example of Logarithms. *Euclid C*, *13*, 1–30, (in Greek).

Tzanakis, C., & Thomaidis, Y. (2000). Integrating the close historial development of mathematics and physics in mathematics education: Some methodological and epistemological remarks. *For the Learning of Mathematics*, *20*(1), 44–55.

Tzanakis, C., & Thomaidis, Y. (2012). Classifying the arguments and methodological schemes for integrating history in mathematics education. In B. Sriraman (Ed.). *Crossroads in the history of mathematics and mathematics education*, (pp. 247–295). Charlotte NC: Information Age Publishing Inc.

Tzanakis, C., Arcavi, A., de Sa, C. C., Isoda, M., Lit, C. K., Niss, M., de Carvalho J. P., Rodriguez M., & Siu, M. K. (2002). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel, & J. van Maanen (Eds.). *History in mathematics education: the ICMI study*, (pp. 201–240). Dordrecht: Springer.

Vagliardo, J. (2006). Substantive knowledge and mindful use of logarithms: A conceptual analysis for mathematics educators. *Focus on Learning Problems in Mathematics*, *28*(3), 90+.

A PRACTICAL STUDY OF USING THE HISTORY OF MATHEMATICS IN A FLIPPED CLASSROOM

Xiaoni LI
East China Normal University, China
370599934@qq.com

ABSTRACT

Flipping classroom is a new teaching mode. It comes from America, changing the traditional teaching mode. Students watch the teaching video produced by their teacher and do some exercises before class. In class, the teacher uses some exercises to test whether students understand the knowledge in the video they have watched, and if they even have some problems, teacher helps them to master and understand this knowledge. On the other hand, in the traditional classes, students learn the knowledge in class and internalize the knowledge after class by doing some exercises. Flipping classroom interchanges the two stages. Students learn the knowledge before classes by themselves using the teaching video, and internalize the knowledge in the classroom.

In China, the practice of using flipping classroom mode began in 2011. Many primary and secondary schools are using flipping classroom mode in their classes now. It has become a hot topic in China. In the process of applying the flipping classroom mode, there are many problems, such as teachers just let students doing much more difficulty exercises in classes. There are teachers, who are confused arguing that since their students have acquired the knowledge before classes, what can they teach in class except doing more exercises?

We believe that the integration of the history of mathematics is a good way to solve these problems. In this context, we started a study in a junior middle school in Shanghai. It is a private junior high school, students of it all having a good level of self-study ability. The school has used flipping classroom for three years. All teachers in the school have the ability to make micro-video.

The lesson "the midsegment of a triangle" is our case study of using the history of mathematics in flipping classroom. Before the class, the teacher produced a micro-video to introduce the definition of the midsegment of a triangle and the property of it, and she gives a way that comes from the textbook to prove the property. Then she gives for homework to students: are there other ways to prove this property? In the class, students give feedback from the micro-video first, then share the other methods they have thought, and finally the teacher guides the students to use the median of the triangle to make different small triangles in the big triangle in order to prove the property (fig. 1). After five minutes discussion in groups, students share their way of proof (fig. 2). The way they used is the same with the method of Euclid. Then, the teacher uses another video to show that we can use the method of calculating the area of triangles given by Liu Hui in the "Nine Chapters" in order to prove the property. Finally, the students do some exercises to remember this property.

After the class, the questionnaire shows that students are very interested in this lesson, can understand the historical material we used, and know the mathematical concept of

transformation. They are impressed by the cut & paste method of Liu Hui, and marvel at the wisdom of the ancients.

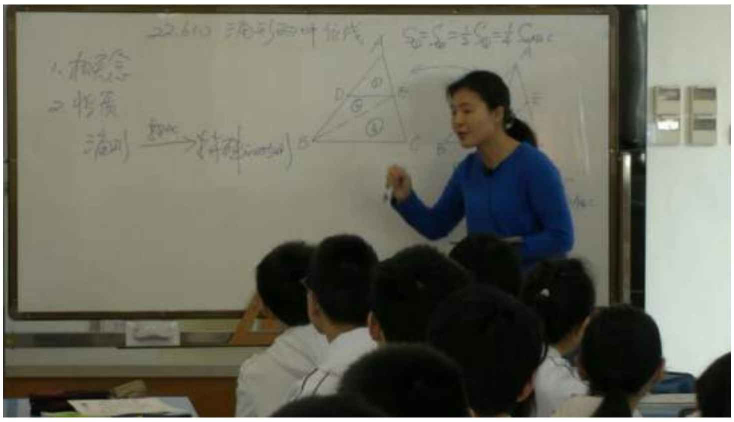


Figure 1: The teacher guides students using areas of small triangles to prove the property

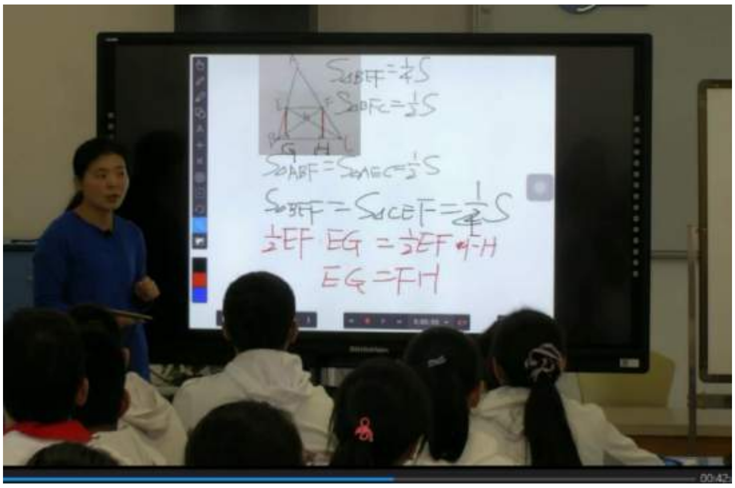


Figure 2: The way students given to prove the property

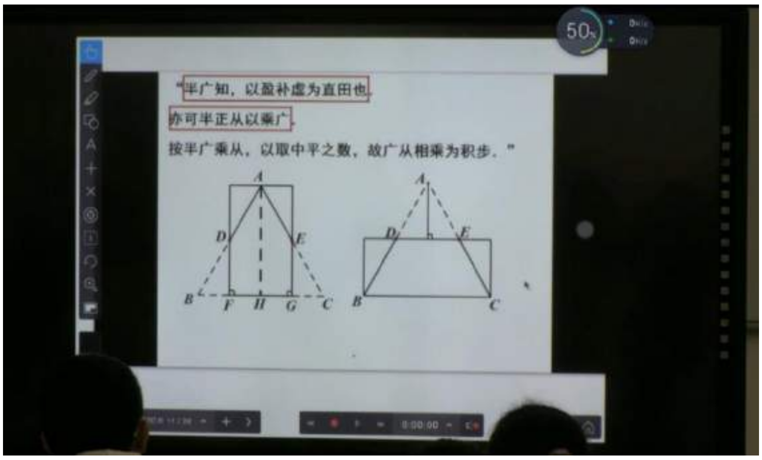


Figure 3: The way Liu Hui used to prove the area of triangles

AN INTERNATIONAL COMPARATIVE STUDY ON HOW MATHEMATICAL CULTURE IS IMPLEMENTED IN THE TEXTBOOKS

Po-Hung LIU
National Chin-Yi University of Technology, Taiping District, Taichung City, Taiwan
liuph@ncut.edu.tw

ABSTRACT

It is claimed in this study that "mathematics in culture" (MiC, a macro-view of the development of mathematical knowledge) and "culture of mathematics" (CoM, a micro-view of the development of mathematical knowledge) are the two main dimensions of mathematical culture. This study has two purposes. First, I attempted to identify the constituents of mathematical culture in terms of the interrelationship between mathematics and culture, particularly the interaction of mathematics with Eastern and Western culture. Second, I selected three sets of high school mathematics textbooks from Taiwan, China, and the United States respectively to investigate how, if any, mathematical culture is implemented. A quantitative analysis revealed that: (1) Taiwanese and Chinese textbooks have many features in common. Both sets of textbooks stress applications of math in daily life and provide several opportunities for exploring, but ignore ethnic features, mathematical dialogue, and the evolution of mathematical concepts. In contrast, the US textbook employs a lot more problems of applications of math in daily life and in nature, with nearly half of the examples and exercises devoted to promoting students' problem-solving abilities to resolve authentic problems. However, as with the other two sets of textbooks, a treatment of ethnic features is absent.

Keywords: International comparison, textbook analysis, mathematical culture

1 Introduction

The United Nations Educational, Scientific and Cultural Organization (UNESCO) proposedthe years of 2003~2012 as the United Nations Literacy Decade (UNLD) (UNESCO, 2003). Its slogan "Literacy as Freedom" aims to increase literacy levels and to empower all people everywhere. The original meaning of literacy is being able to read and write. But in the modern sense, "Literacy is about more than reading and writing – it is about how we communicate in society. It is about social practices and relationships, about knowledge, language and culture." (UNESCO, 2003, p.1).School is not a place for cultivating all kinds of literacy, but it is well-recognized that one of the best approaches of empowering students' literacy, particularly for those that are knowledge-based, is school education. Tarr et al. (2008) indicated that though teachers' teaching practices may be influenced by their beliefs, knowledge, and students' responses, the textbook is still one of key factors determining their instructional decisions. Since mathematical culture is an essential component of mathematical literacy, this study aims to investigate and compare how mathematical culture is implemented in the high school mathematics textbooks of Taiwan, China, and the United States.

2 Constituents of Mathematical Culture

Historically, mathematics education in various countries initially focused on elite education and vocational education, and rarely considered the relationship among mathematics, society, history and philosophy. However, due to the increasing popularity of higher education in the late twentieth century, the advocacy of "mathematics for all" has arisen. Mathematics is no longer viewed just a practical instrument, but also as a discipline for lifelong learning. The cultural facet of mathematics thereafter begins to receive increasing attention as well. However, the scope of mathematics culture is broad, and it is necessary to clarify the content of literacy for mathematical culture.

Based on their review of 164 definitions of culture, Kroeber and Kluckhohn (1952) proposed the following definition of culture:

> Culture consists of patterns, explicit and implicit, of and for behaviour acquired and transmitted by symbols, constituting the distinctive achievement of human groups, including their embodiments in artifacts. (p. 181)

According to Kroeber and Kluckhohn, the essential core of culture consists of traditional ideas and attached values, and culture systems may, on the one hand, be considered as the products of human action, on the other as decisive factors of further action. Following Kroeber and Kluckhohn's conception, this study defines mathematical culture as follows:

> Mathematical culture consists of patterns, explicit and implicit, inductive and deductive, logical and illogical, of and for problem solving behavior acquired and transmitted by symbols, constituting the distinctive achievement of human groups in general, the mathematician in particular.

This definition stresses the inductive and deductive features of mathematical knowledge in the making, and the logical and illogical development of mathematics. While searching for the cultural basis of mathematics, Wilder (1950) reminded us that mathematics is a part of, and is influenced by, the culture in which it is found. In this manner, the culture dominates its elements, and in particular its mathematics. For instance, a Chinese mathematician living about the year 1200 C.E. would mainly focus on computing with numbers and solving equations without paying attention to geometry as the ancient Greeks understood it. In contrast, a Greek mathematician of 200 B.C.E. would focus more on geometrical proofs than on algebra and numerical computation as the Chinese practiced it. This depicts the mathematics in culture. On the other hand, mathematicians in different cultures share some common methodological views and paradigms for working on mathematics that ensure their creations can be recognized by other mathematicians. These common methodological views and paradigms shape a working academic culture, which is the culture of mathematics. This study therefore proposes two major constituents of mathematical culture: mathematics in culture and culture of mathematics.

Mathematics in culture (MiC) is a macro-view description about how mathematical knowledge as a whole has expanded in various cultures, and includes three components:

- *Historical development*: mathematics developed throughout the history to establish its distinguished features.
- *Social needs*: mathematics grew along with the society to meet various demands.
- *Ethnic features*: mathematics evolved over time with the influence of its host culture and gradually established distinct ethnic-characteristics.

On the other hand, the culture of mathematics (CoM) refers to the emergence and construction of mathematical concepts, which is a micro-view description about the dialectical methodology by which a mathematical idea is created and validated through conversation among mathematicians. This methodology includes the following three components:

- *Inductive conjecture*: This component is the very beginning of how a mathematical idea is revealed and created, which can be seen as a thought experiment (Polya, 1954).
- *Deductive validation*: An isolated fact may not be seen as a generalized truth without a logical verification. Deduction is a significant feature of mathematics.
- *Social construction*: It has been generally held that mathematical knowledge not only is a product of self-construction, but also a polished outcome of public dialectic (Ernest, 1998).

Note that a clear distinction between MiC and CoMis impossible and both are intertwined with each other. This suggests the framework for the constituents of mathematical culture shown in Figure 2.1.

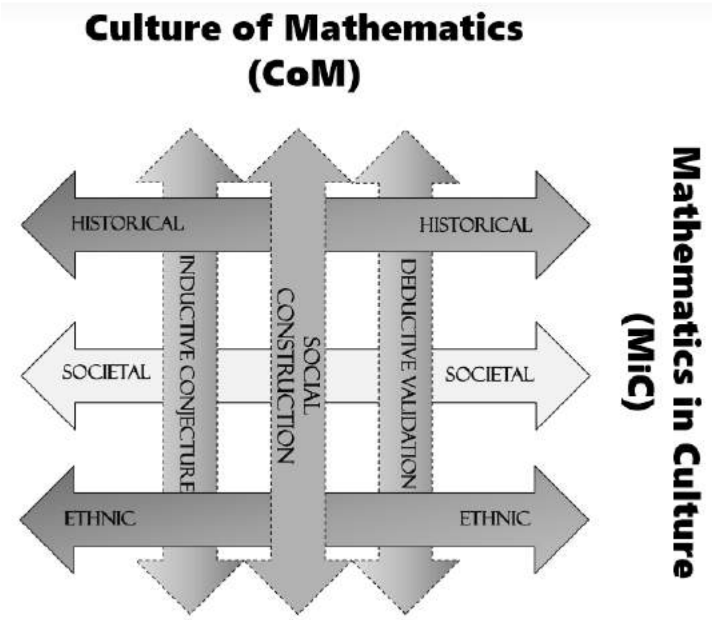


Figure 2.1: The constituents of mathematical culture

Furthermore, based on theoretical consideration, several indices for MiC and CoM were created for coding the components of mathematical culture in the textbooks. There are 13 indices for Mic (Table 2.1) and 8 for CoM (Table 2.2). The validity and appropriateness of each index has been checked and validated by a historian of mathematics and a HPM researcher.

Table 2.1: Indices for MiC

MiC	Index	Definition
History	Concept (H-C)	The origin of concepts
	Method (H-Me)	Different methods in history
	Problem (H-P)	Famous problems in history
	Episode (H-E)	Significant events in history
	Mathematician (H-I)	Distinguished mathematicians in history
	Nature (S-N)	Applications of math in nature
	Living (S-L)	Applications of math in daily life

Society	Economy (S-E)	Applications of math in economy
	Politics (S-P)	Applications of math in politics
	Arts (S-A)	Applications of math in arts
Ethnic	Context (E-C)	Concepts different cultural context
	Difference (E-D)	Approaches in different cultures
	Philosophy (E-P)	Concepts and their philosophical background

Table 2.2: Indices for CoM

CoM	Index	Definition
I & G*	Survey (I & G-O)	Providing opportunities forobserving examples or data
	Pattern (I & G-P)	Leading students to look for patterns
	Conjecture (I & G-C)	Encouraging students to make conjectures
D&P**	Intuition (D & P-I)	Explaining properties by intuitive observation
	Example (D & P-E)	Explaining properties by particular examples
	Logic (D & P-L)	Proving properties by logical deduction
C&D***	Community (C & D-C)	Mathematical dialogue among mathematicians
	Evolution (C & D-E)	Evolution of mathematical concepts

*I & G: Induction and Guessing, **D & P: Deduction and Prove, ***C & D: Community and Dialectic

3 Target Textbooks

Three high school mathematics textbook series produced inTaiwan (T-textbook), China (C-textbook), and the United States (US-textbook) were selected for review. T-textbook and C-textbook are the most popular textbooks in Taiwan and China respectively, and US-textbook is purchased by the National Academy for Educational Research in Taiwan, and is therefore accessible by the researcher. T-textbook has 6 volumes and C-textbook has 5 volumes. US-textbook consists of 2 volumes of algebra and 1 volume of geometry. All examples and exercises in these textbooks were coded.

4 Inter-Rater Reliability

To insure the reliability of the analysis, in the pilot stage, two raters with mathematics and mathematics education background were trained to do the categorization of each index. Their analyses were not regarded as reliable until the agreement rate was above 90%.During the subsequent analysis stage, the third rater (a high school mathematics teacher with a master's degree in education) and the researcher made final decision if there was any inconsistency between the two raters.

5 Results

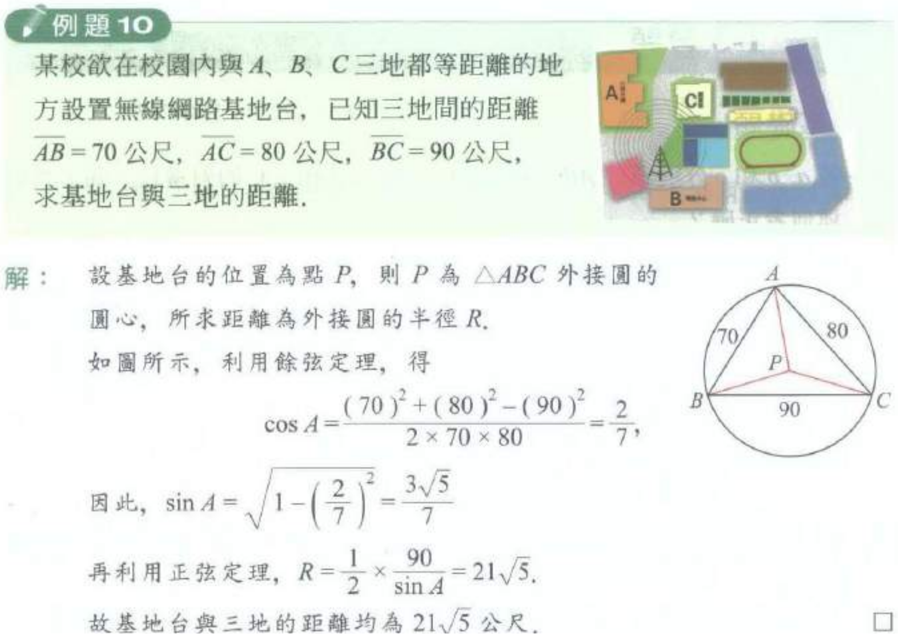
5.1 T-textbook

Table 5.1 shows the frequency and percentage for each index in the Taiwanese T-textbook textbook. It appears that, for MiC dimension, S-L (applications of math in daily life) is the most common index (26.7%) among all. For instance, following the introduction of Law of sine and Law of cosine, students are asked to determine the location of a cellular base station by applying the two laws such that the cellular base station has equal distance from three campus buildings A, B, and C. In another example, lengths of the football (i.e., soccer) field and football net are given, and a player on the point P that is 35 meters away from the bottom edge of the diagram is set to kick the ball. The student is then asked to find the tangent value of the kicking angle APB. Though the two examples can be solved by applying trigonometric identities, they are not realistic because it is impossible to build a cellular base station in the campus and the football player's main concern is the kicking angle itself, but not the tangent value of that angle. As for CoM dimension, I&G-O (providing opportunities for observing examples and data) is most widely used (23.9%).

Note that the History index (H-C) receives less attention, with only 15.7% of the examples and exercises related to it. For instance, an example related to the application of Apollonius Circle is given to find the running trajectory of two hunting dogs. Furthermore, both the Ethnic features and the C&D index (mathematical dialogue and evolution of mathematical concepts) are totally absent in the text.

Table 5.1: Frequencies and percentages for each index in T-textbook

Index	**S-L**	**I&G-O**	**D&P-E**	**H-I**	**D&P-I**	**I&G-P**	**H-C**	**S-N**
Frequency	56	48	34	17	16	12	8	7
Percentage	26.7%	23.9%	16.2%	8%	7.6%	5.7%	3.8%	3.3%
Index	**I&G-C**	**H-P**	**H-E**	**H-M**	**S-A**			
Frequency	3	3	3	2	1			
Percentage	1.4%	1.4%	1.4%	0.95%	0.47%			


例題 10

某校欲在校園內與 A、B、C 三地都等距離的地方設置無線網路基地台，已知三地間的距離 $\overline{AB}=70$ 公尺，$\overline{AC}=80$ 公尺，$\overline{BC}=90$ 公尺，求基地台與三地的距離.

解： 設基地台的位置為點 P，則 P 為 $\triangle ABC$ 外接圓的圓心，所求距離為外接圓的半徑 R.

如圖所示，利用餘弦定理，得

$$\cos A=\frac{(70)^2+(80)^2-(90)^2}{2\times 70\times 80}=\frac{2}{7},$$

因此，$\sin A=\sqrt{1-\left(\frac{2}{7}\right)^2}=\frac{3\sqrt{5}}{7}$

再利用正弦定理，$R=\frac{1}{2}\times\frac{90}{\sin A}=21\sqrt{5}$.

故基地台與三地的距離均為 $21\sqrt{5}$ 公尺. □

Figure 5.1: Determine the location of cellular base station

Figure 5.2: Find the tangent of kicking angle

5.2 C-textbook

Table 5.2 indicates the frequency and percentage for each index in the Chinese C-textbook. Similar to the T-textbook, S-L (applications of math in daily life) is the most common index (25.7%) among all MiC indices. Of the CoM indices, I&G-O (providing opportunities for observing examples or data) and D&P-E (explaining properties by particular examples) are two generally adopted strategies. Even though the percentage of S-L in the C-textbook is almost the same as that in the T-textbook, S-L problems in the C-textbook are more realistic in nature. This appears to be due to a greater use of mathematical modeling as a means for increasing students' problem solving ability. For instance, in the beginning of 'The Concept of Functions', an example is given below:

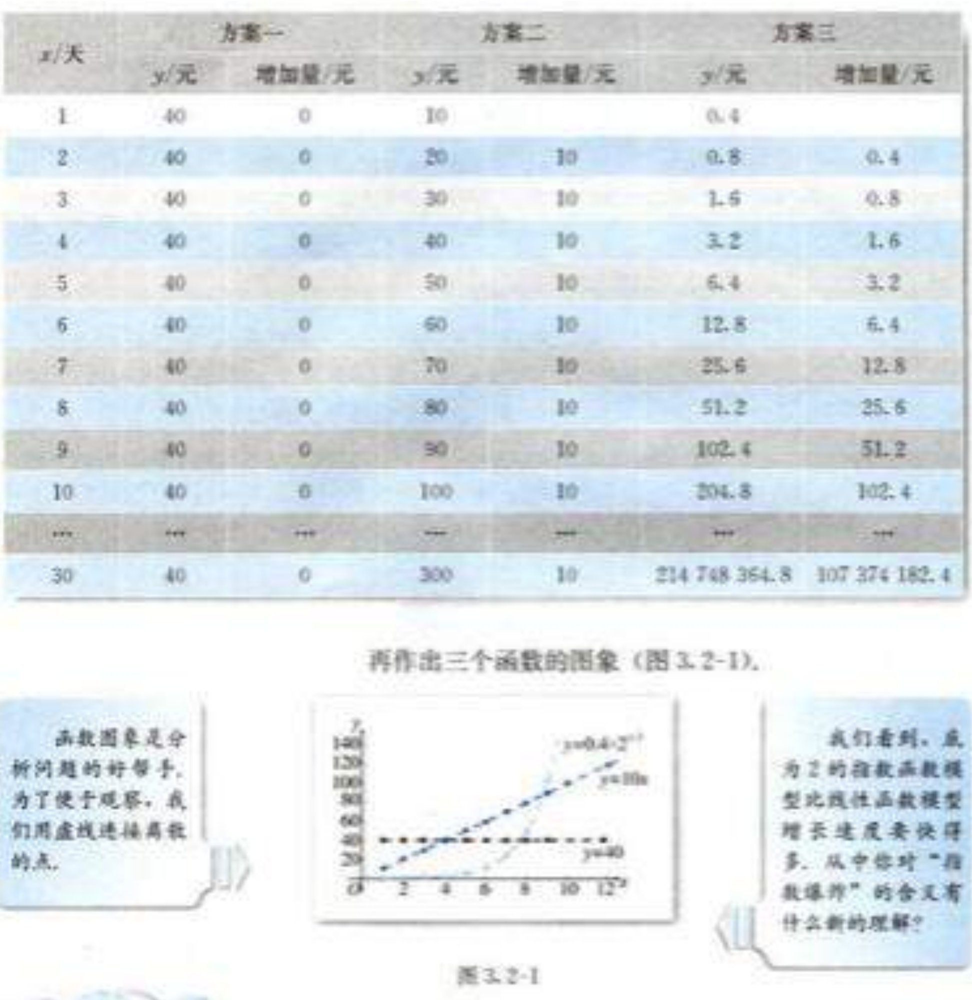
> If you plan to do investment and there are three different proposals.
> 1. Proposal A gets $40 reward every day.
> 2. Proposal B gets $10 reward on the first day, $20 on the second day, $30 on the third day, and so on.
> 3. Proposal C gets $0.4 reward on the first day and double reward thereafter.
>
> Which one do you prefer?

This example is followed by a table indicating the reward for each proposal in 30 days and a graph showing the tendency for growth of each proposal. Students are then led to an algebraic representation for each proposal, thereby fully demonstrating the rule of four (verbal, numerical, graphical, and algebraic representation) for introducing the concept of functions.

Analogous to the data for the C-textbook, it was found that only 14.3% of the examples and exercises are related to the History index, and that the Ethnic features and the C&D indices are also lacking.

Table 5.2: Frequencies and percentages for each index in P-version

Index	S-L	I&G-O	D&P-E	S-N	D&P-I	H-I	H-C	H-P
Frequency	54	36	34	28	16	10	7	6
Percentage	25.7%	17.1%	16.2%	13.3%	7.6%	4.8%	3.3%	2.9%
Index	**H-M**	**S-E**	**I&G-P**	**I&G-C**	**H-E**	**D&P-L**		
Frequency	5	5	4	2	2	1		
Percentage	2.4%	2.4%	1.9%	0.95%	0.95%	0.47%		



x/天	方案一		方案二		方案三	
	y/元	增加量/元	y/元	增加量/元	y/元	增加量/元
1	40	0	10		0.4	
2	40	0	20	10	0.8	0.4
3	40	0	30	10	1.6	0.8
4	40	0	40	10	3.2	1.6
5	40	0	50	10	6.4	3.2
6	40	0	60	10	12.8	6.4
7	40	0	70	10	25.6	12.8
8	40	0	80	10	51.2	25.6
9	40	0	90	10	102.4	51.2
10	40	0	100	10	204.8	102.4
…	…	…	…	…	…	…
30	40	0	300	10	214 748 364.8	107 374 182.4

再作出三个函数的图象（图3.2-1）.

图3.2-1

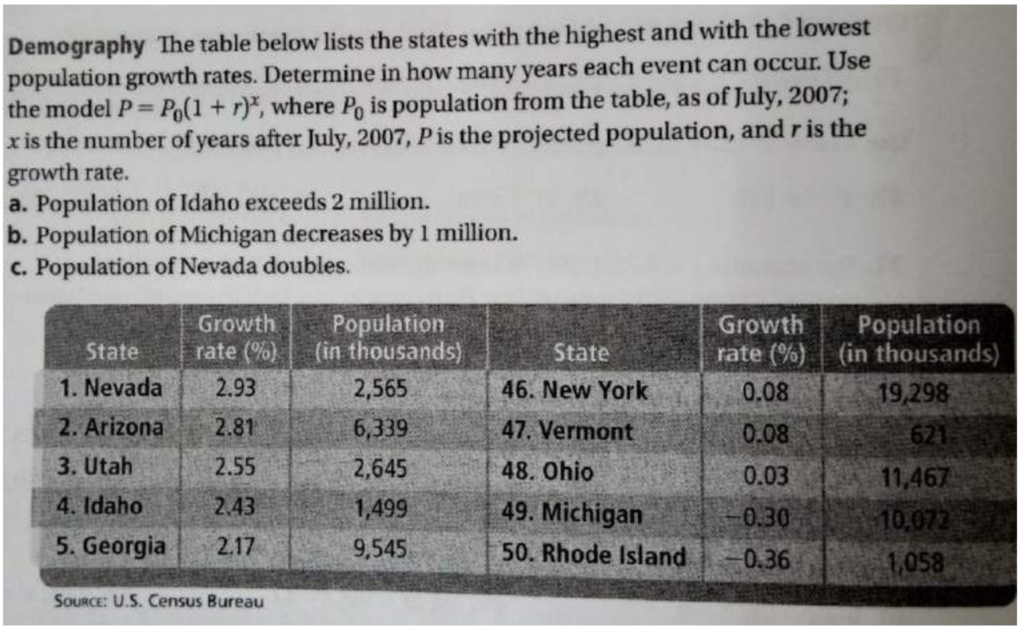
Figure 5.3: Table and graph of the reward for each proposal

5.3 US-textbook

Similar to the cases of the C-textbook and T-textbook, the US-textbook employs many S-L (applications of math in daily life) problems (35.4% as shown in the Table 5.3), intended as a meansto promote students' interest and increase their understanding. This percentage is significantly greater than that of either the C-textbookor the T-textbook. It was further noted that the S-L problems in the US-textbook are more realistic in nature. For instance, in a demography problem (Figure 5.4), the population in the year 2007 and growth rate of several states are given, and students are asked to determine in how many years it will take for each state to reach a specified population.

Table 5.3: Frequencies and percentages for each index in US-textbook

Index	**S-L**	**D&P-I**	**I&G-O**	**S-N**	**D&P-E**	**I&G-P**	**D&P-L**	**H-P**
Frequency	258	143	116	84	68	12	12	12
Percentage	35.4%	19.6%	15.9%	11.5%	9.3%	1.6%	1.6%	1.6%
Index	**I&G-C**	**S-P**	**S-A**	**S-E**	**H-E**	**C&D-E**		
Frequency	6	6	4	3	2	2		
Percentage	0.8%	0.8%	0.55%	0.41%	0.27%	0.27%		



Demography The table below lists the states with the highest and with the lowest population growth rates. Determine in how many years each event can occur. Use the model $P = P_0(1 + r)^x$, where P_0 is population from the table, as of July, 2007; x is the number of years after July, 2007, P is the projected population, and r is the growth rate.

a. Population of Idaho exceeds 2 million.

b. Population of Michigan decreases by 1 million.

c. Population of Nevada doubles.

State	Growth rate (%)	Population (in thousands)	State	Growth rate (%)	Population (in thousands)
1. Nevada	2.93	2,565	46. New York	0.08	19,298
2. Arizona	2.81	6,339	47. Vermont	0.08	621
3. Utah	2.55	2,645	48. Ohio	0.03	11,467
4. Idaho	2.43	1,499	49. Michigan	−0.30	10,072
5. Georgia	2.17	9,545	50. Rhode Island	−0.36	1,058

Source: U.S. Census Bureau

Figure 5.4: The population and growth rate of several states in the US

Furthermore, the US-textbook apparently stresses the role of observation while working on the problems since D&P-I (explaining properties by intuitive survey) and I&G-O (providing opportunities for observing examples and data) percentages are19.6% and 15.9% respectively. In one “looking for a pattern” exercise, the first five rows of Pascal Triangle are given (Figure 5.5), and students are asked (*a*) to predict the numbers in the seventh row and (*b*) to find the sum of the numbers in each of the first five rows and predict the sum of the numbers in the seventh row. Another particular feature of this textbook series is that, as compared to the other two textbooks, the US-textbook tends to emphasize the S-N index (applications of math in nature). A sample problem about Exponential Functions that is related to archaeology is shown below:

Archaeologists use carbon-14, which has a half-life of 5730 years, to determine the age of artifacts in carbon dating. Write the exponential decay function for a 24-mg sample. How much carbon-14 remains after 30millennia?

However, the percentage of History index of the US-textbook is extremely low at only1.87% total.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Figure 5.5: Pascal triangle

6 Conclusions and Discussions

This study aimed to investigate how mathematical culture is implemented in high school mathematics textbooks from Taiwan, China, and the United States. We defined mathematical culture and created a framework consisting of mathematics in culture (MiC) and culture of mathematics (CoM) to serve as a guideline for the analysis. Results show that the T-textbook from Taiwan and the C-textbook from China have many features in common. Both sets of the textbook stress S-L (applications of math in daily life), use I&G-O (providing opportunities for observing examples and data) widely, and totally ignore Ethnic features and the C&D index (mathematical dialogue and evolution of mathematical concepts). Additionally, it was found that the C-textbook is likely to use mathematical modeling as a means for increasing students' problem solving ability. In contrast, S-L problems in the T-textbook are more unrealistic. Compared to the T-textbook and the C-textbook, the US textbook employs more problems of Society index (35.4% of S-L and 11.5% of S-N). Nearly half of the examples and exercises in the US-textbook are devoted to promoting students' problem-solving abilities to resolve authentic problems. However, as in the other two versions, the index Ethnic features is absent. Introducing in what ways an identical mathematical concept was implemented in different cultures may trigger learners' critical thinking and mathematical understanding. Though ethnic features are also related to History index, examples and exercises would not be counted as the Ethnic index if a cultural comparison was not made.

Owing to the ways in which mathematical culture reflects the true nature of mathematics, it should receive consistent attention from teachers and students, and the mathematics textbook is an appropriate agent for achieving the purpose. However, the present study found that certain components of mathematical culture are not employed very widely or profoundly in the high school mathematics textbooks from Taiwan, China, and the United States that were analyzed. The three sets of textbooks overwhelmingly emphasize S-L index, which is expected, but overlook the evolution of mathematical concepts, ethnic features, and the social construction of mathematical knowledge, which are all related to the history of mathematics. In this manner, mathematics is treated more like an instrument than a particular kind of cultural wisdom. As a preliminary stage for entering college, high school students are further supposed to realize that mathematics not only can be applied to resolve daily practical problems, but may be advanced to an abstract level for its own sake. For instance, a historical and inductive approach may connect the Fibonacci sequence with the golden ratio. Further, while introducing the concept of infinite series, the problem of determining the length of coastline can be associated with geometrical fractal. Unfortunately, the three sets of textbooks in the present study also fail to meet that purpose. The role of history of mathematics in teaching has been advocated for decades (Jankvist, 2009, 2011; Liu & Niess, 2006; Liu, 2009; Radford, 1997). Only history can address the ways in which concepts were created and polished through the ages, reveal the distinctive mathematical characteristics of different cultures, and demonstrate the dialectical nature of mathematical knowledge. When studying the textbook, students usually spend more time on doing examples and exercises than reading the text. To show the full essence of mathematical culture, a mathematics textbook should therefore include examples and exercises that refer to history of mathematics as a means to shed more light on the nature of mathematical thinking and knowledge construction.

REFERENCES

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. Albany, NY: State University of New York Press.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education.*Educational Studies in Mathematics*, *71*(3), 235-261.

Jankvist, U. T. (2011).Anchoring students' metaperspectivediscussions of history in mathematics.*Journal for Research in Mathematics Education*, *42*(4), 346-385.

Kroeber, A. L., & Kluckhohn, C. (1952). *Culture*: *A critical review of concepts and definitions*. Cambridge, MA: The Museum.

Liu, P.-H. & Niess, M. (2006). An exploratory study of college students' views of mathematical thinking in a historical approach calculus course. *Mathematical Thinking and Learning*, *8*(4), 373–406.

Liu, P.-H. (2009). History as a platform for developing college students' epistemological beliefs of mathematics. *International Journal of Science and Mathematics Education*, *7*(3), 473-499

Pólya, G. (1954). *Mathematics and plausible reasoning*. Princeton, NJ: Princeton University Press.

Radford, L. (1997). On psychology, historical epistemology, and the teaching of mathematics: towards a socio-cultural history of mathematics. *For the Learning of Mathematics*, *17*(1), 26–33.

Tarr, J. E., Reys, R. E., Reys, B. J., Chávez, Ó., Shih, J., & Osterlind, S. J. (2008). The impact of middle-grades mathematics curricula and the classroom learning environment on student achievement. *Journal for Research in Mathematics Education*, *39*(3), 247–280.

Wilder, R. L. (1950). Cultural basis of mathematics. *Proceedings of International Congress of Mathematicians*, 258-271. Retrieved from
http://www.mathunion.org/ICM/ICM1950.1/Main/icm1950.1.0258.0271.ocr.pdf on 3rd November, 2012.

USING A HISTORICAL PROBLEM IN A MATHEMATICS PROBLEM SOLVING CLASS

Yiu-Kwong MAN
The Education University of Hong Kong, Department of Mathematics and Information Technology, Tai Po, Hong Kong
ykman@eduhk.hk

ABSTRACT

The jug problem is a famous historical problem, which can be found in both the Eastern and the Western literature. The problem is also called the Tartaglian vessel puzzle, Poisson's jug problem or Han Xin's Oil dividing problem, etc. There are various ways to solve this problem, such as the heuristic methods, the search methods or the graphical (or billiard) methods, which can often be found in the textbooks or reference books of problem solving, discrete mathematics, recreational mathematics, number theory, computer programming or artificial intelligence, etc. In this presentation, we will report how to introduce this interesting problem in a mathematics problem solving class, as well as the strategies adopted in finding the optimal solution of the problem, in the sense that the number of steps involved will be the least possible.

REFERENCES

Ball, W. W. R., & Coxeter, H. S. M. (1987). *Mathematical Recreations and Essays*. New York: Dover.

Coxeter, H. S. M., & Greitzer, S. L. (1967). *Geometry Revisited.* Washington, DC: The Mathematical Association of America.

Dudeney, H. E. (1958). *Amusements in Mathematics*. New York: Dover.

Friesen, D. D. (2016). Using the water jugs puzzle in management science class, *Journal of Higher Education Theory and Practice*, *16*, 31-37.

Harvey, B. (1997). *Computer Science Logo Style*: *Symbolic Computing.* MA: Massachusetts Institute of Technology.

Kasner, E. & Newman, J. (1958). *Mathematics and the Imagination*. New York: Dover.

Man, Y. K. (2012). Solving the water jugs problem by an integer sequence approach. *International Journal of Mathematical Education in Science and Technology*, *43*(1), 109-113.

Man, Y. K. (2015). On computing the measurable amounts of the two jugs problem. *Mathematics in Computer Science*, 9, 453-459.

Man, Y. K. (2016). On optimal solution of the general two jugs problem. *Advances in Theoretical and Applied Mathematics*, *11*(2), 137-144.

McDiarmid, C. J. & Alfonsin, J. R. (1994). Sharing jugs of wine. *Discrete Mathematics*, *125*, 279-287.

Ore, O. (1988). *Number Theory and Its History*. New York: Dover.

Polya, G. (1945). *How to Solve It*. New Jersey: Princeton University Press.

Steinhaus, H. (1999). *Mathematical Snapshots*. New York: Dover.

Takenouchi, O., et al (2000). *Jinkoki* (English Translation). Tokyo: Wasan Institute.

Thomas, G. P. (1995). The water jugs problem: solutions from artificial intelligence and mathematical viewpoints. *Mathematics in School*, *24*, 34-37.

Tweedie, M. C. K. (1939). A graphical method of solving Tartaglian measuring puzzles. *Mathematical Gazzette*, *23* (255), 278–282.

Yoshinobu, S., Wilkins, J., Cuoco, A., & Goldenberg, E.P. (2007). The jug problem reloaded. *Mathematics Teacher*, 101, 229-232.

THE USE OF HISTORICAL MATERIALS IN MATHEMATICS TEACHING

The case of logarithms

Xin WANG
College of Teacher Education, East China Normal University, Shanghai, China
wangxinecnu@163.com

ABSTRACT

Dividing the level of curriculum helps us to analyze the issues that arise in the implementation of the curriculum. Mathematics curriculum can be divided into four levels: the "intended" curriculum, the "intended implemented" curriculum, the "implemented" curriculum and the "attained" curriculum (Garden, 1987). This paper mainly studies the "intended" curriculum, the "intended implemented" curriculum and the "implemented" curriculum of the history of mathematics in mathematics curriculum.

In Mainland China, the "intended" curriculum is the goals and requirements in mathematics curriculum standard. Reflecting on the cultural value of mathematics is one of the basic ideas. The curriculum standard states: "Mathematics is an important part of human culture. Mathematics curriculum should properly reflect the history, application and development trend of mathematics, the role of mathematics in promoting social development, the social needs of mathematics, the ideological system of mathematical science, the aesthetic value of mathematics, and the innovative spirit of mathematicians. Mathematics curriculum should help students to understand the role of mathematics in the development of human civilization, and gradually form the correct view of mathematics."

In order to achieve the "intended" curriculum mentioned above, the "intended implemented" curriculum sets up some teaching materials in textbook, which aims to provide teachers with abundant teaching resources, and help students to construct a better understanding of mathematics knowledge. Among the most widely used math textbook in China, there are 24 teaching materials, 12 of which are about the history of mathematics.

Table 1: Historical Materials

	English Title
1	The development of function
2	The invention of logarithms
3	Solving equations in history in China and foreign countries
4	Descriptive geometry andMonge
5	Euclid's *Elements* and axiomatic methods
6	Descartes and analytic geometry
7	Cyclotomic method
8	Probability and cryptology
9	Trigonometry and astronomy
10	The origin of vector and vector notation
11	Helen and Qin Jiushao
12	Fibonacci sequence

Below is the main content of the material on "The invention of logarithm".

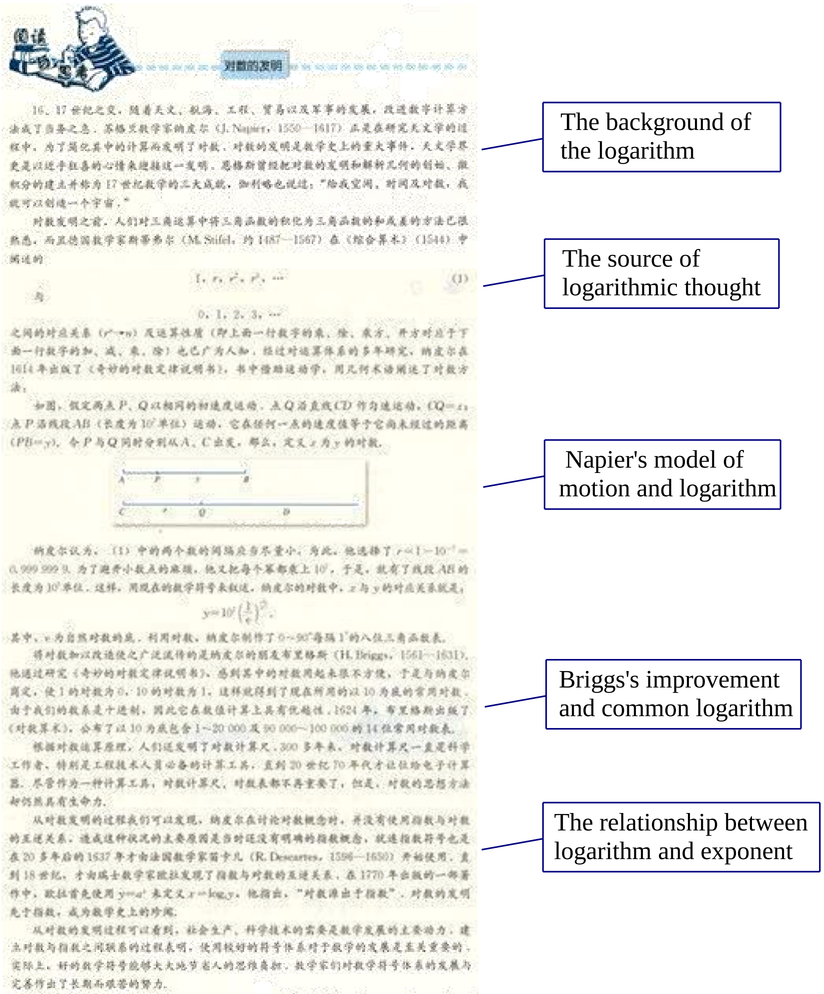


Figure 1: Teaching Material of Logarithm

The “implemented” curriculum refers to the implementation of curriculum in practical teaching. What is the use of historical materials in textbook since 2004? Taking the historical material about logarithms as an example, this study surveyed 315 teachers from secondary school in China and interviewed 8 of them.

Question 1: How much attention do teachers pay to historical materials in teaching?

In the survey result, 1.9% of teachers are always concerned about historical materials, and 23.5% of teachers often concerned, 49.5% of teachers sometimes concerned, 21.9% of teachers seldom concerned, while 3.2% of teachers never concerned.

Question 2: How do teachers use historical materials?

The use of historical materials can be divided into three categories: positive, neutral and negative.72.4% of teachers are positive; they will integrate the appropriate historical materials into teaching to create a situation or introduce the process of the development of relevant mathematical concepts. 25.4% of teachers are neutral; they will ask students to read, understand and think after class. Unfortunately, there are 2.2% of teachers who have a negative attitude towards historical materials. They have no requirement for materials, and even think that integrating them into teaching will impede progress.

Question 3: Why do teachers use or not use historical materials?

In the interview, teacher T1 said: “Historical material about logarithms can stimulate students’ interest in learning and help students understand the invention of logarithms”. Teacher T2 said: “Since class time is limited, there is no time to arrange the teaching of historical materials. And there is a lot of supplementary work to do, so I have no choice but to let students read by themselves”. Teacher T3 said: “Mathematics is so difficult. Integrating the historical materials into class will greatly influence the progress of teaching.” Therefore, the factors that influence teachers’ use of historical materials mainly include the correlation between materials and examinations, the correlation between materials and mathematics, the progress of teaching and the basis of students’ knowledge.

The survey result of the “implemented” curriculum shows that the use of historical materials is not ideal; teachers are less concerned about them. Even if some teachers tried to use such materials, the implementation mostly stayed at a fairly low level, such as “complementation” or “replication”, and failed to fully realize the requirements in the “intended” curriculum. From the perspective of historical material itself, we need to develop more resources to improve the use of it. Here are some suggestions: (1) To focus on the interesting stories that happened to Napier to make the material more interesting; (2) To make more natural the transition from discrete arithmetic progressions and geometric progressions to a continuous motion model, so as to improve learning and meet the students’ cognitive basis; (3) To reveal the significance and practical value of logarithms in order to improve the effectiveness of teaching. (4) To show Napier’s persistence, responsibility, modesty and cooperation in order to enhance the humane nature of mathematics.

REFERENCES

Garden, R. A. (1987). The second IEA mathematics study. *Comparative Education Review*, *31*(1), 47-68.

Ding, R., Huang, Y., Lin Z., & Ma, Y. (2011). The intended curriculum and implemented curriculum based on constructivism. *Educational Science Research*, *06*, 56-61.

WHAT SPECIALIZED CONTENT KNOWLEDGE DO SENIOR HIGH TEACHERS HAVE ABOUT TRIGONOMETRY FROM THE PERSPECTIVE OF HPM?

An exploration and case study

Qi Chun YAN
School of Mathematical and Computational Sciences, Lingnan Normal College,
Guangdong Zhanjiang, China
476890742@qq.com

ABSTRACT

This study reports on the level of trigonometry *history-based specialized content knowledge* (HSCK) of senior high mathematics teachers (n = 153) at Yi Xing senior high school in China. The aim of the study was to determine if this sample of senior high school teachers has an appropriate level of HSCK to teach trigonometric concepts. The development of an instrument to measure senior high school teachers' HSCK of trigonometry with respect to teaching trigonometric concepts is detailed in this report, and the subsequent findings from its administration to the sample are discussed. The findings indicate that the sample has gaps in their HSCK of trigonometric concepts that are on higher mathematics syllabi and are underprepared to teach those concepts for understanding. This study reveals that the majority of the sample does not have enough HSCK to teach trigonometry concepts of the senior high school. The strengths and shortcomings of the sample in relation to their HSCK of trigonometry are presented. The author concludes that senior high school trigonometry content should be taught in initial mathematics teacher education in order to deal with this issue.

RESEARCH ON FACTORS AFFECTING MATHEMATICS TEACHERS' HPM LESSON STUDY

Zengcheng YUE[1,3], Zhongyu SHEN[1,4], Xin WANG[2,5], Jiachen ZOU[2,6],

[1]Department of Mathematics, East Normal University, Shanghai, China
[2]College of Teacher Education, East China Normal University, Shanghai, China
[3]yuezengcheng@126.com, [4]344080566@qq.com, [5]wangxinecnu@163.com
[6]jczou@math.ecnu.edu.cn,

ABSTRACT

A Lesson Study (LS) is a practice-based, research-oriented, collaborative mode of professional development (Huang & Shimizu 2016), so we usually use Chinese LS to conduct HPM lessons at elementary and secondary schools. However, even though teachers conduct LS with support from a professional learning community (PLC), they usually encounter some difficulties from other aspects. As the problems teachers meet in teaching are important issues to pursue an investigation on the effectiveness of history in the classroom (Barbin et al., 2000), it is necessary to investigate the issue deeply. Tzanakis et al (2000), Clark et al (2016) sum up arguments against the integration of history: at least two types of difficulty are included; philosophical and practical. Clark and Lisa (2013) think the greatest challenge in planning lessons informed by history of mathematics is to deal with the resources available. Poh and Dindyal (2016) describe similar difficulties to Clark and Lisa's difficulties in developing a lesson package infused with the historical development of calculus by using an LS approach. But they do not mention factors coming from different members of PLC; especially in China, where the community, consists of a school-based group, a teaching experts group, and an HPM academic group. So, the research question in this paper is: what are the factors that influence teachers' HPM LS?

We use Wang, Qi and Wang's (2017) procedure of HPM LS, constructed from Gu and Wong' Keli Model (2003; one popular type of LS in China), and built on theories of HPM in China to conduct teaching "Double digit divided by multiple digit", "Parallel" in grade 4 in Shanghai. Four procedures are included in this model: Selecting a Topic & Preparing, Discussing & Designing, Implementing & Evaluating, Analyzing & Writing, and there are cycles among the former three procedures based on feedback of an implementation of teaching. The implementation of teaching before demonstrating is called rehearsal teaching, and rehearsal teaching is repeated multiple times until teachers involved feel satisfied with the goal they set out to achieve (Huang & Bao, 2006). We collect lesson plans, videotaped lessons, post-lesson debriefing, the discussions among members in PLC, the demonstrating teacher's and HPM experts' design feedback worksheet and teaching feedback/reflection worksheets, the pre- and post- questionnaires about students' learning, interview after two lessons, and use 3-level coding to organize data and use them as evidence to descriptive research. Based on the data after coding and interview, we form a report with 6 parts. The following gives a sketchy description of each part:

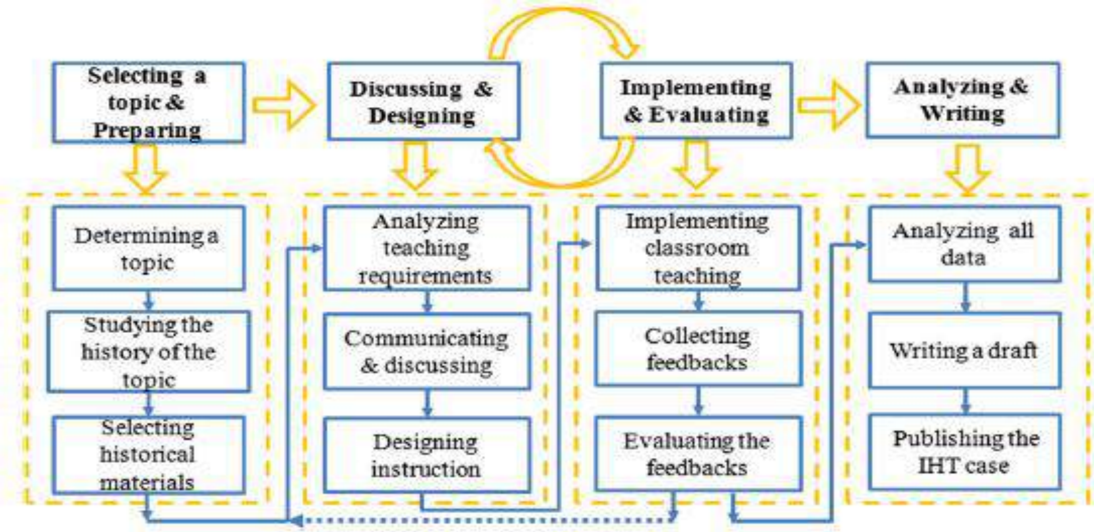


Figure 1: Wang, Qi and Wang's procedure of HPM Lesson Study

Part 1: The case teacher M firstly designed teaching herself. The historical resources she used are "Shang Shi Fa" in Ancient China (Fig. 2 5984÷16) and evolution of long division in western society since 17th century (Fig. 3 732÷6), and M regarded them as exercises after students learned how to divide. The purpose of the former is to make students understand culture in ancient China; the latter is to improve students' ability to understand. After being provided with other resources and advice, M slightly revised her design, adding the idiom "*Ban Jin Ba Liang*" (be six of one and half a dozen of the other) in background of weight units in book "*Yu Zhi Shu Li Jing Yun*" (Empire Kangxi, 1722), and attended a plan meeting to communicating & discussing. In the meeting, she accepted with pleasure the advice put forward by the HPM academic group (namely, to use two lines in the teaching design: the one is the relationship between comprehending, and building the relationship between "quotient is one-digit" and "quotient is two-digit"; the other is the background of using "*Ban Jin Ba Liang*" under a theory of "variation").

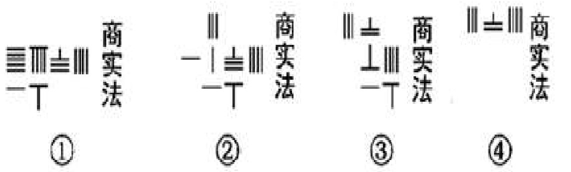


Figure 2: "Shang Shi Fa" in ancient China

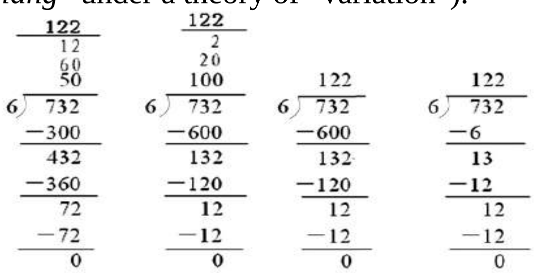


Figure 3: The evolution of division (Eugene,1982)

Part 2: After communicating with a teaching expert in her school, she revised the design to cater to the HPM academic group and the teaching expert. At the beginning of the design, M used "*Zhi Chu Fa*" (a method used in ancient China; namely, using subtraction to calculate division), and did not use the background "*Ban Jin Ba Liang*" in the whole process, as the teaching expert mentioned "Since the weight units used today (1 Jin =10 Liang, 2 Jin = 1 kg) are different from these (1 Jin = 16 Liang, so half 1 Jin = 8 Liang) in ancient practice, students would mix them up. I was surprised to read the design, which was different from what we had discussed, so I invited M and one expert in the

HPM academic group to fill in a design feedback worksheet, and send the expert's worksheet to M to make her change the design, but failed".

Part3: M conducted rehearsal teaching using her design, but the outcome was not good, and the teaching expert's and her colleagues' comments in post-lesson debriefing made her sad. Since M had not adopted the HPM academic group's advice, she again slightly revised her design and taught again, but the outcome was still not good.

Part 4: Finally, M decided to adopt the HPM academic group' advice. Firstly, she used idioms to build links between Chinese and mathematics, since it is thought that no or few such links exist. Secondly, she introduced an exploratory task, which is adapted from problems in "*Yu Zhi Shu Li Jing Yun*", in order to explore the relationship between Jin and Liang in ancient China. Then, assigning the task of finding the links between Zhu and Liang, which is also adapted from problems in "*Yu Zhi Shu Li Jing Yun*", to introduce idiom "*Zi Zhu Bi Jiao*" (haggle over every ounce 1 liang = 4 zi, 1 zi = 6 zhu). In exercises, M used True or False Questions (third calculation method in fig. 2 is included) in order to introduce the development of division (fig. 2), played micro-video to introduce "*Shang Shi Fa*", and put all methods together to compare their advantages and disadvantages.

Part 5: Through "Double digit divided by multiple digit", M gained some experience and tried to transfer these experience to teaching the concept of "parallel lines". Firstly she attended a plan meeting, in order to introduce her analysis of the textbook, the historical source and her students' cognition. Through communicating and discussing with the HPM academic group, she decided to integrate into her design, the definition of parallels in Euclid's *Elements* (c.300 BC; Definition 23: Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction) and in Mohism "*Mo Jing*" (388 BC; "*Ping, Deng Gao Ye*"; that is, if the distance between two lines is equal, the lines are parallel), and the idiom "No straightedge and compasses, no square and Circle" (meaning "Nothing can be accomplished without norms or standards"). Secondly, M finished her design. We were surprised by her good work after reading the design, although we gave some advice. Her design included 6 parts: (1) To recall from knowledge already acquired on "intersection or no-intersection of two lines"; (2) to use intuition in order to introduce the definition of parallels in Euclid *Elements*, and visual illusion of graphs to trigger students' consideration into the limitations of Euclid' definition, and introduce the definition given in the textbook: "If two lines are perpendicular to one same line, they are parallel"; (3) to use many tools in order to draw two parallel lines and students' method to introduce Mohism's definition, and play micro-video in order to introduce drawing tools in ancient China and the idiom "No straightedge and compasses, no square and Circle "; (4) before introducing notations of parallel in history (fig. 4), students take part in an activity of creating a notation for parallels, and through comparing the notations in history and notations they created, students know about the required characteristics of notations – for instance uniqueness and universality; (5) to experience the usefulness of parallel lines in real life. (6) Summary.

Figure 4: The notations of parallel in History (Cajori,1993)

Part 6: Through four trial teachings and post-lesson debriefings, discussions with members in PLC, revision and re-revision, M demonstrated her lesson successfully.

Combining the report of descriptive research with the interview after the lessons, we find that the teacher's self-efficacy and teaching feedback are the decisive factors influencing the HPM practice. Besides, teaching experience, teaching experts, teaching researchers, colleagues and the HPM research group also have different impacts on individual teacher's HPM practice.

REFERENCES

Barbin, E., Bagni, G. T., Grugnetti, L., Kronfellner, M., Lakoma, E., & Menghini, M. (2000). Integrating history: research perspectives. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study*, New ICMI Study Series, vol. 6 (pp. 63–90). Dordrecht: Kluwer.

Clark, K. M., & Lisa, G. P. (2013). "I was Amazed at How Many Refused to Give Up": Describing One Teacher's First Experience with Including History. In B. Ubuz, Ç. Haser & M. A. Mariotti (Eds.), *Proceedings Of the Eight Congress of the European Society for Research in Mathematics Education,* (pp. 1980-1989). Ankara: Middle East Technical University.

Clark, K. M., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang, X. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting – HPM 2016* (pp. 135–179). Montpellier: IREM de Montpellier.

Gu, L., & Wong, J. (2003). Jiaoshi zai jiaoyu xingtong zhong chengzhang [Teacher development in education action]. Kecheng. Jiaochai. Jiaofa [Curriculum.textbook.pedagogy], 2003(1), 9–26.

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*, 279–298.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: an international perspective, *ZDM*, *48*(6), 393-409.

Poh, W. B., & Dindyal, J. (2016). A historical perspective for teaching calculus: the development of a lesson package. Paper presented at the *TSG 25: The role of history of mathematics in mathematics education* of the *13th International Congress on Mathematical Education* (ICME-13), Hamburg, Germany.

Tzanakis, C., Arcavi, A., de Sá, C. C., Isoda, M., Lit, C.-K., Niss, M., de Carvalho, J. P., Rodriguez, M., & Siu, M. K. (2000). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study*, New ICMI Study Series, vol. 6 (pp. 201–240). Dordrecht: Kluwer.

Wang, X., Qi, Ch., & Wang, K. (2017). A categorization model for educational values of the history of mathematics. *Science & Education*, *26*(7-9), 1029-1052.

RESEARCHING HIGH SCHOOL STUDENTS' STRATEGIES FOR SOLVING THE CHINESE RINGS PUZZLE

Tung-Shyan CHEN
National Chin-Yi University of Technology, Taiwan
chents@ncut.edu.tw

ABSTRACT

The Chinese nine linked rings puzzle (Chinese rings puzzle) are one of the oldest known mechanical puzzles. It consists of a long loop with a handle on one end that is interlocked with nine rings (cf. fig.1). It is thought to have been invented originally in China. The origins are obscure. Stewart Culin records a story that the Chinese rings puzzle was invented by the famous Chinese hero Hung Ming (A.D. 181 - 234) in the book "Games of the Orient" (Culin, p. 31).

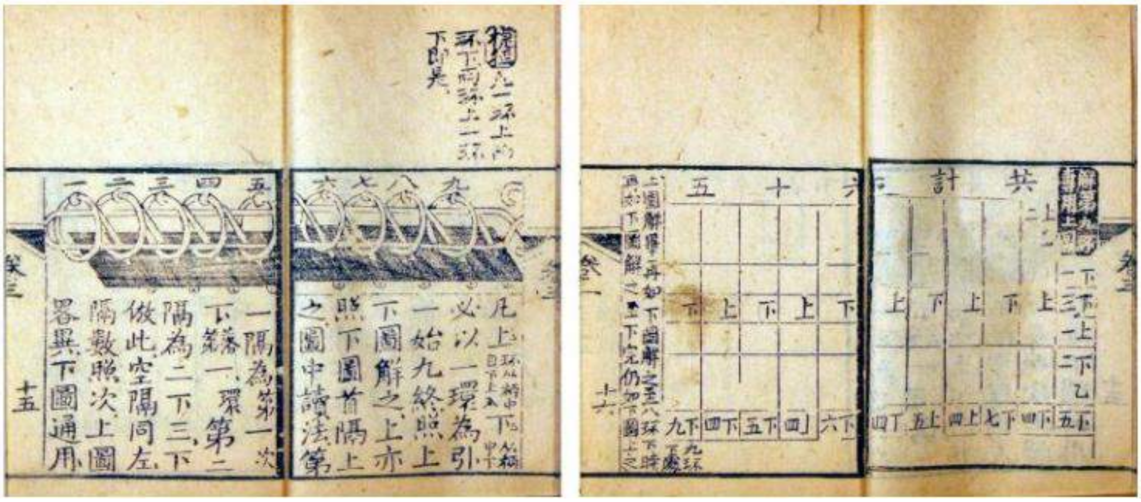


Figure 1: Nine linked rings solution in Bits of Wisdom by Zhu Xiang Zhuren, ca. 1821

The purpose of this study is to investigate high school students' strategies for solving the Chinese rings puzzle. We design a lesson plan to evaluate students' strategies and mathematical statements. In this study, 15 Taiwanese vocational high school students (1 female and 14 males) were involved. We assigned students into eight groups. For a Chinese ring to be removed it needs to meet certain requirements, which involve constantly taking the rings off and on. Students need to disentangle the long loop from all nine rings, and the solution takes 341 moves, so lots of patience is required. We expect students can find the procedure goes through when one of the steps of the procedure involves invoking the procedure itself using recursion and derive the minimum number of moves to solve an *n*-ringed puzzle[1].

[1]The minimum number of moves to solve an n-ringed puzzle is $\dfrac{2^{(2+n)}-3-(-1)^n}{6}$

After this lesson, all 15 students can take the rings off and on. Each group can find the minimum number of 5 rings move (fig. 2).

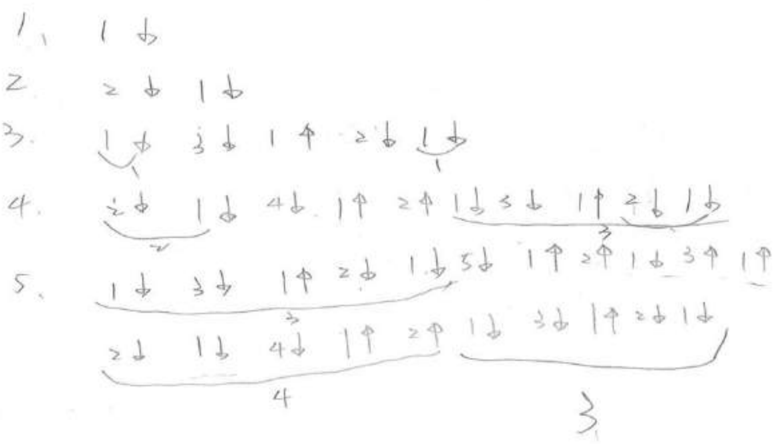


Figure 2: Example of finding the minimum number of 5 rings move

Our results demonstrate five groups can find the procedure goes through when one of the steps of the procedure involves invoking the procedure itself using recursion (fig. 3). Among these five groups, three groups can find the correct recursion with a_0=1, a_1=2,

$$a_n = \begin{cases} 2a_{n-1}+1, & n=3,5,7,\cdots \\ 2a_{n-1}, & n=4,6,8,\cdots \end{cases}.$$

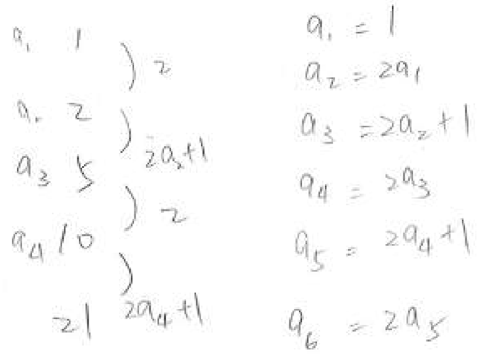


Figure 3: example of involves invoking the procedure itself using recursion

According to the worksheets, two groups find the recursion with wrong mathematical statements (fig. 4). Indeed, it is highly likely that students did not used to write mathematical expressions. We need to strengthen the practice of students in mathematical statements.

Figure 4: example of wrong mathematical statements

In summary, our studies show that puzzle solving is a good way to treat mathematical theories, but students' mathematical statements needs to strengthen. Ultimately, what is at stake here is learning by doing an active learning method for the teaching of mathematics to students.

REFERENCES

Culin, S. (1958). *Games of the Orient.* Tokyo: Charles E. Tuttle Company.

POTENTIALITÉS DE L'HISTOIRE DES MATHÉMATIQUES DANS LA FORMATION DES ENSEIGNANTS DE MATHÉMATIQUES

Ana LEMES
Université Lille 1, France
jimenalemes@gmail.com

RESUMÉ

L'Histoire des Mathématiques (HM) a été considérée comme une ressource attirante pour l'enseignement des mathématiques avec des arguments du type: contextualiser et problématiser les concepts, stimuler la lecture, humaniser les mathématiques, etc. À partir de l'étude ICMI *History in Mathematics Education* (Fauvel & van Maanen, 2000), une communauté de chercheurs dans le domaine de l'éducation, de l'histoire, de la philosophie et de l'épistémologie des mathématiques s'est consolidée, et est devenue une référence internationale. Avec le renforcement de cette communauté, nous voyons des efforts pour systématiser les recherches autour des cours de mathématiques dans lesquels l'HM est utilisée comme principale ressource didactique (Clark, 2006; Arcavi et Isoda (2007); Dorier (2008); Jankvist (2009); Smestad (2012); Matthews (2014); Barnett, Lodder et Pengelley (2014); Smestad (2017)). En parallèle, il est possible d'identifier certains travaux (Guacaneme (2016); Fried, Guillemette et Jahnke (2016); Jankvist, Mosvold et Clark (2016)) qui cherchent à définir des cadres conceptuels et théoriques qui soutiennent l'étude de l'intégration de l'HM dans l'enseignement des mathématiques. Dans cette recherche nous nous sommes intéressés à identifier et étudier les compétences professionnelles essentielles à l'exercice du métier, afin de mieux comprendre le potentiel de l'HM dans la formation d'un enseignant des mathématiques.

Dans ce cas là, nous présenterons le sous-domaine du *Pedagogical Content Knowledge* de Shulman (1987), proposé par Ball (2009) comme *Horizon Content Knowledge*, essayant d'établir une connexion entre un tel sous-domaine, l'HM et les compétences professionnelles nécessaires pour la tâche d'enseignement. En établissant cette connexion, il est possible de faire l'hypothèse que l'HM influence le sous-domaine *Horizon Content Knowledge*, et que cela peut trouver son expression dans la culture mathématique de l'enseignant de mathématiques, autrement dit, dans ses conceptions et ses attitudes.

Il suit que les premières questions qui guident cette recherche sont: Comment les conceptions se développent-elles ? Quels sont les leviers susceptibles de les faire évoluer? L'HM, en tant que discipline, influence-t-elle les conceptions des futurs enseignants sur les mathématiques ? Ces conceptions, influencent-elles les compétences professionnelles d'un enseignant ? Parmi les différentes compétences professionnelles, quelles sont celles sur lesquelles un cours d'HM peut avoir une influence dans le cadre de la formation d'un enseignant de mathématiques?

Cette recherche a pour objectifs principaux d'effectuer une analyse des conceptions des futurs enseignants de mathématiques, à *priori* et à *posteriori* d'un cours d'HM; d'identifier l'influence de ce cours sur les conceptions des futurs enseignants; d'étudier les liens entre conceptions de la discipline et conceptions de son enseignement, par rapport au

développement des compétences professionnelles. Les analyses seront faites à partir de la mise en œuvre des questionnaires et des entretiens, par rapport au cours d'HM. Le cours d'HM est donné de manière magistrale par l'historienne Rossana Tazzioli de l'Université Lille 1. Les objectifs du cours sont de promouvoir la réflexion sur des sujets tels que: le besoin de différents systèmes de numération; l'activité mathématique en tant que partie d'une société; et la relativisation de la démonstration.

REFERENCES (10pt fonts)

Arcavi, A., & Isoda, M. (2007). Learning to listen: From historical sources to classroom practice, *Educational Studies in Mathematics*, *66*(2), 111–129.

Ball D., Bass H. (2009) With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures. *Universitätsbibliothek Dortmund*.

Barnett, J. H., Lodder, J., & Pengelley, D. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education*, *23*(1), 7–27.

Clark, K. (2006). *Investigating teachers' experiences with the history of logarithms: A collection of five case studies* (Unpublished doctoral dissertation). University of Maryland, College Park. Available online at http://drum.lib.umd.edu/handle/1903/3379

Dorier, J. L. (1997). *Recherches en histoire et en didactique des mathematiques sur l'algebre lineaire - perspective theorique sur leurs interactions*. Université Joseph-Fourier, Grenoble I. Disponible en ligne à https://tel.archives-ouvertes.fr/tel-00338400/document

Fauvel, J., & van Maanen, J. (Eds.). (2000). *History in mathematics education: The ICMI Study*, New ICMI Study Series, vol. 6, Dordrecht: Kluwer.

Fried, M. N., Guillemette, D., & Jahnke, H. N. (2016). Theoretical and/or conceptual frameworks for integrating history in mathematics education. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting – HPM 2016* (pp. 211–230). Montpellier: IREM de Montpellier.

Guacaneme, E. (2016). *Potencial formativo de la historia de la teoría euclidiana de la proporción en la constitución del conocimiento del profesor de matemáticas* .(Doctoral dissertation). Universidad del Valle, Colombia.

Jankvist, U. T. (2009b). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Jankvist U., Mosvold R. & Clark K. (2016) Mathematical knowledge for teaching teachers: The case of history in mathematics education. In L. Radford, F. Furighetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (pp. 441-452). Montpellier, France: IREM de Montpellier.

Matthews, M. R. (Ed.) (2014). *International Handbook of Research in History, Philosophy and Science Teaching*. Dordrecht: Springer.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard educational review. 57*(1), 1-23.

Smestad, B. (2012). Teaching history of mathematics to teacher students: Examples from a short intervention. In É. Barbin, S. Hwang, & C. Tzanakis, (Eds.), *Proceedings of HPM 2012 – The HPM Satellite Meeting of ICME-12* (pp.349-362). Daejeon, South Korea: Korean Society of Mathematical Education &Korean Society for History of Mathematics.

Smestad, B. (2017). Design research with history in mathematics education. *CERME 10*, Dublin, Ireland. Disponible en ligne à https://hal.archives-ouvertes.fr/hal-01938803/document

Original historical sources in teaching and learning of

and about mathematics

BETWEEN WORDS AND ARTEFACTS

Implementing history in the math class from kindergarten to teachers training

Frédéric MÉTIN
University of Burgundy, ESPÉ & IREM, Dijon, France
frederic.metin01@u-bourgogne.fr

ABSTRACT

This article presents several ways to introduce a historical perspective into mathematical activities at school level. First highlighting usual difficulties encountered by beginners in implementing a historical approach to their mathematics sessions; it then suggests different modes of appropriation of a variety of sources, from original texts to artefacts described in these texts. For instance it revisits the use of counting tokens of the medieval period as well as two-colored tiles various combinations of which were studied by Sébastien Truchet in 1704. Going back to texts, it is finally suggested another use of old pages as support for marginal writings, allowing comments and appropriation of their contents.

1 Introduction

For the first time in France, history of mathematics becomes a significant part of the national high school curriculum in 2019. As the general introduction of the curriculum suggests: "It may be wise to enlighten the course with elements of historical or epistemological contexts. History can also be considered a fruitful source of problems clarifying the meaning of certain notions". Thus, many items are followed by examples of possible original texts that can be used in the classroom, along with mentions of famous mathematicians' biographies. My aim in this paper is to convince the readers of the relevance of an historical approach to mathematical notions from primary school to university, despite many obstacles faced by anyone using original sources in the classroom.

Sometimes we are led to use attractive anecdotes at the expense of rigor, in order to give a historical perspective to our course on mathematics. Then the usual aridity of our mathematical discourse gives way to a pleasant coloration. Isn't it worth this slight sacrifice? Historians themselves cope with an evolving historical truth. Even in mathematics, contemporary myths die hard. For instance, many stories about the golden ratio persist (Neveux, 1995), and the use of the so-called thirteen-knot-rope still prevails when medieval geometry is discussed. Between historical rigor and pedagogical benefits, I try to find my way focusing on objects, artefacts which can be proposed for study. I will first point out usual pitfalls faced by beginners, and then suggest several ways of using original sources in the classroom, balancing between historical texts and artefacts we can find in these texts.

2 Common pitfalls of the historical approach

The historical approach in mathematics is not common. It even presents several impediments, which are often put forward by the teachers who practiced it: time required to prepare activities that must often be completely created; lack of historical culture about the content to be taught; fear about additional difficulties for students who are not interested in these “useless” subjects: mathematics and history. But these impediments are often offset by a real light shed on students by the use of old resources. Moreover, reading original texts generates a change of view on mathematics, which is discovered much more human than it was thought.

For instance recalling the historical context of the invention of decimals helps to strengthen the link between mathematics and the real world, since Stevin addresses his *Thiende* (Stevin, 1585) to a whole range of crafts that are likely to use it: astrologers, surveyors, tapestry measurers, gaugers, and so on. But as a consequence of this link to the real world, reading the text of the *Thiende* raises questions outside the mathematical field as Peltier and Briand have shown in their study of Stevin's text (Briand & Peltier, 2003). But another problem arises from this study: the authors date the *Thiende* to 1582, while its first edition dates back to 1585. Despite deep bibliographic researches, I have never been able to find any 1582 edition of this book. The confusion may come from Georges Ifrah’s *Universal history of numbers* (Ifrah, 1998), which mistakes the *Thiende* for Stevin’s *Tafelen van Interest* (Stevin, 1582). This is a tiny detail. It shows however the difficulty of being perfectly rigorous, even for specialists of the subject. Let us raise some other difficulties posed by the practice of the historical approach.

2.1 *Words, words, words*: the dangers of storytelling

One of the most common uses of the history of mathematics in the classroom consists in humanizing the concepts through anecdotes from the great inventors’ lives. More and more often, it is also done by showing and reproducing daily use of mathematical notions at various times by ordinary people, and not only geniuses. However, simple anecdotes out of context would not have a great impact, so it is essential to take time to contextualize the concept and to promote a multidisciplinary activity. This supposes a certain historical and epistemological culture for teachers as storytellers.

It is partly the reason why the new EFEC[1] Bachelor’s degree of the University of Burgundy includes a teaching unit on history and epistemology of mathematics. In this unit, I privilege a narrative hook, a form of storytelling, in order to attract the attention of my second year students, most of whom have a very small scientific background[2]. For me, presenting history of mathematics as part of the history of humankind was the ideal means to give a general perspective on the notions of number, size, geometric shapes, as well as on measuring and solving problems.

But the reception of these great stories by my audience did not comply with my objectives. Most students do not have a solid knowledge in both mathematics and history, and the wealth of information provided during the sessions was likely to blur the overall message. A revealer of the difficulties associated with this “grand narrative” approach is obviously the moment of evaluation. Let us give a typical example, which shows how vain it may be to imagine that a great saga in the manner of *the Lord of the Rings* facilitates the mastery of mathematical notions.

One of the questions of an MCQ test was: *Who invented algebra, when, and in which geographic area?* I admit now that these questions were ambiguous, but I thought at that moment that I could gather unequivocal answers. To the contrary, the anthology of results

[1] Acronym for *Education/Training/Teaching/Culture*. It is based on the former Science of Education Bachelor’s degree.
[2] One of the teaching units in Year 1 is devoted to scientific culture.

surpassed expectations: both for the people (Archimedes, Plato, Aristotle, Egyptians, Babylonians, Pythagoras, Diophantine, Fibonacci, etc.) as for the period (from ancient Egypt to the 17th century) or places (Western or Eastern Europe, Middle East, China), everything seemed mixed up to the point of absurdity. My conclusion would be: telling math stories to young adults lacking historical landmarks is not always fruitful.

2.2 *Wingardium Leviosa*: the dangers of magic wands

The magic wands of teachers do not have the powers of Hogwarts wizards' wands. The first one is not rigid, but flexible: it is called the thirteen-knot-rope, or the arithmetic rope. For unexplained reasons, this modern myth has taken root in the teaching and teacher training worlds. It has even met there with some success, fueled by online documents which claim the historical reality of this object without great hindsight.

Let's first explain what this artifact is. It is a cord divided into twelve equal parts by thirteen regularly spaced knots. Both ends of the cord carry a knot. By superimposing these two knots, then stretching the string by the 5th and the 8th knots, you obtain a triangle whose sides measure 4, 3 and 5 equal intervals respectively. The Pythagorean Theorem ensures that the triangle is right-angled and consequently you have constructed a right angle!

This thirteen-knot rope gives substance to the theorem. This is probably what seduces the teachers (and others) who peddle this myth. The corresponding Wikipedia page ("Arithmetic Rope") shows that there are still unattended leaflets on that non-academic encyclopedia. Actually, the main illustration comes from a now missing manuscript in which it was captioned *Arithmetica* (Figure 2.1, truncated on Wikipedia).

Figure 2.1

But the original picture shows the use of a counting device for calculation with tokens. On the webpage it is also stated, without any serious documentation, that the thirteen-knot rope was widely used in Egypt on the building sites of the pyramids, and in the Middle Ages by cathedrals builders. This unverifiable piece of information is taken up by several French pedagogues on their personal websites.

Anyway, the wondrous aspect of this artifact makes it an educational attraction, since it materializes a Pythagorean triplet in a minimalist and elegant way. So why cast doubt on its existence? For a long time, many French IREM groups on history of mathematics have worked on practical geometry of the Middle Ages and Renaissance, up to the modern era. The list of perused books and manuscripts is very complete, even if it can't claim to be exhaustive. It turns out that so far none of the works studied by these groups for years mentions any knotted rope whatsoever, while the use of other instruments (false square, geometric square, astrolabe, ropes and stakes, etc.) is very frequently described. By the way, why would cathedral builders have bothered creating thirteen knots in a rope when only the fifth and eighth ones were used? Or that the use of removable flags or simple marks on the rope would have been equally effective? There are very old testimonies of the use of ropes for practical geometry, in Egypt by arpedonapts (literally "rope stretchers”, the surveyors) or in the *Sulbasutras* in India (Sen & Bag, 1983). In the old texts that have survived, we find no traces of knots on the ropes, except possibly at their ends to attach them to stakes.

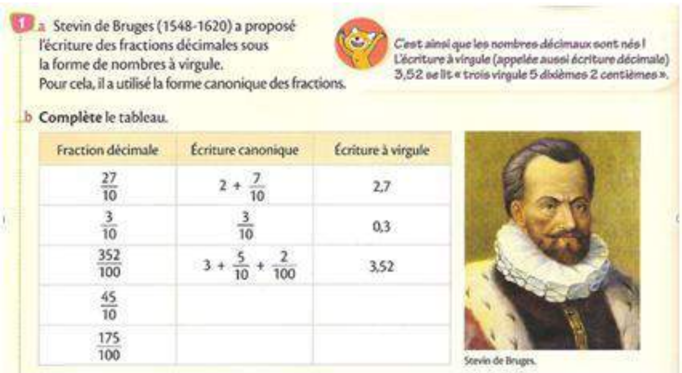
As a modern myth of the educational world, the knot rope does not require any philosophy of history, contrarily to the second magic wand, which is another theory that teachers also echo without precaution: the obscure role of the Golden ratio.

Once again, Egypt is the imaginary place of a mysterious science of the Great Old Ones. The fascination with the secrets of making pyramids began at the end of the 19th century with the publication of a study (Smyth, 1864) who founded pyramidology. One of the great finds of pyramidologists is the discreet presence of the Golden number (Φ) or π in the architecture itself. These values were obtained after playing with measures of certain lengths, even sometimes setting the value of the units arbitrarily. Many publications have shown the vacuity of such assumptions (Markowsky, 1992), but the golden number and its cosmic virtues retain some credibility among the general public.

At school, the Golden number usually arises when it comes to cross-curricular activities between mathematics and the arts. How many times have I heard that classical aesthetics is regulated by the divine proportion invented by Fibonacci but used long before in architecture? Almost always, secondary school teachers invoke a statistical truth: when you present to the general public rectangles of various shapes by asking them which one is the most “beautiful”, the most often chosen rectangle would be the one whose dimensions are in proportion to the Golden number. This unfounded assertion has been denied more than once by actual polls (Jacquier & Drapel, 2005). Unfortunately it seems that Φ doesn’t play any special role in beauty.

2.3 The dangers of handbooks inserts

Let’s go back to Stevin’s *Thiende*. No doubt about its popularity more than four centuries after its publication, since a recent French primary school handbook quotes it in its chapter on decimal fractions and decimal numbers (Briand & Peltier, 2016, 170). The sidebar is likely to be a pleasant illustration of the subject and to anchor Stevin’s portrait in the students’ imagination (Figure 2.2).


1 a Stevin de Bruges (1548-1620) a proposé l'écriture des fractions décimales sous la forme de nombres à virgule.
Pour cela, il a utilisé la forme canonique des fractions.

C'est ainsi que les nombres décimaux sont nés ! L'écriture à virgule (appelée aussi écriture décimale) 3,52 se lit « trois virgule 5 dixièmes 2 centièmes ».

b **Complète** le tableau.

Fraction décimale	Écriture canonique	Écriture à virgule
$\frac{27}{10}$	$2 + \frac{7}{10}$	2,7
$\frac{3}{10}$	$\frac{3}{10}$	0,3
$\frac{352}{100}$	$3 + \frac{5}{10} + \frac{2}{100}$	3,52
$\frac{45}{10}$		
$\frac{175}{100}$		

Stevin de Bruges.

Figure 2.2

Beyond aesthetics, the sidebar can useful by the link given between three parallel forms of decimals, very precisely found in the original text and justified through it. In this sense, it is an example of a very relevant use of the original resource. But it reveals some blunders. For example, you could be surprised by the choice of a fanciful portrait of Simon Stevin to illustrate the inset when there exists an engraved portrait of the author's life, which correctly mentions his identity (Simon Stevin) while he is presented in the box as “Stevin de Bruges” which might suggest to the students that this is a nobleman named Stevin. What is more annoying is that it can be read that Stevin “proposed writing fractions as numbers with commas” and that “that is how decimals are born!” which is incorrect. Finally the recommended spelling of number 3.52 in the form “three point five tenths two hundredths” seems inappropriate to us.

Since this type of sidebar is not uncommon in secondary school textbooks, it can be assumed that the new high school programs will generate new productions in response to their explicit references to history of mathematics. Let us hope that the contents will not only be chosen as nice images, and that they will generate activities based on original resources.

3 Beyond dangers: the use of original materials in the classrooms

The pitfalls I mentioned about introducing a historical perspective into the classroom might confuse and distract teachers from the use of ancient resources left to specialists in history of mathematics. As we have seen above, using ancient texts in class offers no secure way from the point of view of historical accuracy, but it would be a shame to deprive anyone of the light shed by the introduction of a historical perspective on mathematical notions. A multidisciplinary approach contributes to a better understanding of possibly tough concepts and it stimulates the connections between several fields of knowledge for the learners.

For the moment there are but few examples of using original texts in primary school mathematics. It is a recent but more and more frequent concern for teacher trainers, especially historians of mathematics who ponder on the implementation of primary school activities based on a reading of original sources. This is evidenced by the recently published *Passerelles* (Moyon & Tournès, 2018), which offers nine chapters on various approaches to mathematical concepts in reference to their story. The first activity paths that are presented below are in the same perspective as the chapters of the book. These activities have been mainly practiced in the last two years (2016-2018) in various classrooms, mostly with students of the Faculty of Education (ESPÉ) or elementary schools, as well as larger public or teachers during training courses or Popular Science events[3]. For activities at various levels there are many interesting HPM papers (Chorlay 2016, Métin 2012,).

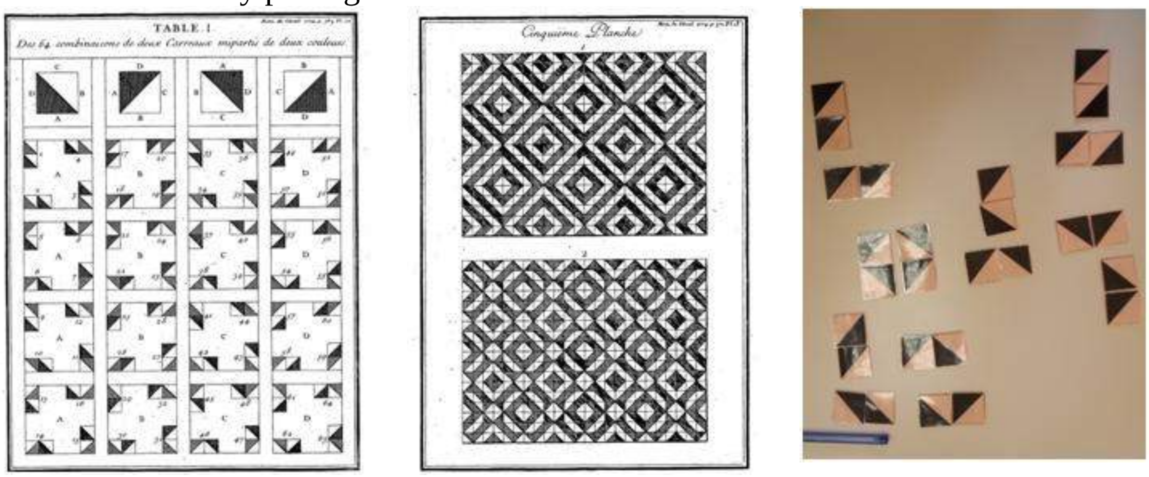
3.1 Floor tiling! Truchet’s patterns

Our first activity has been tested both in nursery school and teacher training. Readers will agree that it is difficult to undertake activities based on reading original documents for kindergarten. But it is possible to take the content of a text both as a resource for an activity (in kindergarten) and as a problem subject in itself (at the university), insofar as this text exhibits materials which can be manipulated even outside their original context. First the historical context of the document is set, and then its content is described. Finally I show the way I used it with children and university sophomores.

[3] For activities at various levels there are many available HPM papers (for instance Chorlay 2016 or Métin 2012).

3.1.1 Sébastien Truchet's Memoire

Father Truchet was born in Lyon in 1657. Naturally focused on mechanics, he was quickly noticed by the king's entourage for his clockmaking skills. He was appointed honorary member of the Academy of Sciences after having probably participated in many hydraulic works in the parks of the castles of Versailles and Marly. As he recounts in his *Memoire* presented to the Académie des Sciences (Truchet, 1704), it was during an inspection visit of the canals in the Orleans area that he became interested in pavements. He had been lodged in a domain whose owner had decided to renew the chapel floor using simple two-colored square tiles separated in two parts by their diagonal (Figure 3.1(a)). In his *Memoire*, Truchet searches for all possible combinations of two such tiles (Figure 3.1(b)), and then he deals with compositions that can be made by putting two such combinations next to each other.



(*a*) (*b*) (*c*)

Figure 3.1: Truchet's tiles

3.1.2 The use of the *Memoire* at University level

I found that this material was suitable for combinatorial exercises including experimentation. The interest of Truchet's two-colored tiles lies in their material aspect that allows manipulation like "let's do it and see what we get", instead of trying to think first. In fact, most undergraduate students would have been unable to determine all the possibilities of contact of two sides of tiles.

The group having been divided into four sub-groups, the students had a large number of two-colored squares of laminated cardboard. By assembling these squares in pairs, each group had to reconstruct one of the four columns of the original plate (Figure 3.1(a)), whose first cells only were displayed. So, the original material wasn't left untouched. On the contrary, the words had totally disappeared, as well as a part of the original plates.

What appeared clearly in this activity are the predominant places of visual observation and the faculty of manipulating objects. Furthermore the use of speech is at first counterproductive. For example, on the production of Figure 3.1(c), how can we determine the missing configurations? This question was a headache for the students of the group in question, who had to classify the configurations already present according to visual criteria, and name the classes obtained to be able to communicate with each other.

In the second part of this activity, the aim was to reduce the number of cases to its minimum by matching similar configurations. Students still used visual evidence, but it was sometimes difficult for them to identify the same configurations when in different positions. Note, however, that students tended to assemble tiles of the same color, as I gave indifferently black-and-white or orange-and-white tiles. I have not studied the impact of colors on students' success, as they did not initially plan to mix tiles of different colors.

3.1.3 The activity in kindergarten

Of course, there is no space for combinatorics in kindergarten. I nevertheless tried to use the cardboard material manufactured for the occasion. The proposed activity had to be supervised by students from the Faculty of Education. It consisted in the duplication of simple configurations inspired by those of Truchet's *Memoire*. The main learning goal of this activity was to develop the children's skills of spotting things in plane geometry without coordinate system, and determining relative positions of geometric shapes. The second objective was to lead pupils to feel the necessity and usefulness of an appropriate geometric descriptive vocabulary.

In some groups, there were harsh discussions because many pupils did not agree on the success of their partners even if they had difficulty explaining the mistakes. Generally speaking, the troubles came from the figures' orientation. University student helped the groups reconcile by leading configuration analysis and requesting expressions to characterize misplaced squares. The necessary manipulations were based on simple transformations: rotation and bilateral symmetry. It should be noted that two symmetrical assemblies were often taken as identical. This is perhaps one of the major disadvantages of two-colored tiles.

I am far from having exploited the full potential of these artifacts for the study of geometric shapes in the classroom. It is even possible outside the classroom, as two-colored tiles still exist today in real life; it is possible to buy them in DIY superstores, for example the "Dément" (that is: mentally ill…) collection in the French brand Leroy-Merlin. The website of this big brand even offers a visualization of some arrangements for bathrooms, in which I had the surprise to find the configurations already present in Truchet's *Memoire*.

No school so far has accepted to host a restoration of the soil of its classes, but the computer generated images of the aforesaid store website[4] let us hope that a virtual renovation project can be considered.

3.2 Counting with tokens

The second chapter of *Passerelles* (Moyon & Tournès, 2018) highlights the relevance of using token charts in primary school, in order to introduce pupils to the history of calculus techniques, from the practice using artifacts to the use of written signs. The authors set out the history of counting tables and their possible pedagogical use. The counting frames manufactured by Dominique Tournès and his IREM group are made of lines drawn on paper. The lines carry the tokens according to their order of units, tens, hundreds, etc. My practice of calculating tokens is quite different, as I draw it from books (esp. Anonymous, 1509 and Anonymous, 1551) which present the

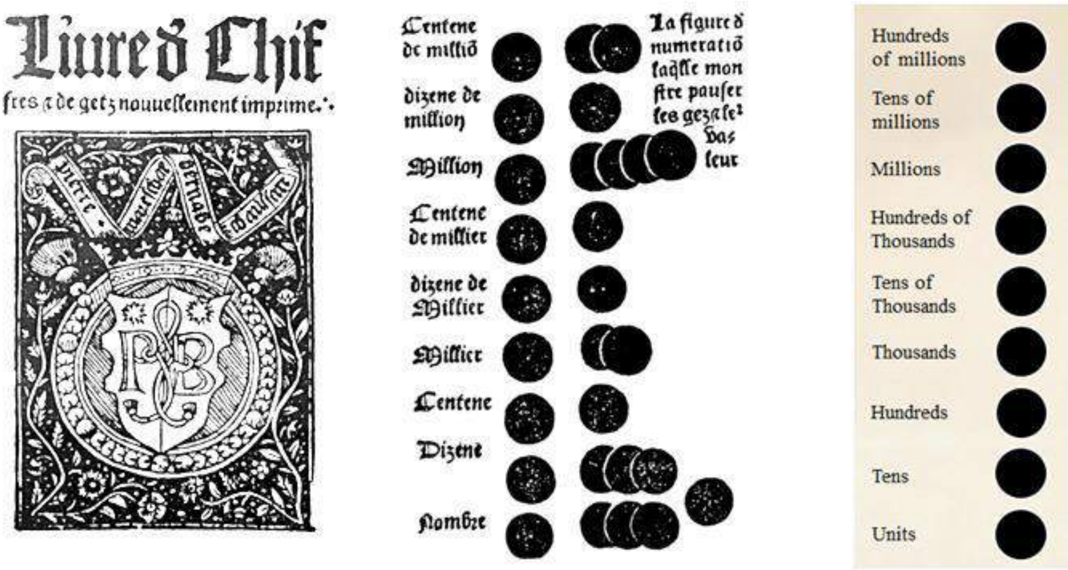
[4] https://www.leroymerlin.fr/v3/p/produits/carrelage-sol-et-mur-noir-blanc-effet-ciment-dement-l-20-x-l-20-cm-e1500488011. Accessed: 15 November 2018.

calculations without lines. Moreover, I do not use tokens with beginners in calculus, but with pupils, students and adults already familiar with the decimal position system and usual operations. Confronting these audiences with that forgotten practice allows them to revisit their representations of numeration and calculations.

It is often for pleasure when it comes to adults. However, it seems also conceivable to propose calculating tokens as remedial teaching in case of learning difficulties with counting and operations. As I will show, when you do not impose a particular technique, letting people appropriate the device, it allows them to question their personal representations of the decimal position system.

3.2.1 Resources

There are many arithmetic books printed at the beginning of the 16th century, mainly in Lyon, which at that time was a major center for both trade and printing. My main reference is a very rare book, named *Livre de Chiffres et de Getz*[5] (Anonymous, 1509), of which remains only one known copy, owned by the Méjanes Library of Aix-en-Provence. Its author is not identified, but the content is not entirely original since it is found partly in 15th century manuscripts and other 16th century printed works. The title page of the book bears only the identity of the printers, Pierre Mareschal and Barnabé Chaussard. As the printers's mark is damaged down on its right side, we can guess that the book was printed between 1508 and 1510 (i.e. when the association between the booksellers stopped), that is the reason why I chose to date it back to 1509.



(*a*) (*b*) (*c*)

Figure 3.2

Except for its rarity, the *Livre de Chiffres et de Getz* follows the same pattern as other books of its time. The first part explains how to manipulate tokens, for the representation of integers (*Numeration*) as well as the four operations. The second part deals with problem-solving based on the rule of three. In my classroom activities or during public sessions, I only present the title page and the caption entitled “The figure of numeration, which demonstrates how to set down the tokens and their value” (Figure 3.2(c)). The first

[5] Which means: *Book on Numerals and tokens.*

exercise consists of understanding how to put the tokens and find the value of the given number. Since no line is engraved on the board, the orders of the quantities are therefore indicated by a column of tokens, called “the tree”, to which the printers took care to add a legend that the practitioners of arithmetic obviously did not have at hand.

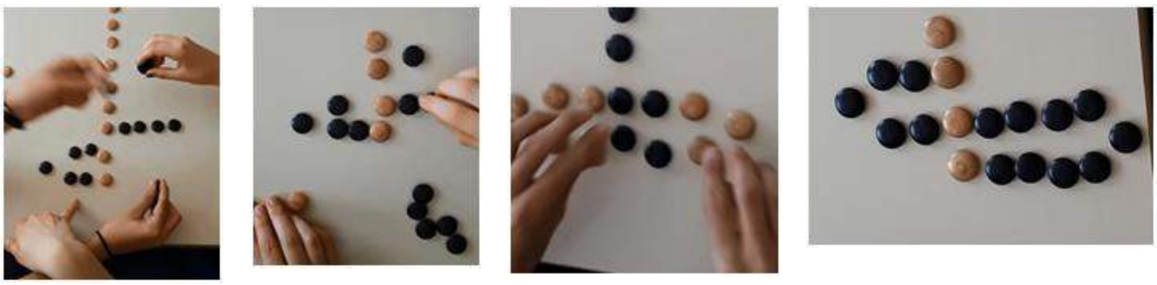
3.2.2 Manipulating tokens in the classroom (and elsewhere)

One of the greatest interests of this abacus without line lies in its very easy manufacturing and installing wherever wanted: just have a good amount of tokens (in my case, tokens from a game of draughts) and a flat surface. The participants in the activities themselves build their computational tree. In this respect, I advise beginners to have tokens of different colors, as can be seen on the pictures in Figure 3.2.

First of all, participants must find the value of the number given as an example in the book. They have but little difficulty in associating each group of tokens with the magnitude of the “branch” it belongs to, and thus to reconstruct the number (which is 214,112,138). Some numeration exercises are an opportunity to explain the role of the so called *quinary* token, placed in an intermediate position between two branches and representing five tokens of the lower branch. One possible explanation comes naturally: beyond five, there is little chance of being able to immediately perceive quantities without counting. In the context of monetary accounts, it would be easy for a virtuoso (but dishonest) arithmetician to take a token here and there and steal his customers. The quinary token allows all the protagonists a visual control of the operations.

Simple operations, namely addition and subtraction, do not need any explanation: in order to add any two numbers in the tree, all you have to do is cumulate the tokens of the same order and, when a sum reaches ten, to place a single token on the branch immediately above them.

When it comes to multiplication, the custom is to place the multiplicand on the left of the tree and the product on the right. The multiplier is not represented, and this is precisely the reason for the diversity of possible techniques. My first example is the multiplication by 20, often used in the old texts for conversions of money because a pound corresponds to twenty shillings (and a shilling to twelve pence). The task illustrated in Figure 3.3(a) is the conversion of 27 pounds (on the left) into shillings. The participants perform it very simply: each token on the left gives birth to two tokens on the right on the next branch up above. The quinary token will thus give two tokens between the tens and hundreds branches, replaced by a token on the hundreds branch. The main difference between the manipulators is their choice of leaving the multiplicand intact or, on the contrary, eliminating each token as it is replaced by the other two on the right.



(*a*) (*b*) (*c*) (*d*)

Figure 3.3

After this rather easy first exercise, I ask the participants to convert 27 shillings into pence, which is equivalent to multiply 27 by 12. Now, with this multiplier, we encounter different techniques. Let's describe three methods illustrated by Figures 3.3(b), 3.3(c) & 3.3(d) respectively:

- 3.3(b): Replacing each token on the left with twelve tokens on the right, that is two tokens in the same level branch and one token in the next branch up above. The quinary token makes no exception and it is only after placing all the tokens on the right that the simplification is carried out.
- 3.3(c): Number 27 is identically copied on the right, then all its tokens are raised by a branch ("we do: times ten"), finally each token on the left is doubled and moved to the right into the branch of the same order. The quinary token is simply raised to the branch of tens. The third pupil of the picture will find 374, because in his first raising, the quinary token was erroneously placed on the branch of the hundreds instead of being in the intermediate position.
- 3.3(d): The tokens on the left are reproduced on the right and doubled (but the student forgets to double the quinary token). Then, instead of raising the right stack of tokens from a branch, the students operate this move on the left tokens. Then they stop, puzzled (this is where the picture is taken). The strictly gestural aspect of the procedure led to a dead end. Resuming the procedure step by step, with discussions and comments, allows them to rectify the manipulation, by correcting the error on the quinary.

These various techniques are guided by different decompositions of multiplier 12, either as a block (Figure 3.3(b)), as 10 + 2 (Figure 3.3(c)), or as 2 + 10 (Figure 3.3(d)). The most interesting moment for the participants is the verification of their result and search for a possible error. For that purpose it is useful to film them after asking them to restart manipulations for the camera provided that they comment their actions, though it is not natural for them to speak while operating.

I have observed many times that this tokens calculus activity can be conducted without using language, because the essence of the practice lies in gestures which anyone can reproduce in front of other people without a word. It could almost happen in complete silence. However, I systematically ask for the explanation of the manipulations, whether the operators are beginners or experts, because without the verbalization the exchanges would only be visual and they would remain at a technical level. Since practice is not science, neglecting words would condemn participants to act only as performers.

3.3 Experiencing math oddity

Let's briefly mention classroom experiments based on arithmetic texts of the late Middle Ages and Renaissance. These texts do not use any algebraic formalism, but they expose problem solving algorithms without the use of letters. One of the works I have used the most is the *Oeuvre Tressubtile et Proufitable*[6] by Juan de Ortega, the first printed Spanish arithmetic (Ortega, 1515).

The simplest way to share the experience of math oddity with readers is to give them the original text of one of the reading / interpretation activities offered to high school students (Ortega, 1515, fol. XCIXr -our translation from French-)

[6] *Very Subtle and Profitable Work.*

A man makes his will and he has 3000 crowns left and he leaves his wife pregnant. And so he orders that if he dies and after his death his wife gives birth to a son, the said son will have the three parts of his property and the mother the other part. And [if] she gives birth to a daughter, the mother will have the three parts of his possessions and the daughter will have the other part. What happens is that after the death of the father, the woman made two children together, namely, son and daughter. It is asked how the possessions of the deceased person shall be divided so that the will of the father be observed.

This kind of problem is called "problems of will", and it is often found in old books on commercial arithmetic. The popularity of these problems may come from the need to turn to arithmeticians or even algebraists for inheritance disputes, because the laws on inheritance were extremely complicated for ordinary men.

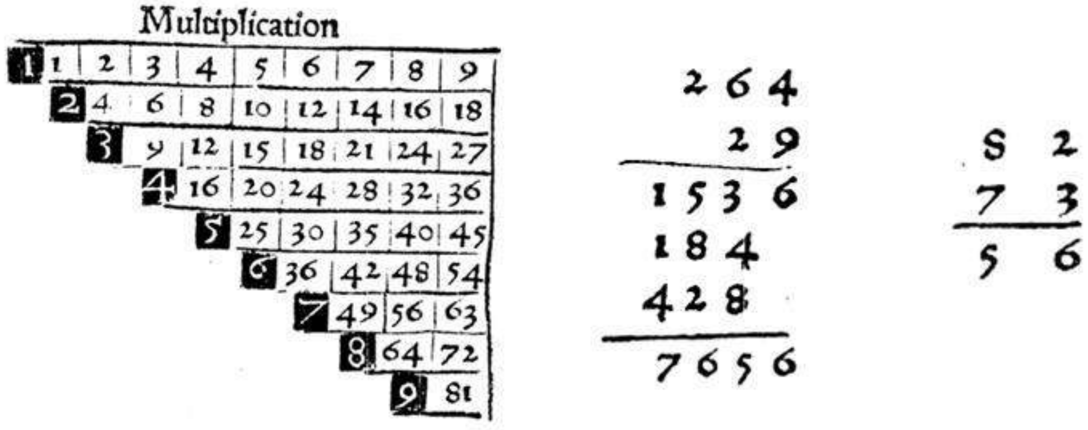
In the classrooms, the first obstacle to understanding comes from typography. It is however surprising to realize that generally no students are upset by the fate of the daughter. Anyway, once the language barrier has been overcome, understanding Ortega's solution can be quite uneasy:

And you will do that way: start with the daughter because if the daughter has a part the mother must have the three parts. For this pose 1 for the daughter and 3 for the mother, and the son must have three times as much as the mother and it will be 9. Now, add these three sums that is to say 1, 3, 9 & are 13 for the divider. Now, say by the rule of three: if 13 gives me 3000, what will give me 9? Multiply and divide as the rule of three requires & you will find 230 crowns & $\frac{10}{13}$*, which is the share of the daughter. And to the mother 692 crowns and* $\frac{4}{13}$*, and to the son 2076 crowns and* $\frac{12}{13}$ *of a crown.*

Mathematics teachers will quickly grasp that these are proportional shares, but you must remember that these notions are far from the concerns and skills of contemporary French high school students. These stumble particularly on the coefficients 1, 3, 9 and 13, the role of which in solving the problem they hardly understand. Here the numerators and the denominator are not distinguished by their positions in a fraction but by the word "divider", which qualifies the number 13.

It is certainly not the most difficult text that I have proposed to high school students. As a rule, the initial lack of understanding, frustrating as it may be, nevertheless leads to the satisfaction of perceiving the meaning of the texts through their mathematical contents before grasping the words. For us, the best work in this perspective remains Valentin Mennher's *Arithmetique Seconde* (Mennher, 1556), with its old fashioned algebraic resolutions of equations. For a more detailed study of that question, I will refer readers to another article (Métin, 2012).

Even without words, old practices in mathematics can turn out to be strange for contemporary readers. For instance, Figures 3.4(a) to 3.4(c) are visual excerpts from Robert Recorde's *Ground of Arts* (Record, 1543). Figure 3.4(a) shows a multiplication table that our contemporary students consider incomplete or halved. On Figure 3.4(b), the same students can't recognize their multiplication algorithm, but it is their algorithm, with the exception of the carried numbers. Figure 3.4(c) is the most puzzling. It shows the method promoted by Recorde to avoid learning the multiplication table for numbers over 5. For instance, if you want to know the product of 8 multiplied by 7, just evaluate the differences between each of these two numbers and 10: the “rests” are 2 and 3 respectively; firstly multiply 2 by 3 and you get 6; secondly calculate the difference between each initial number and the “rest” of the other one: 8 – 3, equal to 7 – 2, is 5. Concatenate the two results and you obtain 56, which is the result of the multiplication 8 × 7. Amazing! After this element of surprise, the students invariably go through two steps: How (does it work)? Why (does it work)?



(*a*) (*b*) (*c*)

Figure 3.4

To favor investigation and questioning, it is worth removing all words from the original excerpt, leaving only the operations. The surprise comes from the lack of information (“I don’t know the reason why it works”), so don’t provide the students with too much information. Even a simple picture can sometimes be useful, as I show now.

3.4 Back to Louis XIV’s Versailles in primary school

At primary school level, the history curriculum gives prominence to great French figures of the past. Many aspects of the Grand siècle can give birth to original activities about symmetry or geometrical construction programs. In the context of a lesson study workshop, I had to find documents adapted to young pupils who are in the process of discovering symmetry. As my homeland of Dijon is renowned for its culinary culture, I was prone to look for them in the feasts.

3.4.1 The initial document

An impressive table plan is displayed in the kitchens of Chantilly castle, which were the quarters of the famous cook Vatel (Figure 3.5). This is the plan of a table for King Louis XIV at the castle of Marly-le-Roi, as it had been submitted to him in 1699. At Marly, the king took his meals with the royal family and particularly distinguished courtiers. The single rectangular table had been replaced by two oval tables and the plan shows the layout chosen on a special occasion. This arrangement was surely guided by symmetry, not complete however since there are seventeen covers. Perhaps symmetry was not desirable? The king having no counterpart, he could only sit in a unique, distinguishable place, which may correspond to the isolated cover at the bottom center of the ellipse.

The arrangement of dishes on the table is part of the aesthetics of the whole. The plates of the guests are represented by disks distributed around the entire circumference of the oval. At the center, a curvilinear hexagon must correspond to a centerpiece carrying spices and herbs. Between plates and centerpiece, the "Roast dishes" materialize an axis of quasi-horizontal symmetry, while the circular soup tureens and the "Pots-à-oille"[7] are arranged in a symmetrical way compared to the central one. The round plates of hors d'oeuvre complete the set to occupy a maximum of space on the table.

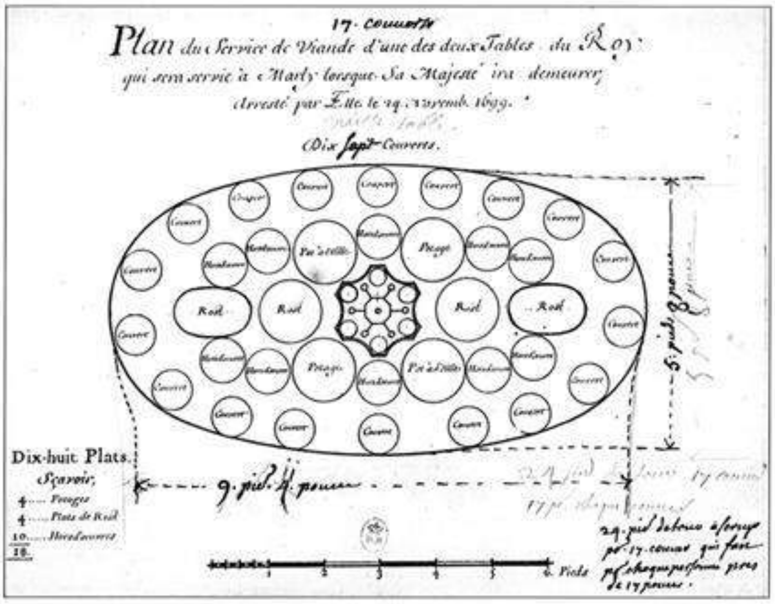


Figure 3.5

This document, strongly linked to the history of France (as far as butlers are concerned), was a useful starting point for discovery and deepening activities on symmetry in primary school.

3.4.2 Activities in the classroom

The colleague who welcomed me in his classroom was a highly qualified user of interactive whiteboard and computer technologies in general. After a short time of exchange on the historical context of Versailles in the time of the sun-king, the table plan of 1699 was dissected, insisting on the organization of the reception and the placement of the guests and dishes on the table. When a first "mirrored" idea of placement was formulated, the pupils were invited in turn to make this property appear by coloring: a first pupil chose an object which they painted, the next one must use the same color to paint one of the objects that corresponds to it and the result was submitted to the entire class.

Introducing the notion of axis of symmetry was facilitated by the interactive coloring task and the final request: "Where could you cut this table to obtain two exactly identical parts?"

The plenary activity generated but few errors as the exchanges within the class were successful and the incorrect proposals were immediately corrected. However it should be noted that a second axis of symmetry, this one being horizontal, had been accepted by all (including the teachers) on the basis of visual evidence, while the number of guests around the table did not allow it. It would have been necessary to agree on a partial symmetry, but

[7] These tureens were both containers for meat in sauce and prestigious elements of decoration.

the adults in the classroom preferred not to go too far with this notion which could cause some trouble among the pupils.

Further activities have consisted in reconstituting incomplete figures by symmetry, all the symmetric objects chosen according to the royal theme: chandeliers, gardens, windows, facades, and so on. I even launched a more complicated activity on napkin folding, which proved to be as promising as this new field of research in history of mathematics (Friedmann & Rougetet, 2017). But to keep the promises, I still have to find a better way to use it in the classroom.

In a simpler follow-up activity, I had the pupils reconstruct the king's table of which they had only one half and the tracing paper of the other half to color symmetrically. Some of them adopted a simplifying strategy of returning their tracing paper, reproducing identically the image given on the tablet, and finally restore the initial orientation of their paper. This strategy was clever in this case because I hadn’t been careful of creating table layouts with only one axis of symmetry. In addition, the pupils’ ability to estimate the success of their work at a glance made all point-by-point examinations useless. I had to find a trick to force them to analyze their production.

Thus I decided to create brigades of waiters, butlers and wine waiters: the pupils would set the table for a royal banquet. Lacking porcelain and silver tableware, they prepared half-tables with plates and plastic cups, placemats and paper napkins, aluminum dishes, all usual school material. A red tube from the gym serving as an axis of symmetry, the brigades were ready to set the table. Given the length of the table (about ten meters) and the height of the pupils, it was not possible for them to supervise the work from above, especially since it was a collective task evaluated by a single Controller of the *Menus-plaisirs* of the king.

By themselves, pupils tend to perform the task in two stages, successively considering the position and then the orientation of the objects to be placed. They first assign a location to the various elements. Only in a second time they pay attention to the orientation of these objects, which generates discussions and even a contestation of the authority of the Controller. Some pupils consider that the first stage was sufficient for the task to be completed, thus focusing on the functional aspect of cutlery and dishes, while the aesthetic aspect was emphasized by the verifiers.

4 Texts as artefacts

We may consider old book as objects as well as sources of knowledge. As in most of school topics, many teachers and students refer to the written word as inarguable. So when you propose a text for reading, it is always taken as the honest truth. Fortunately, there are at least two manners of contradicting that postulate.

First, you can choose obscure writers from the past, as I did with Juan de Ortega, Samuel Marolois or Jean Bullant (Métin, 2006) whose names and works are unknown to our French students. There are so many now forgotten texts, that we have a variety of true errors and mistakes close at hand.

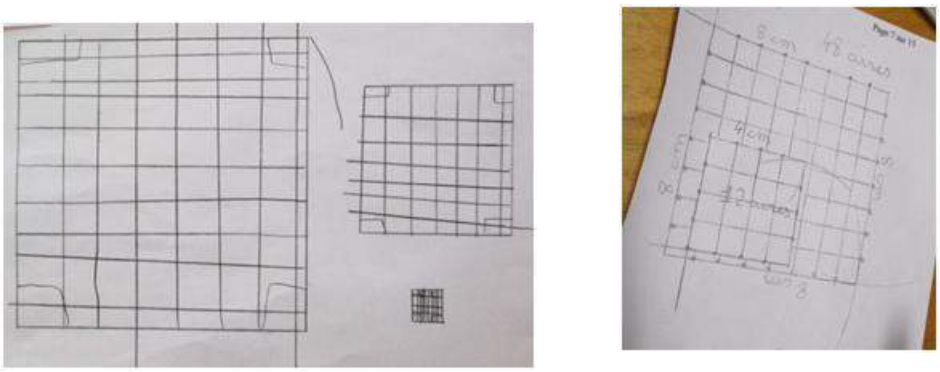
Secondly, you can partly remove texts or images from the original pages, even when the texts are perfectly understandable by themselves. For instance in 17th century books on geometry, the names of points were often displayed before these points were actually defined. The readers were then supposed to follow the construction steps on fold-up plates. The complete figure being under the reader’s eyes, the points were used both as mathematical points and as parts of an image. This way of describing figures before defining their components was also in use in manuscripts, whose writers sometimes set drawings apart. When the plates or illustrations are lost, then our task is to reconstitute the whole. But as this reconstitution leads us to dig deeper in the subjects, and offers the opportunity of an exciting investigation, why not provoke the absence?

4.1 Doubling the square with or without Plato

In Plato's famous dialogue Meno, when Socrates puts forward the duplication of the square, he does it to support his theory of reminiscence. The dialogue was the subject of an interdisciplinary mathematical / French research conducted by a Paris IREM team, and written by Renaud Chorlay as the fifth chapter of *Passerelles* (Moyon & Tournès, 2018). There are no figure dating back from the ancient times in the dialogue, it is not even sure that they might have existed at that time. The figures have for a long time been reconstructed by Plato's readers, but it is an exciting exercise to try to understand the text without them and to reconstruct them in the course of the reading. Despite the difficulties of the text, the chapter highlights the interest of the activity in French as in mathematics. In the Paris IREM group project, words and speech are at the center of learning objectives.

It turns out that I had also worked on the duplication of the square, but in a different perspective. After discovering fractions, pupils in an elementary school in Dijon had troubles with dividing quantities. The very nature of fractions didn't go without saying. I therefore took advantage of this moment of doubt to suggest a “little exercise”, namely the construction of a square whose area was double the area of a given square.

As in the Paris experiment and as in Plato's dialogue, the pupils' first reaction was to double the side of the square, either numerically from its measurement, or by using the compass. How do you explain them that the area of the square they obtained is not twice the original area? They found by themselves a way of testing their solutions in subdividing each of the obtained squares into squares of equal or nearly equal areas (see Figure 4.1(a)), a process they called “areas technique”.



(*a*) (*b*)

Figure 4.1

When the subdivision was accomplished according to the standards, it is the calculation that left something to be desired (Figure 4.1(b)): the preteen here considered that a square of side 4 had a “12 areas” area and that its double, whose side is 8, had an area of “48 areas”. He could not explain his calculations, but he was happy with his result since, like his friends, he had obtained four times the area of the initial square!

Where were we, between text and artefact? I must say that no word by Plato was ever revealed to the pupils. Nevertheless, their own research directions followed the pattern of Meno's slave's ones. Of course, the teacher played the part of a modern Socrates, questioning preteens in the right direction. Meno's initial square had served has an artefact source for a research activity. Here I had removed the text, but other experiments lie on the removal of pictures.

4.2 Removing pictures

Our main research field in history of mathematics is 17th century military architecture. Of course, fortification is strongly linked to geometry, as it is a matter of lines and angles. But constructions are made under constraints and early modern fortifiers had to be able to compute the measures of lengths and angles, which implied a solid knowledge in Euclidean geometry.

Even for pupils without great knowledge of Euclidean theorems, it is possible to propose mathematical activities based on old fortification treatises. Readers will find such activities in my other article in the present volume. The basic activity focuses on drawing the shape of a polygonal fortress according to the construction programs we find in original sources, as for instance *La Fortification reduicte en art et demonstree* by Jean Errard of Bar-le-Duc (Errard, 1620).

During my researches on military architecture, I discovered a very special handbook written by a mathematics teacher of a French Artillery Academy (Famuel, 1684). What was special in this book was an almost white page, with an empty space facing the detailed construction program of the fortified hexagon. I realized that all illustrations in this handbook were handmade. It was used by his author as a pedagogical support: the students were given (or certainly must buy) the booklet with the complete text, but no illustrations at all, and they had to complete the empty spaces with the figures corresponding to the construction programs. It seems to be a trend in French Artillery Academies at the end of the 17th century, because I found two other books of the same type from the same period and the same context.

The natural challenge for use was to reconstitute the figure. As a teacher, I suggested my students to do it themselves, in order to complete the copy in the public library. This was such a funny and interesting activity that I decided from that day on to systematically remove pictures from fortification books. I already have mentioned the supplementary obstacle for contemporary students to follow instructions quoting names of points not yet defined. The advantage here is that the teacher from the past has created the text with full knowledge of his students' blindness. As the original pages were intended as supports for drawing, I do the same now, turning the geometrical books of the past into a kind of modern coloring book (without colors, just lines).

In case of dissatisfaction due to the lack of colors, you can turn to Oliver Byrne's edition of Euclid (Byrne, 1847). If you do so, your task will be to remove the colors from both text and illustrations, and to invite students to reconstitute the colored illustrations along with the colored parts of the texts. It appears that this is not easier than reconstruct missing black-and-white figures in 17th century fortification books.

4.3 Mathematics in the margins

Our last work lead about the use of original sources in the classroom deals with teacher training. I have used written resources as writing artifacts, that is to say, as objects designed to serve as supports for written commentaries about the texts they carry. It was not for students to undertake a textual exegesis of obscure passages of ancient texts, but rather to question the mechanisms they use to understand the contents of these texts.

Writing in the margins when studying a book is a well-established tradition in mathematics. Let's mention at least the most famous example: Pierre de Fermat annotating his copy of Diophantine's Arithmetic and leaving a marginal mention that would become his famous conjecture. According to Fermat himself the margin was too small for his clever demonstration. During my wanderings in libraries, I discovered numerous marginal notes, certainly of smaller impact than Fermat's ones, but witnessing the real interactive aspect of traditional books made of paper. In addition, whether written by well-known scholars or obscure readers, marginal mentions are manifestations of mathematics as human activities.

One of my latest research projects at Dijon IREM deals with written traces of student activity. In its didactical part this project focuses on the mathematics rough book seen as a private journal for students. In its academic part, it relies on the marginal annotation of original texts facsimiles, ranging from commercial arithmetic to algebra, geometry and probabilities. My analysis of the productions is centered on three lines of research: the necessity of rewriting or not, the need for translation in familiar terms or usual notions, and the place of the diagrams.

During University teacher training sessions, I propose students to work on photocopies, with instructions to jot down in their margins anything they had in mind (questions, reflections, astonishment) which they considered part of their learning mechanism.

I gave for example this extract of a curious book on practical arithmetic (Cathalan, 1566):

> *A man has 3 windmills, one of which grinds 5 bushels a day, while the second one grinds 7 bushels, and the third one 8. Comes the seller who wants to grind 100 bushels of wheat. I ask: how should the miller divide wheat between the windmills, so that all three windmills take the same time to complete the task?*

It is not that easy for young students to understand that a bushel is a measure of capacity (of eight gallons). Moreover, the French text mentions a specific measure, the *setier*; or "septier" as Cathalan writes it, which is a mystery to them. They finally hardly understand that this is a proportional sharing problem.

This is probably the reasons why students need to rewrite the text. Unsurprisingly, the words are a first obstacle, and may need a complete reformulation of the text, even if some students neglect the crucial point of the question, the fact that the three mills must have finished at the same time. Productions often show the students' efforts to stress the meaning of words, even if the text is not rewritten. Students use the photocopies to highlight words and redefine them. We must not be surprised by some students' lack of understandings. I purposely indicated that it was not a matter of perfect understanding, but of writing down all the elements of their search.

5 Conclusion

I have presented various activities related to the use of original texts in the classroom with the concern not to sacrifice rigor on the altar of the beauty of tales. Focusing on artefacts effectively saves us from controlling the historical context and precise circumstances. This could be summed up by this formula: not let oneself be locked in a historical horse collar when resources are relevant to mathematics, but allow oneself to explore the historical and social context of these resources. Let us try to summarize the lessons drawn from the various experiments I described.

5.1 What is doing history of mathematics?

In my opinion, doing history of mathematics is first of all practicing mathematics, provided that these mathematical activities are inspired by original sources that can be directly exploited with students or not. Therefore, it is not a matter of strictly historical methods. The questions are mainly aimed at understanding the concepts, even if the process of understanding includes domains off the beaten scientific track. However, calculating with tokens without questioning the medieval period and the link between objects and numbers would be a really dry activity.

In the case of the tiles, there is such a small difference between 18th century objects and the contemporary model that it was quite possible not to refer to the Truchet's text for the activity. However, the search for combinations of tiles is legitimized by the Memoire of 1704, whereas it could seem artificial or useless if it was simply asked as a research subject. This combinatorial exercise will be easily presented as a reconstruction of an ancient text, the existence of which generates verifications and comparison with the students' solutions.

5.2 An epistemological *e(n)strangement*

The concept of an epistemological *dépaysement* was recently re-presented by David Guillemette (Guillemette, 2015*a* & 2015*b*). What is at work here is the transformation of a familiar object (mathematical knowledge) into another less familiar object. The history of mathematics disrupts the usual view of mathematics by confronting us with writings and practices in which we do not recognize our own knowledge. What ordinarily went without saying becomes surprising.

In the examples I gave, the study of arithmetic texts of the Renaissance best allows the epistemological *dépaysement* of high school students and in training teachers. A major interest of the change of scenery here is to allow questioning new questioning by placing student in the dark in front of a text they do not understand at first glance. It happens then a triple effect for people in this situation:

1° as students are forced to deepen their look into matters they do not appreciate much they can first say "I do not understand a word!"

2° attempts to decode mathematics behind this foreign language leads them to reassure themselves about their own capacity for comprehension. Finally they discover they are able to interpret ancient languages (algebra) and methods; it is therefore a work of translation and commentary, because students have to insert complementary algebraic lines to take full advantage of the contents.

3° after being relieved, students will have to think about the mechanisms that allowed them to rediscover the hidden mathematical notions. In fact, this understanding has been favored by the X-rays of their understanding, which showed them the mathematical skeleton of the text.

The *dépaysement* only makes sense if people change their views. As with travels, this change of view is due to meeting others. In other words, looking for the initial context of a mathematical notion or appropriating practices from another time brings us back to the experience of the unfamiliar. More than a change of scenery, it is a revolution. I had no words for this special concept, so I decided to create the neologism *enstrangement* to significate the ability of making strange what was familiar.

5.3 A non-magisterial context

Using original documents means that teachers have to accept not to be the unique source of knowledge. In fact, neglecting historical references leads teachers to presenting subjects as their own knowledge without mention of any third party. However it is possible to insert mathematics in the history of human progress.

In the case of Truchet's tiles for example, it is pleasant to offer students the opportunity to solve a problem already studied in the 18^{th} century but still relevant and within their reach. Would they be interested in working on tile arrangements without the challenge of solving a question posed by an Academician of the Enlightenment? It is Truchet himself who asks the question to the students. This enquiry is a form of adventure that would not be allowed with the teacher alone.

Teachers must even accept to become simple guides of their students in front of a research put by a third person. No longer the exclusive owners of knowledge, teachers become travel organizers, facilitators of discovery, as they are when they take their students to museums. In this case, mathematical activity can’t be reduced to problem solving. What’s more, students here can both use reasoning skills and non-brain skills to experiment end manipulate objects. Working with artefacts is a matter of gesture as well as reasoning; you can’t limit the scope of authorized skills, or you would restrict your students’ abilities, even in the case of a text study. Favoring the surprise (the How? that leads to the Why?) implies making space for students to choose all available means to succeed in the task. But what may not be usual in mathematics (at least in France) is that we are thus led to leave space for students’ expression.

5.4 Personal expression

It seems important to me to establish a dialogue with students about the studied mathematical concepts. Learners are well aware that they don’t invent the properties of objects or the techniques and methods they discover in their activities. But one of the major difficulties for novice teachers is to listen to what students have to say about what they are studying, and even to encourage questions. In scientific activities, these teachers tend to provide all the answers and do not favor the emergence of questions that could deviate from the already marked path. This often happens to ensure the success of the activity.

In this case, teachers promote knowledge more than learning in general, and particularly methods. My proposal for an analysis of understanding based on marginal annotations of the texts is very recent and my working IREM group is in its early stages. I hope my young fellow teachers will agree with the need to promote a personal verbalization of their own students' path in the appropriation of methods or mathematical practices. By putting my trainee students in the position of exegetes, I try to help them to move away from the model of knowledge holders, an all-powerful posture linked to the status given by diplomas.

My goal would be achieved if I could find with them the pleasure of a Joyful Wisdom and the adventure of the search for knowledge. For me, introducing a historical perspective through the use of ancient resources, texts and artefacts, is a means of achieving this goal.

REFERENCES

Anonymous (1509). *Livre d'chiffres et de getz nouvellement imprime*. Lyon: Pierre Mareschal & Barnabé Chaussard. (The Méjanes Library, Aix-en-Provence, Rés. S. 167, 3).

Anonymous (1551). *L'arithmetique & manière de apprendre a Chiffrer & compter par la Plume et par les getz en nombre entier et rompu, facile a apprendre & tres-utile a toutes gens.* Paris: Jehan Ruelle.

Briand, J., Peltier, M.-L. (2003). Étude de la disme de Stevin de Bruges. *Concertum*, *Carnets de route de la COPIRELEM*. Tome 2, 381-406. Paris: ARPEME.

Briand, J., Peltier, M.-L. (Eds). (2016). *Opération Maths - CM1 - cycle 3*. Paris: Hatier.

Byrne, O. (1847). *The first six books of the Elements of Euclid, in which coloured diagrams and symbols are used instead of letters for the greater ease of learners*. London: William Pickering.

Chorlay, R. (2016) Historical sources in the classroom and their educational effects. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.). *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between History and Pedagogy of Mathematics*. Montpellier: IREM de Montpellier, 5-23.

Errard, J. (1620). *La fortification demonstree et reduicte en art par feu J. Errard de Bar-le-Duc. 3rd edition*. Paris: Pierre Delon.

Famuel, C. (1684). *L'Art héroïque ou l'Architecture militaire, divisé en deux traitez. Le premier apprend à bien fortifier les Places tant regulieres qu'irregulieres : l'autre parle de la maniere de les attaquer.* Metz: Pierre Collignon.

Friedman, M., & Rougetet, L. (2017). Folding in recreational mathematics during the 17th-18th centuries: Between Geometry and entertainment. *Acta Baltica Historiae et Philosophiae scientiarum* 5(2), 5–34.

Guillemette, D. (2015*a*). *L'histoire des mathématiques et la formation des enseignants du secondaire : sur l'expérience du dépaysement épistémologique des étudiants*. PhD dissertation. Montréal: Université du Québec à Montréal.

Guillemette, D. (2015*b*). *A conceptual and methodological framework anchored in sociocultural approaches in mathematics education for the investigation of* dépaysement épistémologique. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.) (2015). *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp.195-204). Copenhagen: Danish School of Education, Aarhus University.

Guyot, P., & Métin, F. (2006). Using history of mathematics and physics in teachers training: The Officer's sciences in the Ancien Regime. In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings of HPM 2004 & ESU 4: ICME 10 Satellite Meeting of the HPM Group & Fourth European Summer University on the History and Epistemology in Mathematics Education,Conference on History in Mathematics Education* (pp. 340-345). Iraklion: University of Crete.

Ifrah, G. (1998). *The universal history of numbers*. London: Harvill Press.

Jacquier, C., Drapel, K. (2005). *Le nombre d'or : réalité ou interprétations douteuses ?* Lausanne: Swiss Federal Institute of Technology in Lausanne.

Markowsky, G. (1992). Misconceptions about the Golden Ratio. *The College Mathematics Journal, 23*(1), 2-19.

Mennher, V. (1556). *Arithmetique seconde*. Anvers: J. Loë.

Métin, F. (2006). Original sources in the classroom: Contribution to Plenary Panel Discussion. In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings of HPM 2004 & ESU 4: ICME 10 Satellite Meeting of the HPM Group & Fourth European Summer University on the History and Epistemology in Mathematics Education, Conference on History in Mathematics Education* (pp.188-189). Iraklion: University of Crete.

Métin, F. (2012). Using history for *Math in EFL* teaching: Original texts and cultural approach in the classroom. In É. Barbin, S. Hwang, & C. Tzanakis (Eds.), *Proceedings of HPM 2012 – The HPM Satellite Meeting of ICME-12* (pp. 313-322). Daejeon: Korean Society of Mathematical Education & Korean Society for History of Mathematics.

Moyon, M., & Tournès, D. (Eds.) (2018). *Passerelles. Enseigner les mathématiques par leur histoire au cycle 3*. Paris: ARPEME.

Neveux, M. (1995). *Le nombre d'or. Radiographie d'un mythe*. Paris: Éditions du Seuil.

Ortega, J. (1515). *Oeuvre tressubtille & profitable de lartz science de aristmeticque & geometrie translaté nouvellement despaignol en francoys…* Lyon: Etienne Balland.

Recorde, R. (1543). *The ground of artes : teachyng the worke and practise of Arithmetike, moch necessary for all states of men. After a more easyer & exacter sorte, then any lyke hath hytherto ben set forth: with dyvers newe additions, as by the table doth partly appeare*. London: R. Wolfe.

Sen, S. N., & Bag A. K. (1983). *The Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava, with Text, English Translation and Commentary*. New Delhi: Indian National Science Academy.

Smyth, C. P. (1864). *Our Inheritance in the Great Pyramid*. London: A. Strahan.

Stevin, S. (1582). *Tafelen van interest, midtsgaders de constructie der selver*. Antwerp: Christoffel Plantijn.

Stevin, S. (1585). *De Thiende Leerende door onghehoorde lichticheyt allen rekeningen onder den Menschen noodich vallende, afveerdighen door heele ghetalen sonder ghebrokenen*. Leiden: Christoffel Plantijn. English translation and commentaries in Struik, D. J. (1958). *The principal works of Simon Stevin, vol. II: Mathematics*, 371-455. Amsterdam: Swets & Zeitlinger.

Truchet, S. (1704). Mémoire sur les combinaisons. In *Histoire de l'Académie royale des sciences. Année MCCIV. Avec les Mémoires de Mathématique & de Physique, pour la même Année. Tirés des Registres de cette Académie*. Paris: Martin, Coignard et Guerin.

QUELLE RIGUEUR POUR ENSEIGNER L'ANALYSE?

Ce que nous apprend le calcul des différences (1696- 1768)

Sandra BELLA[1,3], Mònica BLANCO[2,4]
[1]Université de Nantes Laboratoire de Mathématiques J. Leray Laboratoire SPHERE (Paris) France,
[2]Universitat Politècnica de Catalunya, Spain
[3]bellusky@hotmail.com, [4]monica.blanco@upc.edu

ABSTRACT

À travers l'analyse et la comparaison de textes de mathématiciens, nous proposons de mener une réflexion sur la manière de présenter les notions d'infini et d'infiniment petit sur lesquelles repose l'analyse. Si le calcul des différences a été abandonné au profit de la notion de limite, nous pensons que la réflexion en amont faite par les mathématiciens est utile pour connaître le contexte dans lequel la notion de limite a été élaborée, car cela offre une possibilité pour aborder les difficultés rencontrées aujourd'hui dans l'apprentissage et l'enseignement de l'analyse. Nous verrons qu'une rigueur *a priori* n'est pas toujours la meilleure stratégie pour faire accéder les étudiants à la finesse de certaines notions et qu'il est de bon a loi dans un premier temps de passer par certains détours.

Dans cet atelier, nous nous tournons vers l'histoire pour comprendre les difficultés qu'ont représentées les fondements de l'analyse à leur début. Par un article paru en juin 1684, Leibniz rend public l'invention de son calcul différentiel en présentant ses règles et quelques-unes de ses applications immédiates. Ce calcul, que Leibniz nomme également «Analyse des infinis», fait intervenir la notion d'infiniment petit. Certains mathématiciens français s'enthousiasment pour le nouvel algorithme et el Guillaume de l'Hospital qui se l'approprie suffisamment pour être à même d'élaborer le premier traité de calcul différentiel qu'il publie en 1696 sous le titre *Analyse des infiniment petits pour l'intelligence des lignes courbes.* Si certains reconnaissent que le traité contient une «sublimité» de découvertes structurées de manière cohérente, ils le jugent cependant d'accès difficile pour les non savants. D'autres condamnent le calcul différentiel qui y est promu car il repose sur la notion d'infiniment petit dont le statut est jugé bien trop obscur pour être accepté en mathématique. Pendant presque un siècle, des améliorations de présentation sont tentées par des savants, pour la plupart des enseignants. Cet atelier portera essentiellement sur les textes suivants:

- *Éclaircissemens sur l'Analyse des infiniment petits* (Paris, 1725), par Pierre Varignon (1654-1722).
- *Analyse démontrée, ou la Méthode de résoudre les problèmes de mathématiques* (Paris, 1708), par Charles-René Reyneau (1656-1728).
- *Commentaire sur l'Analyse des infiniment petits* (Paris, 1722), par Jean Pierre Crousaz (1663-1750).
- *Analyse des infiniment petits, suivie d'un nouveau commentaire pour l'intelligence des endroits les plus difficiles de cet ouvrage* (Avignon, 1768), par Aimé Henri Paulian (1722-1802).

Pourrait-on enseigner les infiniment petits afin de les utiliser dans une pratique mathématique «rigoureuse»? Les questions sont nombreuses et les réponses apportées sont diverses: ces entités sont tantôt interprétées comme des grandeurs, des "évanescents", des fictions, etc. Quelles sont leurs règles de manipulation? Que signifie diviser par zéro? Autant de questions qui résonnent aussi dans notre propre pratique enseignante. La connaissance de la genèse et la progression des idées et des concepts mathématiques peuvent contribuer à l'enrichissement de l'activité enseignante, en particulier l'incorporation de l'histoire permet de connaître quels types de problèmes étaient étudiés et de quelles manières se sont modifiés leurs contenus et les façons de les aborder.

ON EULER'S FORMULA - BETWEEN STANDARD AND NON-STANDARD ANALYSIS

An interpretation of Euler's *Introductio in analysin infinitorum*

Piotr BLASZCZYK
Pedagogical University of Cracow
pb@up.krakow.pl

ABSTRACT

The workshop combines an interpretation of historical sources, specifically (Euler 1755, ch. 3) and (Euler 1748, §§132–134), with methodological investigations on the role of definition in mathematics. The basic truths of modern mathematics – the series expansions of e^x, sin$x$, cos$x$ – are definitions of complex analysis, while in the 18th century they were proved as theorems. Strangely enough, for mathematical definitions are considered to be mere conventions, while theorems are to be qualified as true or false.

We know that it was Euler who proved the identity e^x = cos$x$ + i sin$x$. In his 1748 *Introductio in analysin infinitorum*, chaper VIII, he expanded sin x and cos$x$ into series. In the Workshop, we focus on §§132-134, where Euler expands the function cos$x$. In his development, Euler does not apply any technique of real analysis, not to mention the notion of limit, but rather infinitesimal and infinitely large numbers instead. That is why we provide an interpretation of Euler's proofs in modern nonstandard analysis. Of course, it is out of question that Euler did not know the ultrapower construction that we apply, or any other technique of modern mathematical logic. He also ignored the 19th century real numbers, as well as the 20th century notion of an ordered field. However, we can show that he implicitly applied rules of non-Archimedean fields, and his operations with infinite sum can be interpreted in terms of nonstandard analysis. As an easy introduction into a technique of algebraic interpretation of historical texts, during the Workshop, we will provide an interpretation of Euclid's *Elements*, book V definitions 4 and 5, and Descartes' operations on line segments as introduced via diagrams in his 1637 *La Géométrie*.

This material and the associated interpretation have been used in my teaching, and students found this approach to ordered fields very instructive, especially because of the idea to introduce real numbers as a special ordered field, rather than via a construction. Mathematical curricula are different in different European countries, but it seems that they share a common feature; namely, that the notion of ordered field appears implicitly, rather than explicitly. Now, starting with the notion of an ordered field, one can grasp easily the idea of a non-Archimedean field. However, this idea is commonly overlooked in most educational systems. Therefore, this workshop is focused on the idea of an ordered field and its history. Seen in historical perspective, mathematical definitions are not simply conventions, but rather basic truths that cannot be proved. Thus, sometimes they turn to axioms; vide Euclid's *Elements*, V, def.4-5, or Descartes' definition of multiplication and division of line segments. In this connection, Euler's formula is a very special case, because in his development these expansions are proved. However, when the framework (i.e. when the non-Archimedean field has been replaced by the conventional ε-δ one),

Euler's findings are introduced as definitions.

The general plan of the Workshop, as well as an annotated PDF file containing excerpts from historical texts were made available to the participants on the web (https://esu8.edc.uoc.gr/wp-content/uploads/2018/08/ID167-Blaszczyk-SharedDoc.zip).

UN ÉCLAIRAGE HISTORIQUE POUR L'ENSEIGNEMENT DES NOMBRES NÉGATIFS

Anne BOYÉ[1], Xavier LEFORT[2]
IREM des Pays de la Loire, France
[1]anne.boye@neuf.fr, [2]xavier.lefort@iutsn.univ-nantes.fr

ABSTRACT

À travers l'analyse et la comparaison de textes de mathématiciens, nous proposons de mener une réflexion sur la manière de présenter les notions d'infini et d'infiniment petit sur lesquelles repose l'analyse. Si le calcul des différences a été abandonné au profit de la notion de limite, nous pensons que la réflexion en amont faite par les mathématiciens est utile pour connaître le contexte dans lequel la notion de limite a été élaborée, car cela offre une possibilité pour aborder les difficultés rencontrées aujourd'hui dans l'apprentissage et l'enseignement de l'analyse. Nous verrons qu'une rigueur *a priori* n'est pas toujours la meilleure stratégie pour faire accéder les étudiants à la finesse de certaines notions et qu'il est de bon a loi dans un premier temps de passer par certains détours.

Dans cet atelier, nous nous tournons vers l'histoire pour comprendre les difficultés qu'ont représentées les fondements de l'analyse à leur début. Par un article paru en juin 1684, Leibniz rend public l'invention de son calcul différentiel en présentant ses règles et quelques-unes de ses applications immédiates. Ce calcul, que Leibniz nomme également «Analyse des infinis», fait intervenir la notion d'infiniment petit. Certains mathématiciens français s'enthousiasment pour le nouvel algorithme et el Guillaume de l'Hôpital qui se l'approprie suffisamment pour être à même d'élaborer le premier traité de calcul différentiel qu'il publie en 1696 sous le titre *Analyse des infiniment petits pour l'intelligence des lignes courbes.* Si certains reconnaissent que le traité contient une «sublimité» de découvertes structurées de manière cohérente, ils le jugent cependant d'accès difficile pour les non savants. D'autres condamnent le calcul différentiel qui y est promu car il repose sur la notion d'infiniment petit dont le statut est jugé bien trop obscur pour être accepté en mathématique. Pendant presque un siècle, des améliorations de présentation sont tentées par des savants, pour la plupart des enseignants. Cet atelier portera essentiellement sur les textes suivants:

- *Éclaircissemens sur l'Analyse des infiniment petits* (Paris, 1725), par Pierre Varignon (1654-1722).
- *Analyse démontrée, ou la Méthode de résoudre les problèmes de mathématiques* (Paris, 1708), par Charles-René Reyneau (1656-1728).
- *Commentaire sur l'Analyse des infiniment petits* (Paris, 1722), par Jean Pierre Crousaz (1663-1750).
- *Analyse des infiniment petits, suivie d'un nouveau commentaire pour l'intelligence des endroits les plus difficiles de cet ouvrage* (Avignon, 1768), par Aimé Henri Paulian (1722-1802).

Pourrait-on enseigner les infiniment petits afin de les utiliser dans une pratique mathématique «rigoureuse»? Les questions sont nombreuses et les réponses apportées sont diverses: ces entités sont tantôt interprétées comme des grandeurs, des «évanescents», des fictions, etc. Quelles sont leurs règles de manipulation? Que signifie diviser par zéro? Autant de questions qui résonnent aussi dans notre propre pratique enseignante. La connaissance de la genèse et la progression des idées et des concepts mathématiques peuvent contribuer à l'enrichissement de l'activité enseignante, en particulier l'incorporation de l'histoire permet de connaître quels types de problèmes étaient étudiés et de quelles manières se sont modifiés leurs contenus et les façons de les aborder.

WHY BOTHER WITH ORIGINAL SOURCES?

Renaud CHORLAY
IREM de Paris (Université Paris Diderot),
Laboratoire de didactique André Revuz (EA 4434), UA, UCP, UPD, UPEC, URN, France
renaud.chorlay@espe-paris.fr

ABSTRACT

This workshop reports on a teaching experiment which was carried out in May 2017 in France, in the final year of elementary school (year 5) and in the first year of middle-school (year 6). Its starting is point the most famous passage of Plato's *Meno*, in which Socrates challenges a slave to construct a square twice as large (in area) as a given square. However rich Socrates' problem may be from a mathematical viewpoint (characterization of common quadrilaterals, distinction between length and area, comparison of areas, proportionality), its study does not require any explicit use of original sources. In this experiment, the challenge was not only to make students work on Socrates' *problem*, but on Plato's *text* as well. Before reporting on the outcome of the experiment, we will present tools and concepts which we found in research works bearing on literacy and reading comprehension, and which helped us design the experiment.

Keywords: original sources, area, proportionality, argumentation, literacy

1 Rationale

1.1 Mathematical content

In this paper, we report on a teaching experiment which was carried out in 2017 in Paris, with students in the final year of primary school (year 5, students aged 10) and the first year of middle-school (year 6). It was designed on the basis of one of the most famous problems in the history of mathematics; a problem which is discussed in one of the foundational texts of the Western philosophical tradition. In the dialogue entitled *Meno* (we will refer to the online edition of (Plato, 1967)), Plato (427-348 BCE) staged a discussion between two characters: Socrates – a philosopher, and Meno – a noble Athenian. About halfway through the dialogue, Socrates engages in a side-dialogue with one of Meno's slaves, who is referred to as "boy". Socrates starts from a square (A) (see Figure 1.1), and challenges the slave to determine or exhibit a square twice as large in area. The slave finds the question easy, and suggests doubling the side (diagram (B)). Socrates points out that this answer is incorrect: shape (B) is a square indeed, but its area is four times that of (A), as can be seen in diagram (C). Socrates then joins the four midpoints of the sides of the large square and claims that this new, tilted, shape is a square whose area is twice that of (A). He justifies this by counting the number of half squares: two in (A), four in the tilted square.

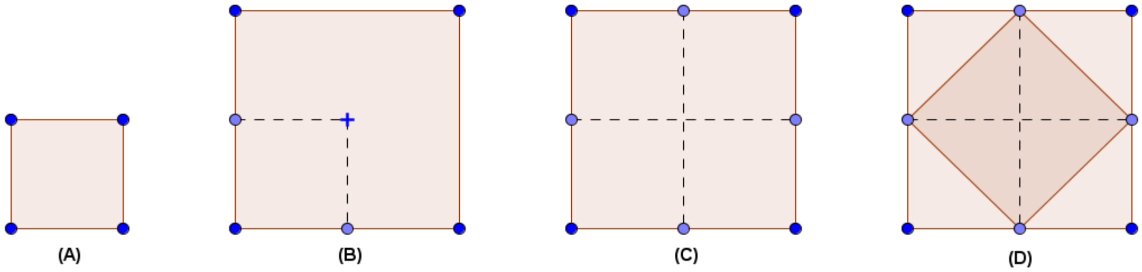


Figure 1.1: The diagrams referred to in Plato's text.

The mathematical content of this passage is rich but fairly elementary, and echoes standard curricular requirements for the end of primary school or the beginning of middle-school (depending on the country): describing rather complex plane geometrical shapes using with a precise vocabulary; characterizing common shapes (in particular when it comes to checking or proving that the tilted quadrilateral is a square); comparing areas, either through a cut-and-paste approach or by measuring (either with an *ad-hoc* or with a conventional unit); distinguishing between situations where proportionality holds, and situations where it does not (here: in squares, the area is *not* proportional to the side).

Of course, the mathematically trained reader would probably see more mathematics at stake in this problem, in particular Pythagoras' theorem. Indeed, Socrates shows that the area of the square on the hypotenuse of an isosceles right-angled triangle is equal to the sum of the areas of the two squares on the sides of the right angle of this triangle, which is a special case of the non-numerical version of the Pythagorean theorem. If we were to use the numerical version, a new feature of the problem would come up: it can be shown that the ratio between the length of the side of the first square and its diagonal (which is the side of the solution square) cannot be expressed numerically using only one whole number, or a ratio of whole numbers. There is probably an allusion to this mathematical fact in the dialogue, when Socrates acknowledges that, if the slave cannot "say" or "reckon" what the side of the solution square is, he can at least try to "show" it (Plato, 1967, 84a). The construction is indeed elementary, whereas the numerical determination is difficult, and depends what you consider to be legitimate "numbers". Since neither the Pythagorean property nor the irrationality of $\sqrt{2}$ are usually studied at this level of the educational system, we chose to leave this completely outside the scope of our experiment. Other choices would have been possible, for instance to engage in a numerical approximation of the measure of the side of the solution-square by trial and improvement, using decimals (Kosyvas & Baralis, 2010).

1.2 A difficult problem, in a difficult text

The study of this problem involves two well-known epistemological obstacles. The first one lies in the difficulty in distinguishing between two different magnitudes associated with one plane shape, namely the length of its border and the area of its surface. The second one lies in the force of the linear model, which leads most students (and probably most adults) to believe that when two magnitudes depend on one another, proportionality holds.

These two sources of difficulty do not play equivalent parts in our experiment, partly for curricular reasons. For students of that age (around 10), at least in France, the notion of area is a key target in the curriculum, and our teaching sessions are designed for students who are already aware of the following facts: a plane shape such as a polygon has both a length and an area; the procedures for comparing lengths and those for comparing areas differ; so do the conventional units for both magnitudes. Consequently, we assume that the students will be display some level of expertise when dealing with the length-area aspect of the problem; and that studying this problem will improve their command of these notions. By contrast, even if the French national curriculum requires that some situations where proportionality does not hold be studied, the main target for students of that age is to study situations where it does hold, and to solve linear problems using an ever growing range of techniques. For most of the students in our experiment, the study of Meno's problem was probably the first occasion to come across the fact that, in the enlargement of plane shapes, a scale factor of 2 leads to a multiplication of areas by 4 instead of 2. We do not claim that this isolated encounter with a counter-intuitive phenomenon will enable students to overcome this epistemological obstacle – should such a thing be possible at all. However, this encounter with a tricky and surprising phenomenon could be used later on in the year, in particular to justify the rules for changing units in areas (e.g. there are 100 cm in a meter, but there are 10 000 cm2 in a square meter).

Since we wanted students to study Plato's text and not only Socrates problem, the mathematical content along with its didactical and cognitive properties was not all we needed to pay attention to. The text is rather long – we used a 6-page excerpt – and of an argumentative nature (even if in a dialogical form). Moreover, this excerpt is a mathematical digression embedded in a philosophical dialogue whose main focus is not – at all – mathematics[1].The excerpt under study is structured by the interlacing of these two scales: main dialogue / digression, philosophical problems / mathematical problem. Indeed, the dialogue between Socrates and the slave – which bears on shapes and areas – is regularly interrupted by sibylline asides between Socrates and Meno; asides which bear on the teaching/learning process. Moreover, in the study of the mathematical problem, the situations of the three characters are asymmetrical: the slave (and the 10-year old reader) understands in the end that his intuitive solution is erroneous, whereas Socrates and Meno (and the teacher) know it from the start. For Meno and Socrates, what is a stake is not a geometrical problem, but rather the true meaning of "believe", "know" and "learn".

For these reasons, the text is not only long but objectively difficult to understand. Clearly, making sense of the text requires that shapes and magnitudes be studied; but it also calls for a continuous work of explicitation and reformulation of sibylline or ambiguous statements. For lack of some key information (what is really at stake for Meno and Socrates in their little "experiment" with the slave? What is the correct answer to the mathematical puzzle?), the reader has to continually make hypotheses as to what the various characters know and aim for. Many passages are rather obscure upon first reading, since the key to comprehend them is given in a later part of the text. For the reader, this has both a cognitive and an affective impact: one has to agree to go on reading without understanding everything. One has to accept the fact that, at different times along the reading process, the degree of understanding of different parts of the text will evolve.

Beyond these general features which make the reading experience a demanding one, two other specific aspects should be mentioned. First, Plato's manuscripts were transmitted without diagrams, and most contemporary editions chose not to provide visual help. There can be no doubt for the reader that Socrates is discussing and drawing diagrams, however it is left to the reader to sketch them along the way, which is not trivial since the text is occasionally ambiguous. In our design, we thought that making hypotheses about the diagrams mentioned, described, and discussed in the text was a task that could be fruitfully entrusted to students. Second, the mathematical vocabulary used by Plato is not ours. In particular, he used the same name for the units of length and of area (the foot). Although this fact was common in Ancient mathematics – in Greek mathematics but in Chinese or paleo-Babylonian mathematics just as well – it can be confusing for the reader. In our design, we regarded this feature of the text as providing an

[1] The main topics discussed in *Meno* are virtue/excellence (what is virtue/excellence? Are all men equally capable of virtue? Is it inbred or can it be taught?), and teaching/learning.

opportunity for the students to spot this ambiguity, discuss and criticize it, and maybe suggest ways to reduce it.

1.3 Why bother with the original text?

This experiment was designed in the context of a larger research programme on the use of original sources in the classroom. An outline of its theoretical background can be found in (Chorlay, 2016, pp.9-14). To put it in a nutshell, we choose not to focus on history of mathematics in the classroom, but on the use of historical "documents" – be they texts, diagrams, or instruments – as a means to entrust students with tasks of a reflective nature; tasks which bear on a sample of mathematics. These tasks – which we collectively denoted as *meta*-tasks – are usually referred to by verbs such as: reformulate, translate, make explicit, disambiguate; assess, criticize; justify, prove, spot a missing argument and provide one; generalize, assess the generality. With its rather long and sometimes oddly worded list of arguments bearing on reasonably basic mathematical notions, Plato's text seems to lend itself particularly well to this type classroom work.

As a consequence, the classroom sessions reported upon in this paper are to be regarded as part of a research programme, and not as a teaching resource which we would claim should or could be used widely in more ordinary contexts. One reason for this is that the three sessions were designed – over a rather long period of time – by a group of three: the researcher, one primary school teacher2, and one secondary school teacher3. Hence, those who actually implemented the sessions should be considered as associate researchers. A second reason is that, for research purposes, we decided to keep the tasks as difficult, demanding, and challenging as we deemed possible, at the risk of facing classes of nonplussed students supplying irrelevant, senseless or random answers; or no answers at all. This highly demanding format is in keeping with our goal, which is to probe and try to delineate the thin line between the fruitfully demanding, and the altogether impossible (for students of a given age); or, to put it differently, between productive and unproductive struggles.

This general perspective has to be kept in mind in order to understand the many specific choices reported below in the description of the sessions. For now, let us mention three consequences. First, in order to collect data showing what students managed to do when working on their own, we put the emphasis on written tasks, even in cases when we think it would not be necessary or even useful in ordinary teaching conditions. Of course, we also audio-recorded the sessions in order to study the collective phases as well as the interactions with the teacher. Second, the investigation is of a qualitative nature, not only because of the size of the samples (two classes) but also because the aim is to study what is possible in a given educational context; hence we take one instance as a proof of possibility; hence, we focus on the analysis of the qualitative variations in the range of actual answers rather than on their relative frequencies. Third, since we were ready to face "failure" – suggesting a relevant research-result of "impossibility" – we were also ready to let some students fail. At this point, there is tension between the goals of the researcher and that of the teacher.

This report provides an opportunity to discuss a key element of our research programme which was only mentioned in passing in (Chorlay, 2016). Then, the focus was

[2] Dominique Heguiaphal, école primaire Arago, 75013 Paris, France.
[3] Alexis Gautreau, cité scolaire Rodin, 75013 Paris, France.

on a first characterization of meta-tasks. Working on Plato's text made it necessary to take into account didactical problems which are not directly related to mathematics, but to reading comprehension and, more generally, literacy.

For this purpose, we drew mainly on the work of the research team of Roland Goigoux. With a theoretical background in the didactic of literacy and in textual linguistics, these experts developed a research-based teaching resource entitled Lector & Lectrix, which was designed to improve the reading comprehension skills of students from year 3 to year 9 (Cèbe & Goigoux, 2009). According to them, "understanding a text" rests on the interplay between several skills: decoding the written code (the basic meaning of "being able to read"); linguistic and textual decoding skills regarding syntax, lexicon, punctuation, connectors; availability of referential knowledge (knowledge about the world: in this specific case, about shapes and geometric magnitudes), and strategic skills (regulation, control, and assessment – by the student – of his/her reading activity). Following Umberto Eco's Lector in Fabula, they highlight the importance of the latter skills:

> To understand a text, the reader has to simultaneously use all these skills so as to carry out a twofold processing activity: some local processing – which gives access to the meaning of groups of words and of sentences – and some more global processing – allowing for the construction of a coherent mental representation of the whole. (…) The latter process – called semantic integration – is of a cyclic nature: each new input leads the reader to reorganize the representation which he/she constructs step by step, along the way (…). This means that the reader should be flexible enough to be able to acknowledge that his/her first representations are provisional, hence revisable. (Cèbe & Goigoux, 2009, 7. Our trans.)

On this theoretical basis, Goigoux points out that these strategic skills are not often taught and trained explicitly, and that this could account for the persistence of a significant proportion of low-achieving students who can decode written texts but not actually read them as soon as their length or level of complexity exceeds the very basic. To address this issue, his team wrote a series of textbook specifically aiming for an explicitly training of students in reading comprehension. Let us mention some of their "guiding principles":

- Make students more active and able to regulate their own reading activity: avoid long lists of detailed questions; ask students to assess their own degree of understanding ("I'm sure of this", "I'm quite sure", "I'm not so sure"…)
- Ask students to fill the "blanks" of the text: one has to cooperate with the text to go a little beyond what it says explicitly. One should teach the distinction between what the text says, and what it leaves for the reader to infer (and inferring is not the same as imagining or inventing); everyone has his own "way of understanding", but a socially shared understanding is to be aimed for.
- Ask students to reflect on the characters' thoughts, in terms of goals (for the future), of motives (in connection to the past), but also in terms of feelings and emotions; of knowledge and reasoning.
- Learn to memorize and make sense by constantly reformulating and paraphrasing.
- Learn to adjust the reading strategies to a specific goal: Reading strategies are goal-dependent, and there are many possible reading goals; the teacher should point to this variety, and make the current reading goal explicit.

- Pay a constant attention to the lexicon: The meaning of a word can be explained by the teacher before or during the reading. Students should also realize that a reader can make hypotheses as to the meaning of a new word.

In summary, beyond the specific content-related goals (shapes, area, proportionality), we wanted students to experience argumentation in a mathematical context, by reading – but also by reformulating, complementing, assessing, or providing – arguments. Since the opportunity to do this was provided by a long and difficult text, these *meta*-tasks were intertwined with less specific – but just as challenging – text-reading tasks. We thus drew on the guiding principles of the *Lector & Lectrix* teaching programme to design our experiment.

Before describing the three teaching sessions, we need to mention two negative choices. First, we decided not to expatiate on the philosophical meaning of the text; we touch on it when we feel it is necessary to make sense of some passages in the dialogue. This may be frustrating to the educated readers who knows how deep Plato's text is, and who have experienced other ways of using it in their teaching – in particular in teacher-training contexts. Second, we chose not to expatiate on the historical context. However, as preparatory work for the sessions, students were asked to read and summarize basic background information (location of Athens on a map of Europe, short biography of Plato etc.).

2 Outline of the teaching sessions. Samples of students' worksheets

The three 1-hour teaching sessions designed with the two teachers were taught in May 2017 in Paris, in two "ordinary" classes, one in the final year of primary school, and one in the first year of middle-school. All students' worksheets were collected, and the sessions were audio-recorded. For lack of space, we will only present the outline of the sessions (in particular the list of tasks entrusted to students), and discuss a few samples of students' individual worksheet (for a more detailed account, see (Chorlay, 2018)). This implies that this report will be biased, since we will mention only in passing what happened during the collective discussion phases, either among students or with the teacher. We will also focus on the mathematical tasks, at the expense of the reading tasks; lack of space is not the only reason: we also need more time – and probably need to collaborate with researchers working on reading-comprehension – to be able to analyze these aspects at research level.

The outline of the three sessions is the following:

Session 1: Discovery of the text (up to "And might there not be another figure twice the size of this, but of the same sort, with all its sides equal like this one?") ; discovery of the main problem ; questions on the characterization of the square by its sides only ; questions on the meaning(s) of "foot".

Session 2: Reformulation of the duplication problem; comparison between solutions suggested by students and the solution of the slave; assessment of Socrates' criticism of the slave's answer.

Session 3: Discovery and assessment of Socrates' solution. Final look back, and reflection on the meaning of the whole dialogue, in particular with respect to the asides between Socrates and Meno.

2.1 Session 1

Outline of session 1 (we italicized the questions students were asked directly). The text referred to is (Plato, 1967, 82a-82d); the first diagram drawn by Socrates is a square with sides of two feet each (diagram A of Figure 1.1):

- Collective work: correction of the preparatory homework: Plato, citizenship and slavery in Ancient Greece, location of Greece and of Athens on a map of Europe. Short presentation of the goal and format of the three sessions.
- Silent reading of the beginning of the dialogue
- Second silent reading. *Use three colours to sort the sentences or words into three categories: "This I understand" "This, I understand a little" "This, I don't understand*"
- Collective discussion: *Why is this text difficult?*
- *We all agree that part of the difficulty stems from the fact that the characters are discussing diagrams that are not available in our edition of the text. Take five minutes to draw, in the margin, what you think the diagrams are.*
- *The sentence "*The space is twice two feet" *is very important:*
    - *Can you explain what Socrates means? (you can write, draw ...)*
    - *Do you agree with him?*
- Collective discussion
- *In the text, Socrates seems to be saying that a shape with four equal sides has to be a square. Do you agree with him?*

The answers students gave to the final question were not surprising: many remembered that a quadrilateral with equal sides can be a (non-square) rhombus; those who said it had to be a squared were quickly convinced by the collective discussion. The two questions about the contention that "the space is twice two feet" were less standard, and elicited a variety of responses. In figure 2.1A, the student found a geometrical interpretation of the values in the text (since a side is 2 feet, two sides are 4 feet) which does not involve areas.

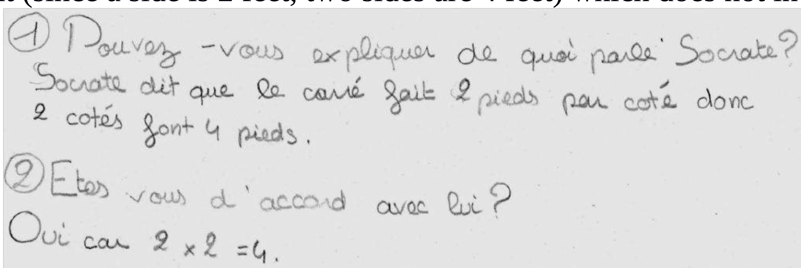


Figure 2.1A[4]

[4] "(1) Can you explain what Socrates is talking about? Socrates says the square is of 2 feet per side, so 2 sides are 4 feet.(2) Do you agree with him? Yes, since 2×2 = 4."

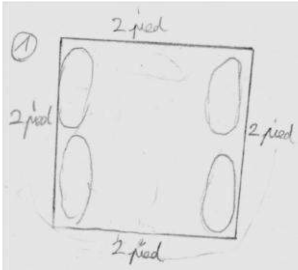


Figure 2.1B

In figure 2.1B, the student drew a clever diagram which accommodates the ambiguity of the text: the schematic feet denote both a unit length, and the location of the four unit-squares. Figures 2.1C and 2.2D show symmetrical successes/shortcomings: student 2C identified multiplication as away to work out the area of the square, but did not spot the need to change units for areas. Student 2D did not clearly mention areas, but chose to express the result in square centimetres. The written trace "two times two cm2" is ambiguous, since it can be interpreted either as "(2×2) cm2" or as "2 × 2cm2". Both answers are correct, the first one being closer to the standard formula, the second one being closer to Socrates explanation (in which he decomposes the square in two rectangles of 2 square-feet each).

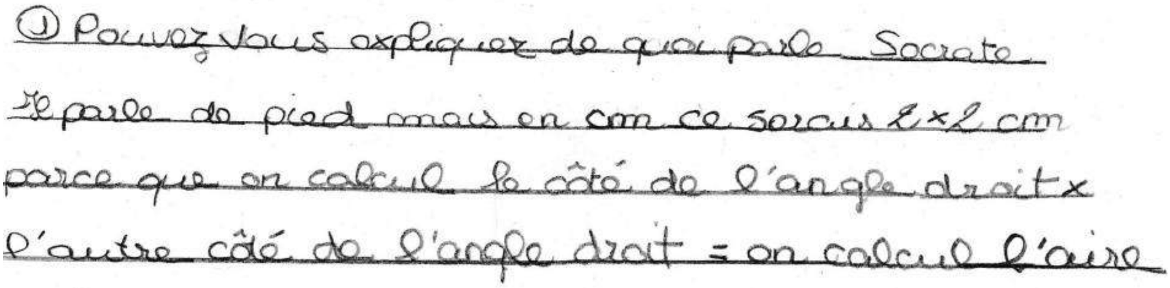


Figure 2.1C[5]

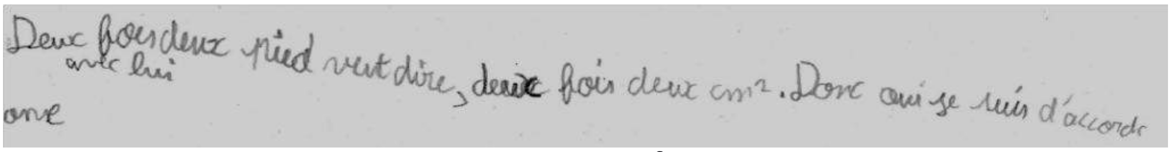


Figure 2.1D[6]

On the basis of the individual answers of the students, it was not difficult for the collective discussion to lead to a consensus on two points: First, in the text "4" refers to an area and not a length; second, given the fact that "foot" is a unit of length, maybe Socrates should have used expressions such as "square foot" or "foot of area" to avoid confusion between length and area.

2.2 Session 2

Session two ran very smoothly, so we will not report on it in any detail. Since our goal was not to design a problem solving session, but a session in which students were to make sense of, and assess arguments from the dialogue, we only gave them 1 minute to show us

---

[5] "(1) Can you explain what Socrates is talking about? He is talking about feet, but in cm it would be 2 × 2 cm since we work out the side of the right angle × other side of the right angle = we work out the area."
[6] "Twice two feet means, twice two cm². So yes, I agree with him."

their intuitive answer to the main problem. As was expected, many suggested doubling the side. Some suggested drawing a congruent square next to the first one, which clearly doubles the area, but results in a non-square rectangle. In one answer-sheet, the student drew arrows to denote the enlargement process in a way which is reminiscent of the use of touch-screens on computers or cell-phones. The outline of the session is:

- (Collective work, without the text) Reminiscing and reformulating Socrates' problem: "And might there not be another figure twice the size of this, but of the same sort, with all its sides equal like this one?"
- Spontaneous answers of students (1 min).
- Silent reading of Socrates' explanation of the incorrectness of the slave's answer. Students are asked to draw the missing diagrams.
- Collective discussion, consensus on the incorrectness of the slave's answer.
- Collective discussion on the meaning of the aside between Meno and Socrates:

  Boy : Clearly, Socrates, double.

  Socrates: Do you observe, Meno, that I am not teaching the boy anything, but merely asking him each time? And now he supposes that he knows about the line required to make a figure of eight [square] feet; or do you not think he does?

  Meno: I do.

  Socrates: Well, does he know?

  Meno: Certainly not.

As far as the philosophical aside between Meno and Plato is concerned, once students have come to the conclusion that the slave's answer is incorrect, they can be asked to spot the verbs in the excerpt. Clearly, Socrates and Meno mean to distinguish between, on the one hand "thinking you know", and on the other hand, "knowing". Thus, they are only willing to use the word "know" in cases where the conviction bears on a true statement.

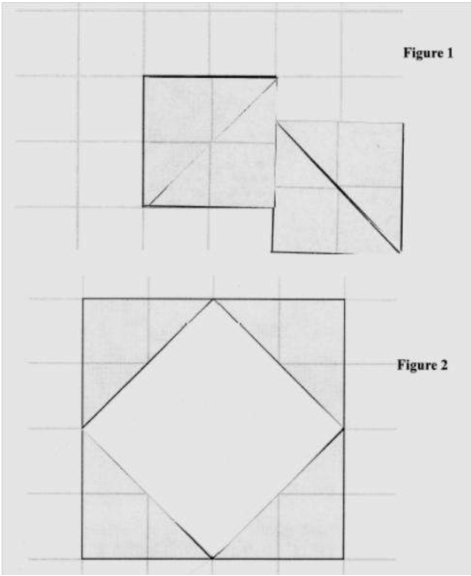
2.3 Session 3

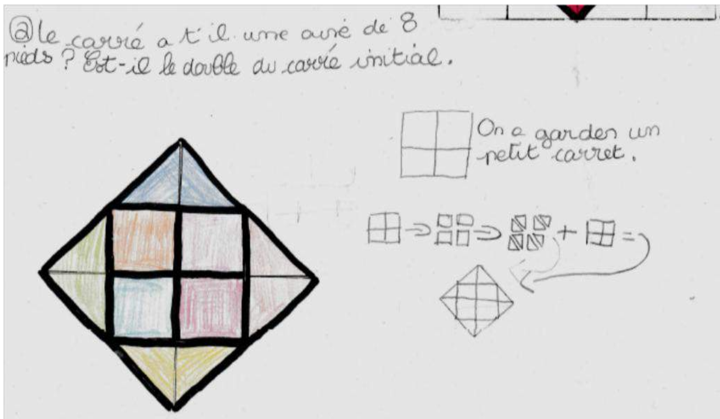
In session 3, Socrates solution was first read by the teacher, who supplemented the missing diagrams along the way (Figure 1.1). The students were then asked to assess Socrates' proposal, without having the text at their disposal. The outline of the session is:

- Reading Socrates' solution. The teacher draws the corresponding diagram on the blackboard.
- *To see if Socrates' answer is correct or incorrect, we need to check two things:*
  - *That the tilted shape in diagram* (D) *is, indeed, a square. Write down what geometrical instruments you need to use to check this*.
  - *That its area is twice that of the square* (A) *from which we started.*
    To do this, you can use either: (1) shape (A), *a marked ruler and a calculator; or (2) two shapes of type* (A) *and one of type* (D), *with scissors and glue; or (3) shape* (D).
- Final look back. Collective discussion on the meaning of the text.
  - *Do you think the goal of Meno and Socrates was to make fun of the slave?*
  - *Does the dialogue between Meno and Socrates bear on squares and areas? If not, what is it about?*

Let us focus on the answers to the area question. We thought that many middle-school students would rather measure lengths on the diagram and use multiplication to find an approximate value of the measure of the area of the tilted square, since the use of formulae is the standard procedure to deal with areas in middle school. It so happens that in the context of this problem, no students did that; all used the cut-and paste approach, usually in very clever and convincing ways.

Figures 2.2A and 2.2B show two solutions using (or alluding to) scissors and glue:



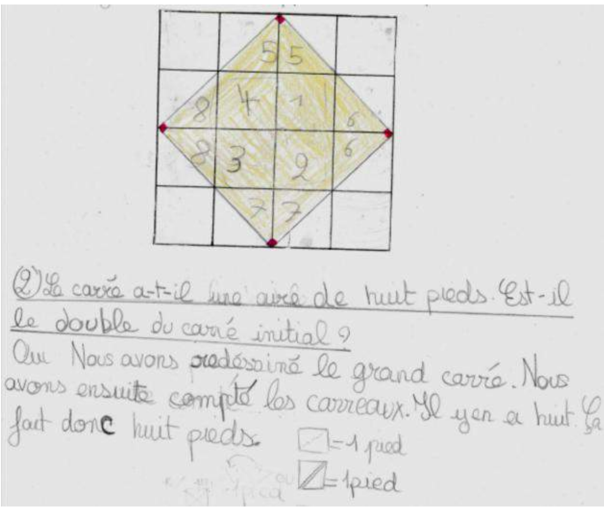


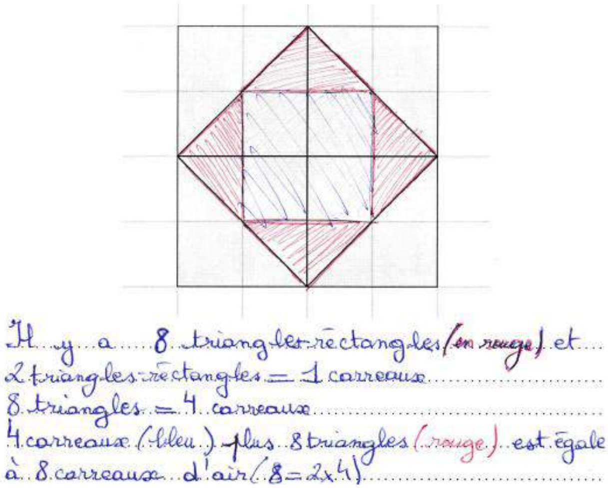
Figures 2.2A (left) and 2.2B[7] (right).

In 3A, the student cut out the tilted square from the diagram at the bottom of the page, cut it in four isosceles right-angled triangles, and used them to make up two copies of the original square. In 3B, the student explained with a mixture of diagrams and words that the tilted square can be decomposed into the original square plus one copy of the original square decomposed into eight halves of the unit-square.

Other students managed to validate Socrates' answer without any instruments:

[7] "(a) Does the square have an area of 8 feet? Is it twice as large as the initial square?" [teacher's questions]. "You can keep a small square."





Figures 2.3A[8] (left) and 2.3B[9] (right).

In 4A, the student counted the number of unit-squares in the tilted square, and added a small diagram to show that the area of one (square)-foot could be found either in a square or in a pair of half squares. 4B shows a variant of this reasoning.

One student provided a correct answer that we had not anticipated:

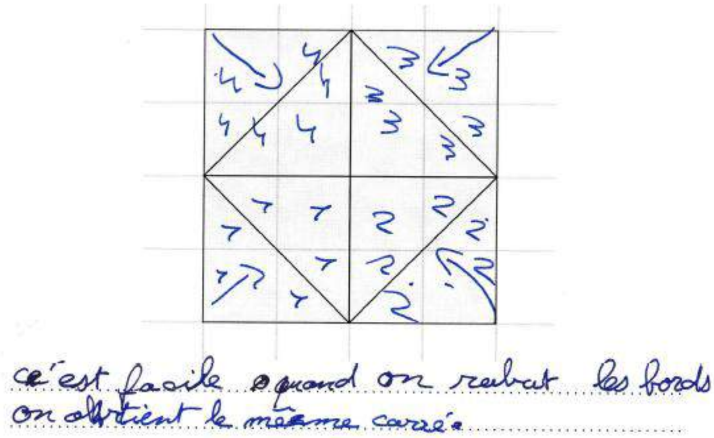


Figure 2.4[10]

In figure 2.4, the tilted square is seen as half the large square (you just need to "fold the corners to get the same [i.e. tilted] square").

When it eventually came to discussing the general meaning of the text, it was not difficult for students to say that Socrates and Meno are clearly not doing this to make fun or humiliate the slave. Students usually interpreted Socrates lengthy explanations as a sign of benevolence, and suggested that, for Meno and Socrates, the dialogue bears on what a good teacher is: someone who explains patiently and in detail; someone with whom even an uneducated slave can learn. They also mentioned the fact that making mistakes is not shameful, and that it is sometimes necessary to make mistakes, in particular if it helps you

[8] " (2) Does the large square have an area of eight feet? Is it twice the initial square? Yes. We redrew the larger square. Then we counted the squares. There are eight of them. So it makes eight feet."

[9] "There are 8 right-angled triangles (in red) and 2 right-angled triangles = 1 small square. 8 triangles = 4 small squares. 4 small squares (blue) plus 8 triangles (red) is equal to 8 small squares of area (8 = 2×4)"

[10] "It's easy, when you fold the edges you get the same square."

realize that what you think is wrong. We are not claiming that is a deep, or even accurate rendition of the philosophical content of the text. However, for lack of additional information on Plato's philosophy and on the content of the rest of the dialogue, we take these elements to be indicative of students' ability to make reasonable hypotheses on the meaning of this excerpt as a whole, and to acknowledge the fact that something is at stake here beyond squares and areas.

3 Conclusion

Even though this experiment centred on a famous mathematical problem, the sessions we designed were not problem-solving sessions in the ordinary sense. Rather, students were to study the problem along with a list of answers (some incorrect, some correct), and the three sessions were designed so that the tasks which were most regularly entrusted to students were: reformulate, disambiguate, make explicit; assess an answer, assess an argument, complement a justification. Consequently, three levels can be distinguished: 1/ the level of the mathematical problem (shapes, areas, proportionality) 2/ the level of argumentation (making sense of the arguments, assessing their validity, their clarity), and 3/ the level of semantic integration (ability to go on reading even if everything is not clear or even makes perfect sense, ability to make hypotheses as to what the characters know and want, ability to revise these hypotheses as the text unfolds). If we were to give *one* answer to the "why bother with the original source?" question, we would say that our initial target was level 2: we regarded the study of this exchange of arguments about a geometrical problem as providing opportunities for *meta*-tasks, and as a means of enculturation into argumentation in mathematics, at a stage of the educational system where argumentation does not usually play a prominent part (if any). However, as we designed this project, we began to take level 3 into account for both practical and theoretical reasons. From a practical viewpoint, we did not want the three sessions to be complete failures because this long and difficult text made no sense to the students! Objective properties of the text – in particular, the interlacing of two dialogues: a mathematical dialogue between Socrates and the slave, and a philosophical dialogue between Socrates and Meno – made it necessary for the design to include scaffolding strategies. From a theoretical viewpoint, the fact that our research programme on the use of original sources in the teaching of mathematics called for reflection on reading-comprehension was highlighted in (Chorlay, 2016, 10), but, then, we gave no indications as to how to do it. The *Meno* experiment gave us a first opportunity to attempt to make use of inputs from research on literacy in the design of a teaching sequence.

As usual, whether or not this experiment was successful depends on the criteria for success. Some global indicators are positive: the engagement of students in the sessions was fair or good, as the written productions and the recordings show. Also, the two teachers found the sessions intense but rewarding, and included them in their teaching in 2018. As far as level 1 is concerned, the mathematical notions at stake in the text are relevant for students of this age: on some occasions, most students gave correct and sometimes creative answers (as for the assessment of Socrates' answer); on other occasions, as had been anticipated, some provided correct but oddly-worded answers, and some made standard mistakes. In the latter case, the variety of answers among students was sufficient for the collective exchange of arguments to lead to a consensual correct answer, under the guidance of the teacher. As far as level 2 is concerned, the extent to which this experiment contributed to an enculturation into mathematical argumentation cannot be assessed, since such a process can only take place over a long period of time. As far as level 3 is concerned, we probably need to work with researchers in literacy so as to specify research questions and methods.

A report on this experiment was published in (Moyon & Tournès, 2018), among a selection of experiments in using historical documents in the mathematics classroom for what the new French national curriculum calls "cycle 3" (final two years of primary school + first year of middle-school). This book is circulated by two professional associations, that of secondary school maths teachers (APMEP), and that of primary school teacher-educators (ARPEME). Thus, it is presented not only as an experiment in the context of a research programme, but also as a resource for training teachers and teaching children. Needless to say, a study of the reception of this resource by educators and teachers who were not associated to its design would greatly contribute to the reflection on the use of HPM in teaching and training, from a different perspective than that of task-design.

REFERENCES

Cèbe, S., & Goigoux, R. (2009). *Lector & Lectrix. Apprendre à comprendre les textes narratifs. CM1-CM2-6e-Segpa*. Paris: Retz.

Chorlay, R. (2016). Historical sources in the classroom and their educational effects. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between History and Pedagogy of Mathematics (HPM 2016, 18-22 July 2016)* (pp. 5-23). Montpellier: IREM de Montpellier.

Chorlay, R. (2018). Doubler le carré avec Platon. In M. Moyon, D. Tournès op.cit.

Kosyvas, G., & Baralis, G. (2010). *Les stratégies des élèves d'aujourd'hui sur le problème de la duplication du carré. Repères IREM , 78*, 13-36. Available online at: http://www.univ-irem.fr/spip.php?rubrique24

Moyon, M., & Tournès, D. (Eds.) (2018). *Passerelles. Enseigner les mathématiques par leur histoire au cycle 3*. Paris: Commission inter-IREM Histoire et épistémologie des mathématiques – ARPEME.

Plato (1967). *Laches – Protagoras – Meno – Euthydemus* (trans. by W.R.M. Lamb). Cambridge (Mas.): Loeb Classical Library. Available online at: http://www.perseus.tufts.edu/hopper/text?doc=Perseus%3Atext%3A1999.01.0178%3Atext%3DMeno

TEACHING MATHEMATICS AND ALGORITHMICS WHITH RECREATIONAL PROBLEMS

The *Liber Abaci* of Fibonacci

Marc MOYON
IREM de Limoges, University of Limoges, CNRS, XLIM, UMR 7252, F-87000 Limoges, France
marc.moyon@unilim.fr

ABSTRACT

I propose an empirical study on theintroduction of an historical perspective onmathematics education at different levels in the French secondary school curriculum (11-18 years old). First of all, I present, in the historical contexts of the twelfth and thirteenth centuries, elements of the biography of Leonardo da Pisa, better known as Fibonacci, and of his mathematical work. I pay close attention to the mathematics of the Islamic countries that had largely fed his thinking. I dedicate an important part of our contribution to excerpts from theLiber Abaci to understandbetterhow his work might contribute to today's classroom. All selected extracts belong to the category of so-called 'recreational' problems. They were chosen for their algorithmic structure that allows us to work with pupils on, among other themes, algebra and algorithmics (including coding). Finally, I give details of mathematical and historical extensions.

1 Introduction

The work presented in this paper[1]is part of a global project introducing an historical perspective on mathematics education developed in the IREM (Institute of Research in Mathematics Education) of Limoges under my supervision[2] and following the works of the French Inter-IREM Commission on history and epistemology of mathematics.[3] With secondary school mathematics teachers, I experimented with activities in classroom and analysed pupils' work.[4] More precisely, my purpose, here, isto take advantage of the algorithmic nature of a 'recreational' problem from medieval mathematics. I think that it is a suitable way to introduce, practice and clarifyalgorithmics in classroom and also to improve pupils' abilities to code computer programs (with Scratch[5]).

I focus on the so-called 'Apple Orchard Problem' as it was presented by Leonardo Fibonacci (13th c.) in his well-known Liber Abaci. Since the historical context is important to understand the genesis and the development of mathematical ideas, I present briefly, in the first part, a bibliographical overview of Fibonacci. Then, I explain the main reasons which make me consider this problem as a recreational one. Furthermore, I

---

[1] I partly base our contribution on (Moyon, 2019). I give here new extensions by taking into account others historical sources and new mathematical developments. Iwould liketo express mysincere thanks to Maurice OReilly for his diligent proofreading of this paper. Nevertheless, I am the only one responsible for the flaws or errors that remain.

[2] http://www.irem.unilim.fr/recherche/algorithmique-histoire-des-mathematiques/ (Accessed: 20 August 2018). For this contribution, I worked with Valérie Fréty (Collège Maurice-Genevoix, Couzeix) and Julie Pousse (Collège Louise-Michel, Saint-Junien). I thank them for their confidence and their availability.

[3] http://www.univ-irem.fr/spip.php?rubrique15 (Accessed: 20 August 2018).

[4] In the same way, the IREM of Limoges participated to the "Passerelles" project with the edition of a book (Moyon & Tournès, 2017) and the construction of a companion website providing complementary documents http://www.univ-irem.fr/spip.php?rubrique505 (Accessed: 20 August 2018).

[5] Scratch is used (https://scratch.mit.edu. Accessed: 20 August 2018) because it is provided free of charge and it corresponds to the most used in French classroom.

analyse[6] several different pupils' work (13 to 15 years old) from mathematical activity to coding. In the last part, I give, for the same problem, mathematical extensions, for more advanced pupils, involvingsequences satisfying a recurrence relation. Finally, I give extracts from another medieval Latin text in the same genre.

2 Fibonacci and his *Liber Abaci*

In this part, I limit my remarks to necessary historical contents[7] because my goal is not to offer a complete work on Fibonacci (even if that were possible[8]) but it is rather an opportunity to put in context the mathematical content that follows. In addition, this part could be useful for mathematics teachers or teacher educators who would like to reproduce my experiments in their own practice.

2.1 Fibonacci: some biographical elements

Leonardo of Pisa was born in the last third of the twelfth century and died after 1241. He belonged to the merchant elite of Pisa, a very important maritime republic on the Tuscan coast. He was the son of Guglielmo Bonacii, hence his nickname Fibonacci, contraction of 'filius Bonacci'. He was also known as Leonardo *Bigollo* (the traveller or the wanderer) for his numerous travels in the Mediterranean Basin. He visited several regions such as Egypt, Syria, Constantinople, Sicily and Provence. His most important journey was to Bugia (now Bejaïa in Algeria) where his father worked as a 'scriberesponsible for Customs' (Caianiello, 2013, 241). In thisMediterranean port city under Pisan rules, he was initiated to Arabic mathematics and, especially, to the Hindu-Arabic numeral system and algebra. Fibonacci largely benefitted from the cultural and scientific contexts of the Mediterranean Basin of the thirteenth century with important commercial and diplomatic relations between the north and the south, in particular with the fourth, fifth and sixth crusades, contemporary with Fibonacci. Probably in his native city, he met the Emperor Frederick IIwho appreciated the sciences in general.[9] It is well-known that the court of the Holy Roman Emperor comprised various multilingual scholars from all over the Mediterranean Basin. As Latin, Greek and Arabic were spoken, numerous written sources circulated. Fibonacci seemed to have scientific and friendly links with many of these scholars such as Michael Scot, John of Palermo and Theodore of Antioch.

2.2 The works of Leonardo

Fibonacci is nowadays well-known thanks to the famous sequence defined such as every number after the first two is the sum of the two preceding ones. Nevertheless, his work should not be reduced to it (Figure 2.1). Two books of fundamental importance were: his*Liber Abaci* for arithmetic and algebra (Fibonacci, 1857) and his

[6] The analysis is based both on the observation of students' activity during the research and on the examination of written documents (taking into accounts general and mathematical abilities). See (Moyon, 2019, 248).

[7]All this part is extracted from (Moyon & Spiesser, 2015) and (Moyon, 2016).

[8]For more details, see (Caianiello, 2012, 2013; Høyrup, 2016).

[9]Fibonacci begins his *Flos* – a short text on algebra with problems involving several unknowns written between 1226 and 1230 (Picutti, 1983) – with: "While I was in Pisa in the presence of your Majesty, very glorious Prince Frederick, Master Jean of Palermo, your philosopher, discussed with me many questions about numbers and among other things proposed me two problems concerning both geometry and number" (Fibonacci, 1854, 2; Moyon, 2016, 7).

Practica geometriae for geometry (Fibonacci, 1862).[10] He authored also three other writings: the *Liber quadratorum*, the aforesaid *Flos*, and the *Epistola ad magistrum Theodorum*.[11] Considering the breadth of his work, Fibonacci seems to have mastered the scholarly mathematics of his time, as well as the practical problems related to it. After the vast movement of translation from Arabic into Latin in the twelfth century, he himself contributed to the spread throughout Europe of Islamic mathematics with their new theoretical ideas and problems. Below are three basic examples.

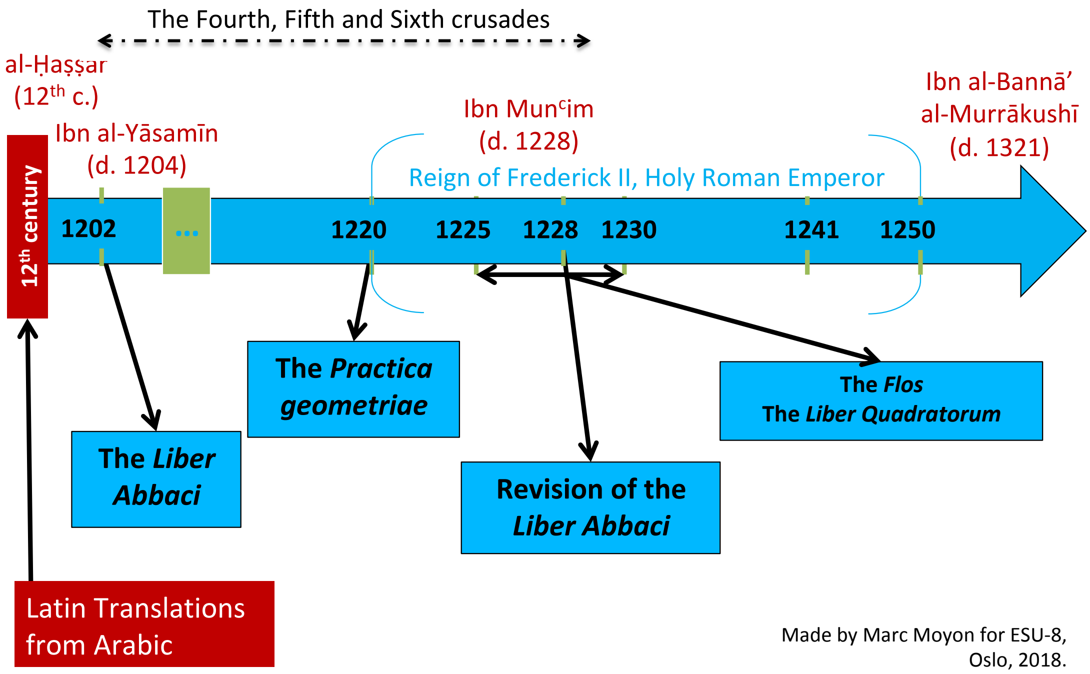


Figure 2.1

The first example is the Hindu-Arabic numerals that are presented at the beginning of the first chapter of the *Liber Abaci*:[12]

> *The nine Indians figures are:*
> *9 8 7 6 5 4 3 2 1*
> *With these nine figures and with the sign 0 which the Arabs call zephyr, any number whatsoever is written.* (Fibonacci, 1857, 2; Moyon, 2016, 12)

And he follows explaining the writing of any number, however big:

> *A number is a collection of units, which increases indefinitely by its own orders. The first [order] is composed of the units from one to ten. The second is composed of the tens, from ten to one hundred. The third is composed of the hundreds, from one hundred to one thousand. The fourth is composed of the thousands, from one thousand to ten thousand and so one for any of the following orders until infinity, each order being tenfold from the previous one.* (Fibonacci, 1857, 2; Moyon, 2016, 12)

An important part of the *Liber Abaci* (chapter 5, 6 and 7) is dedicated to fractions –

[10] Both are translated into English (Sigler, 2002; Hughes, 2008). Nevertheless, the Latin readings are really necessary because both suffer from various failings. See, for example, (Rommevaux 2008; Høyrup 2009) for the *Practica Geometriae*.
[11] Fibonacci mentions two other texts but they are nowadays considered lost.
[12] In the following, all the original extracts from the *Liber Abaci* are framed.

with the horizontal bar – and to computations with them. This part is largely (but not exclusively) inspired by Maghrebian mathematics. Following al-Ḥaṣṣār (12th c.) and Ibn al-Yāsamīn (d.1204), Fibonacci identified fractions to solve various types of problems (Moyon & Spiesser, 2015).

The second example is related to algebra detailed in the last two chapters of his *Liber Abaci* but also widely used in his *Practica geometriae* to solve geometrical problems of measurement (Miura, 1981; Hughes, 2004; Moyon, 2012). This new discipline was born in Bagdad between 813 and 833 with *al-kitāb al-mukhtaṣar fī ḥisāb al-jabr wa-l-muqābala* [The Compendious Book on Calculation by Restoration and Comparison] written by al-Khwārizmī.Fibonacci demonstrates an excellent knowledge of the Arabic corpus, in particular, the seminal text of al-Khwārizmī(d.*ca*.850) and, among others,*al-Kitāb al-jabr wa-l-muqābala* [Book on Restoration and Comparison] of Abū Kāmil (d. *ca*.930). For numerous problems, he proposes both algorithmic and algebraic procedures.

The third example focuses on practical geometry. Inheritor of the Islamic tradition of ᶜ*ilm al-misāḥa* [science of measurement], we must consider him as a major player in the beginning of the geometry of measurement in Latin Europe. With the *Practica geometriae*, the reader is taught, among other things, to measure plane figures, to divide them into different other figures giving geometrical constraintsand to measure inaccessible distances (Moyon, 2017, 111–113, 577–616).

Finally, Fibonacci is one of the greatest medieval Western mathematicians. However, it is also necessary to consider him as a crucial link between Islamic countries and Latin Europe.

3 'Recreational problems' in the medieval history of mathematics

3.1 Generalities

The focus is on problems that I now consider part of recreational mathematics even tough they are not always defined as such by their authors. It is useful to consider Høyrup's comment on this matter when he explains that recreational problems:

> *... are pure in the sense that they do not deal with real applications, however much they speak in the idiom of everyday* [...]. *Nonetheless, their social basis is in the world of know-how, not that of know-why* [...]. *The distinction between these two orientations of knowledge is of general validity but has particular implications for mathematics.* [...]
> *Here, techniques and methods are by necessity primary, and problems are secondary, derived from the techniques which are to be trained. Anybody familiar with schoolbooks on arithmetic will recognize the situation, and scholasticized systems are indeed those where problems constructed for training purposes dominate. Apprenticeship-based systems, for their part, tend to train as much as possible on real, albeit simple tasks*. (Høyrup, 2008, 1252–1253)

Recreational problems (or mathematical riddles) are, in general, easy to state, tempting to work on and, sometimes, annoyingly difficult to solve. What we categorize today assuch problems are very ancient and omnipresent in many mathematical practicesworldwide. They travel between different cultures.[13] The same problem can even be found in Mesopotamian, Chinese, Greek, Sanskrit, Arabic, and Latin sources (Heeffer, 2014). Historians of mathematics, such as Hermelink (1978), consider the analysis of this corpus as an approach to studying the transmission of knowledge from one culture to another. Following this kind of project, Singmaster wrote in 1988:

> *I think that the temporal and geographical distribution of the sources suggests that recreational mathematics owes a much greater debt to China and India than to Greece, and that there must have been earlier and more extensive communication from India to Europe via the Arabs than is presently known.* (Singmaster, 1988, 195)

Recreational problems seem to be more and more important in the history of mathematics, especially in the context of medieval teaching from as far back as the *Propositiones acuendo ad juvenes* written by Alcuin of York in the ninth century.[14] In addition, as Sesiano (2014b) excellently showed, the medieval texts are historically important. They widely feed the later collections edited in the Renaissance. Furthermore, Sesiano mentioned about the *Liber Mahameleth* that "many problems tend to be of a recreational nature" adding that "this had become traditional at the time for mathematical treatises" (Sesiano, 2014a, xvii). As he made clear in his *Introduction to the History of Algebra* that the "area of recreational mathematics [is] a domain that was to grow considerably during the Middle Ages to the point where it became a standard component of works on algebra" (Sesiano, 2009, 25). In this context, Fibonacci is one of the most influential medieval authors thanks to his *Liber Abaci*.

3.2 Recreational Problems in the *Liber Abaci*

The *Liber Abaci* is divided into fifteen chapters.[15] The first seven chapters deal with numeration involving integers, fractions and operations on them. The following chapters, from the eighth to the eleventh, focus on problems linked to commercial rules with special emphasis on conversion of currency, allocations of profit, alloying of currencies where ratio and proportions (with the rule of three among others) are very important. The twelfth and thirteenth chapter of *Liber Abaci*, the longest part (almost half of the book) are devoted to various methods of solving recreational problems.[16] Fibonacci named them with the expression *erraticae questiones*.[17] In the last two chapters, the Pisan mathematician presents computations on radical numbers (based on Euclid's *Elements*) and, as already mentioned, algebra.

The problems of chap. 12 are really different, but they all seem to be common application problems with improbable and even absurd conditions. Moreover they all reveal the Fibonacci's passion for numbers, for algorithms and for mathematical reasoning. Here are some typical problems where, to the reader (or learner)'s amusement,

[13]The most complete reference is (Singmaster 2013) with a detailed annoted bibliography. The reader can also refer to (Tropfke, 1980, 573–660).
[14] The edition of the Latin text is made by Folkerts (1978), the English translation by Hadley and Singmaster (1992). For an extended and corrected version, see (Hadley & Singmaster, 1995).
[15]For details, read (Moyon & Spiesser, 2015, 421–422).
[16]In order to better understand Fibonacci's conception, Hannah (2011) analyses in detail three collections of problems: men giving and taking, men finding a purse, and men wishing to buy a horse.
[17]For a discussion on the meaning of this expression, read (Moyon, 2019).

animals are seen reasoning with integers, fractions and proportionality.[18]

> *A certain lion is in a certain pit, the depth of which is 50 palms, and he ascends daily 1/7 of a palm, and descends 1/9. It is sought in how many days will he leave the pit?* (Fibonacci, 1857, 177; Sigler 2002, 273; Moyon, 2016, 28)

And

> *A certain lion eats one sheep in 4 hours, and a leopard eats one sheep in 5 hours, and a bear eats one sheep in 6 hours; it is sought, if one sheep is thrown to them, how many hours it will take them together to devour it?* (Fibonacci, 1857, 182; Sigler, 2002, 279–280; Moyon, 2016, 24–26)

And finally,

> *Two ants are on the ground 100 paces apart, and they move in the same direction towards a single point; the first of them advances daily 1/3 of a pace and retreats 1/4; the other advances 1/5 and retreats 16; it is sought in how many days they will meet?* (Fibonacci, 1857, 182; Sigler 2002, 280; Moyon, 2016, 27)

It is good to finish any discussion of animal problems with the famous 'Rabbits problem' which gave birth to the well-known 'Fibonacci sequence'. It is a very suitable example of mathematical modelling.

> *How many pairs of Rabbits are created by one pair in one year?*
> *A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.* (Fibonacci, 1857, 283; Sigler 2002, 404; Moyon, 2016, 29–32)

Here, for our purpose – to introduce historical perspective in maths education –, I chose another problem from the twelfth chapter with an algorithmic structure. I call it the 'Apple Orchard Problem'.[19]

3.3 The 'Apple Orchard Problem'

> *A certain man entered a certain pleasure garden through 7 doors, and he took from there a number of apples; when he wished to leave he had to give the first doorkeeper half of all the apples and one more; to the second doorkeeper he gave half of the remaining apples and one more. He gave to the other 5 doorkeepers similarly, and there was one apple left for him. It is sought how many apples there were that he collected.* (Fibonacci, 1857, 278; Sigler 2002, 397; Moyon, 2016, 33–36)

I give the entire problem (with the solution) in the first appendix. Fibonacci, as often, proposed two solutions. The first one can be translated into modern notation as follows:

$$1 \to 1+1=2 \to 2\times 2=4$$
$$4 \to 4+1=5 \to 5\times 2=10$$
$$10 \to 10+1=11 \to 11\times 2=22$$
$$22 \to 22+1=23 \to 23\times 2=46$$

[18]These problems are all extracted from (Moyon, 2016) where I also give solutions with commentary (in French).
[19]Singmaster (2013) mentioned this problem in the chapter 'Monkey and Coconuts problems'under the heading 'Arithmetic& Number-Theoretic Recreation'.

$$46 \to 46+1=47 \to 47\times 2=94$$
$$94 \to 94+1=95 \to 95\times 2=190$$
$$190 \to 190+1=191 \to 191\times 2=382$$

Fibonacci ended with: "this total is the number of apples; and thus reversing the order that was proposed you will be able to solve any similar problem". This first method is based on the inversion of the algorithm: Fibonacci proposed to work backwards and each operation is replaced by its inverse. This 'method of inversion' is well-known and we have, among others, different Indian sources mentioning it. For example, Āryabhaṭa (fifth-sixth century) in the *Gaṇitapāda*, the mathematical part of the *Āryabhaṭīya*, wrote the following verse: "In a reversed [operation], multipliers become divisors and divisors, multipliers. And an additive [quantity] becomes a subtractive [quantity], a subtractive [quantity] an additive [quantity]" (Keller, 2006, 118). Another example comes from Brahmagupta. In his *Brāhmasphuṭasiddhānta*, writtenin the seventh century, this rule can be read: "beginning from the end, make the multiplier divisior, the divisor multiplier; [make] addition subtraction and subtraction addition; [make] square square-root, and square-root square; this gives the required quantity" (Datta & Singh, 2004, 232).

The second solution is an algebraic one. In modern notation, let *x* be the number of apples initially picked, then the linear equation to solve is:

$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}x-1\right)-1\right)-1\right)-1\right)-1\right)-1\right)-1=1$$

or, step by step (or door by door), following the original approach,

After the first door, there remain$\frac{1}{2}x-1$apples,

After the second door, there remain$\frac{1}{2}\left(\frac{1}{2}x-1\right)-1=\frac{1}{4}x-\frac{3}{2}$ apples,

After the third door, there remain$\frac{1}{2}\left(\frac{1}{4}x-\frac{3}{2}\right)-1=\frac{1}{8}x-\frac{7}{4}$ apples, and so one

$$\to \frac{1}{2}\left(\frac{1}{8}x-\frac{7}{4}\right)-1=\frac{1}{16}x-\frac{15}{16}$$
$$\to \frac{1}{2}\left(\frac{1}{16}x-\frac{15}{16}\right)-1=\frac{1}{32}x-\frac{31}{32}$$
$$\to \frac{1}{2}\left(\frac{1}{32}x-\frac{31}{32}\right)-1=\frac{1}{64}x-\frac{63}{64}$$

After the seventh door, there remain$\frac{1}{2}\left(\frac{1}{64}x-\frac{63}{64}\right)-1=\frac{1}{128}x-\frac{127}{128}$ apples, and indeed, it corresponds to only 1 apple, thus, $\frac{1}{128}x-\frac{127}{128}=1$ and so$x=382$.

4 The 'Apple Orchard Problem': Experiments in French Secondary Schools (14-16 years old)

Secondary school mathematics teachers, involved in the experiments with me, judged 'Apple Orchard Problem'and the Fibonacci's original text mathematically and pedagogically interesting for pupils aged 14 to 16 years old. Taking into account several theoretical studies on the history and pedagogy of mathematics (Fried, 2001; Fried, Guillemette & Jahnke, 2016; Jankvist, 2009), my colleague and I decided to base our activity on using original sources in classroom. I know, from my own experience,[20] that the whole enterprise of reading a source is a really difficult and

[20] For example, (Moyon, 2013, 2017).

time-consuming task but we believed that the pupils were able to work with the extract of the *Liber Abaci*. We agree with the idea that “reading a source deepens the mathematical understanding on both levels, on that of doing mathematics and on that of reflecting about mathematics” (Fried, Guillemette & Jahnke, 2016, 219). And, for us, it was a major purpose to make pupils think about the algorithmic structure of the problem, the generalization indicated by Fibonacci[21] and the modern use of coding. Thus, reading an original source (translated into French) adapted to the expected educational level, is another manner of applying new concepts, quite different from usual exercises. Thus we followed the basic guidelines of the hermeneutic procedure such as presented by Jahnke in (Fried, Guillemette & Jahnke, 2016, 216).

To help pupils read and comment the text, we presented the text and the author into their historical context (see part 2 of this contribution) and we gave them a questionnaire in two stages (each of one hour).[22] The aim of the first stage was to read the problem, understand it and work on it so as to solve it in their own words. At this stage, they did not have Fibonacci’s solutions.

1. Individual work: Identify the different steps of the statement in your rough work.
2. Collective work (pupils are separated in small groups of 4): Determine the number of apples sought by a method agreed by the group.
3. On the answer sheet, write your solution to the ‘apple orchard problem’ (one sheet per group) that can be presented to the class.

The second stage consists in reading the algorithmic solution. It was stilla collective work. The supporting questions were:

4. Read Fibonacci’s solution. Did you find the same number of apples?
5. How could you describe simply the solution given by the Pisan mathematician?
6. Translate it into today’s mathematical language.
7. According to you, what does it mean “you will be able to solve any similar problem”?

Several procedures were tested by the pupils. Most are quite explicit. I analyse below the main results (all the works have been anonymized) regarding the method used.

4.1 Mathematical Workings

4.1.1 By an algebraic method

Several pupils (about 30%) used algebra to solve the problem taking the number of apples picked as the unknown. A major error arose: the confusion between the remaining fruits $\left(\frac{1}{2}x-1\right)$ and the number of fruits given at the doorkeeper$\left(\frac{1}{2}x+1\right)$.

[21] “et sic revertendo, secundum quod propositum fuerit, in ordinem retro, **poteris quamlibet similium positionum reperire**” (Fibonacci, 1857, 278), bolded by me.

[22] My colleague and I planned a third stage to work on the algebraic solution but unfortunately we ran out of time. The questions were: 1) Read the second solution proposed by Fibonacci. How is it different from the first? 2) What is, for Fibonacci, the “thing”? What kind of mathematics is he using? 3) Translate this solution into today mathematical language.

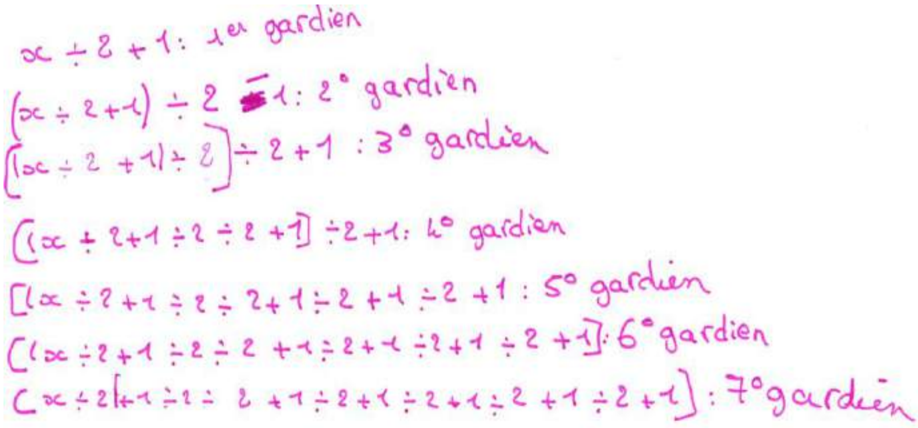


Figure 4.1: Work of Elisa, Nina and Juliette

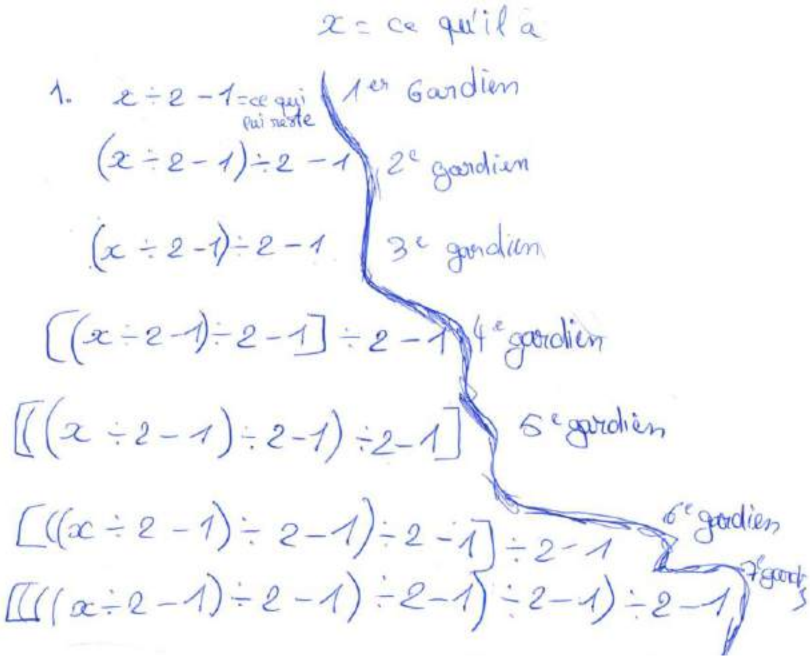


Figure 4.2: Work of Mathilde and Mélissandre

Only half students of this group came up with an equation. And, in this case (Figure 4.3), as they did not reduce the fractions as soon as possible, they gave up because they could not solve the equation.Here, a direct approach involves a complicated equation, by the standards of their mathematical ability.It follows that, algebra seems not to be the better strategy for 14-16 years old pupils.

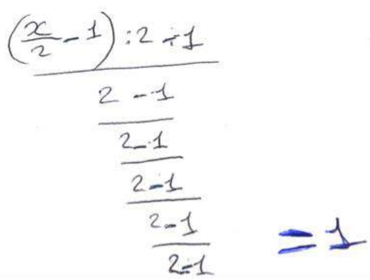


Figure 4.3: Algebraic solution written by Fayza and Germain (first attempt, see fig. 4.5).

4.1.2 By a trial and error procedure

Other pupils (about 20%) tried to solve the problem by trial and error. They took a random number and they executed the program repeating the process until they decided to stop trying. Several attempts were made. They generally stopped when they obtained a negative or a decimal number.

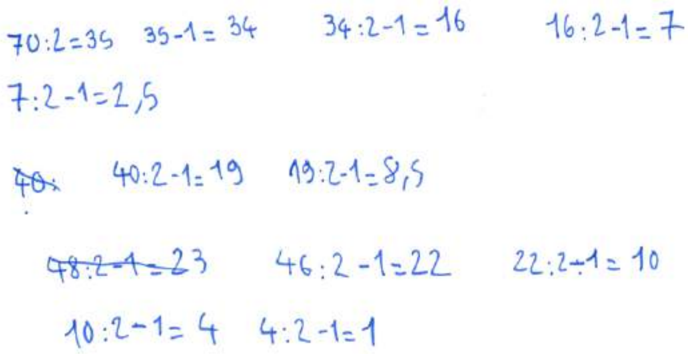


Figure 4.4: Here, in the work of Elvis and Dorian,four different numbers were tried: 70, 40, 48 and 46 (the last ‘successfully’)

Unfortunately, only few pupils (less than a third) reasoned from the results found to improve the choice of the initial number to be tested. No pupil solved the problem completely by following this procedure.

4.1.3 By the method of inversion or the working backwards strategy

The problem proceeds from complex, initially,to simple at the end. And, as it involves a sequence of reversible actions, the work backwards strategy isprobably the best procedure to perform.

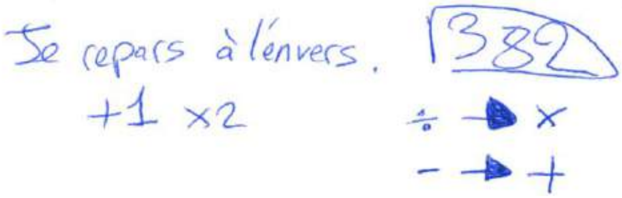


Figure 4.5: Second attempt of Fayza and Germain (see fig. 4.3): explanation of the algorithm of inversion (quite similar to Āryabhaṭa and Brahmagupta!)

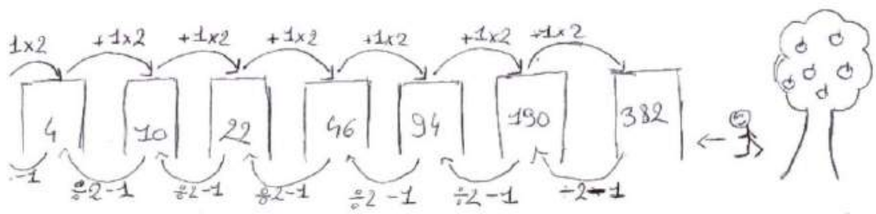


Figure 4.6: The algorithm in both directions: forward and backward (work of Julie, Lisa, Lola and Mélanie)

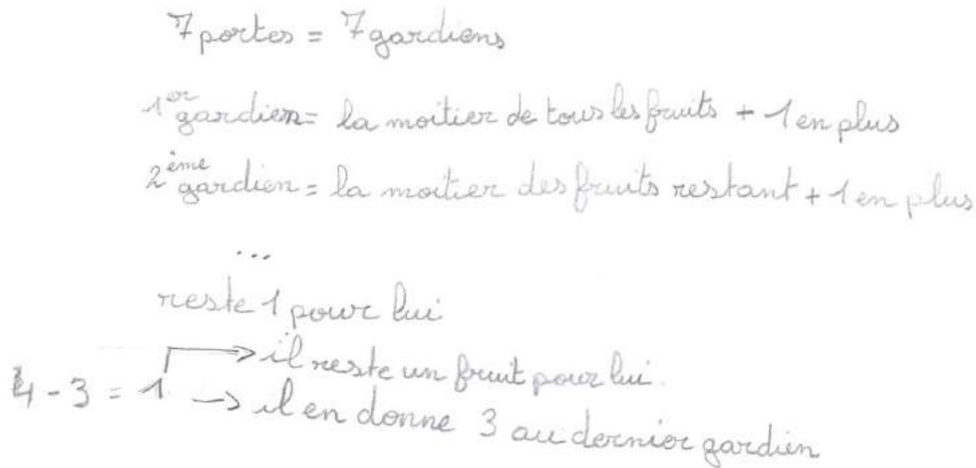


Figure 4.7: Reformulation of the problem (solution of Alice, Mina and Lilou)

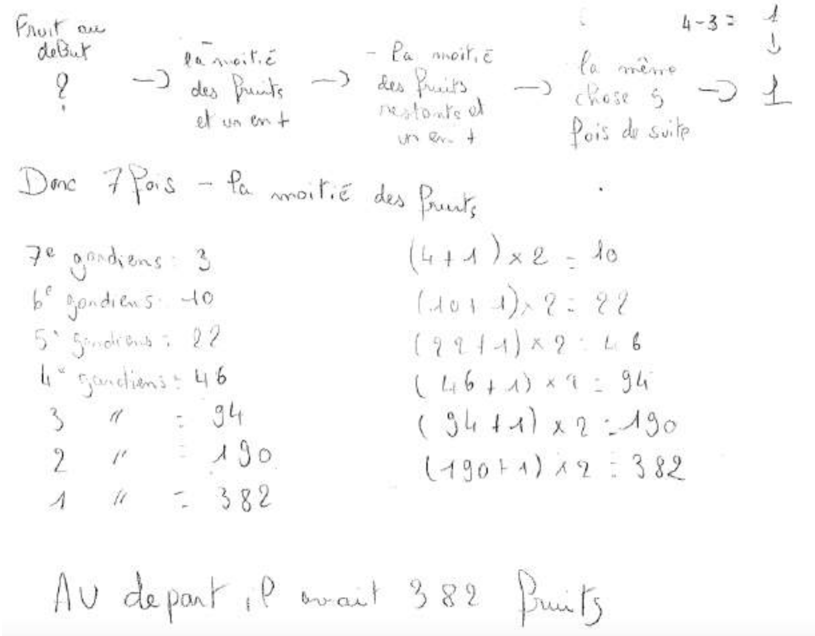


Figure 4.8: Schematization of the problem and complete resolution made by Annabelle and Glwadys.

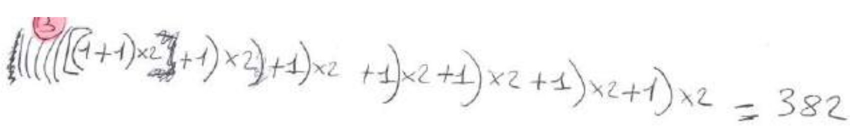


Figure 4.9: With the help of the calculator (final solution of Fayza and Germain, see fig. 4.3 and fig. 4.5)

Nevertheless, several difficulties appeared. The first one is just at the beginning to understand the exact order of the operations, i.e. $(1+1)\times 2=4$ or $1\times 2+1=3$. After that, the main problems arising are the misuse of parentheses or the distributivity of multiplication over addition. Those difficulties were expected by our *a priori* analysis. It was thus necessary to work on it.

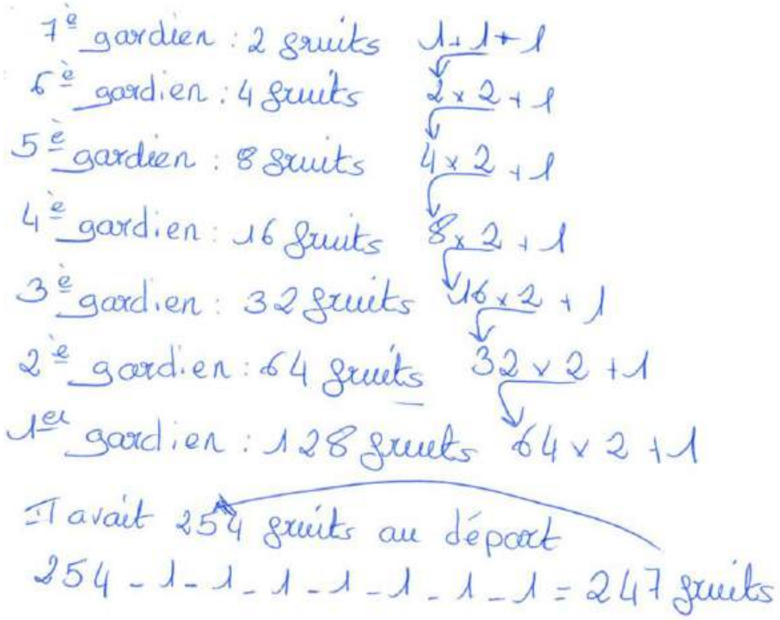
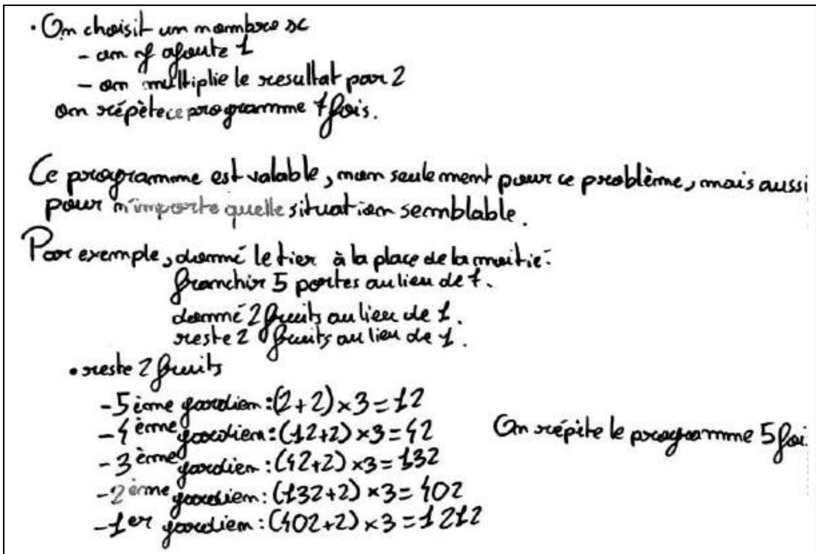


Figure 4.10: Attempt of Baptiste, Manon and Francisco.

4.2 Generalization and coding

You will be able to solve any similar problem. This is Fibonacci's conclusion following his first solution. Pupils understand what kind of similarity (and generality) Fibonacci mentioned. They discuss on the mathematical parameters of the problem: the number of doors, the number of apples given to the doorkeepers, number of apples remaining.For example:



We choose a number x:
– we add to it 1,
– we multiply the result by 2,
We repeat this program 7 times.

This program is valid not only for this problem but also for any similar situation.
For example, give the third rather than half; pass through 5 doors rather than 7; give 2 fruit rather than 1; 2 fruit remain rather than 1. [...]

Figure 4.11: Proposition from the work of Mathéo, Louis, Léo and Maxence.

When pupils all understood the algorithmic nature of the problem and its parameters by following a collective exchange in their proper studyinggroup, we asked to them: "So, how many apples were picked if there remains only one apple after 457 doors?" All (except one girl who wanted to work 'in her own head') experimented with Scratch (in a French environment) to code the solution. They done it alone or in pairs and 73% of our pupils successfully completed this work.

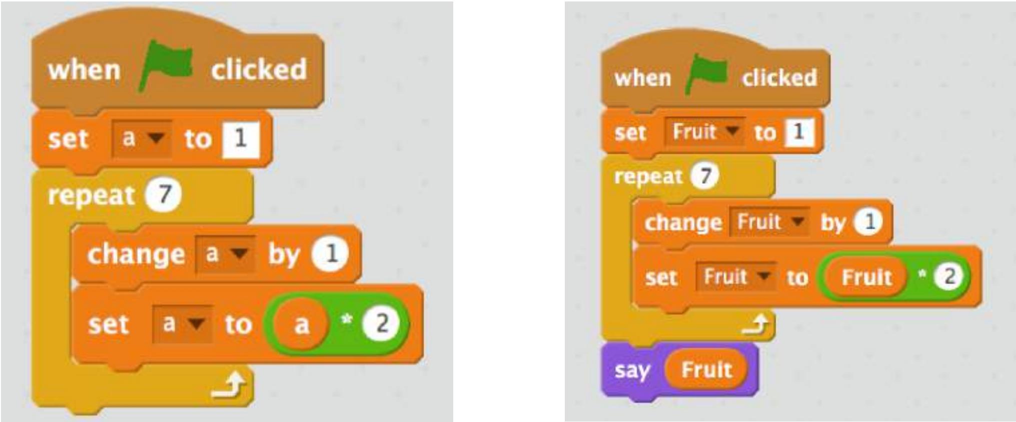


Figure 4.12: First examples – works of Leo and Maxence (see fig. 4.11) and work of Annabelle and Gwladys (see fig. 4.8) –with two successive operations (adding 1 and multiplying by 2)

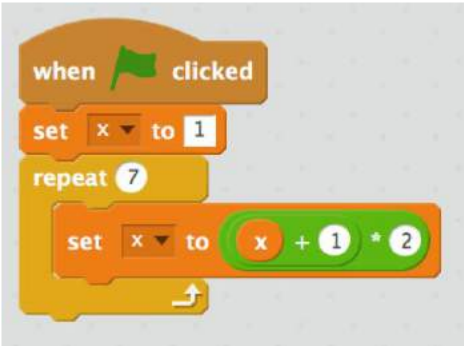


Figure 4.13: The work of Fayza and Germain (see fig. 4.3, 4.5, 4.9) where operations are reduced in one line

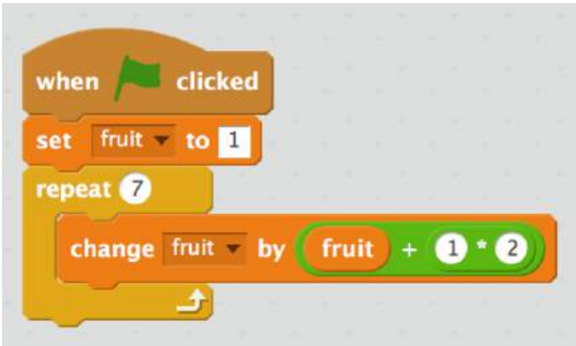


Figure 4.14:Alice's work (see fig. 4.7) : misuse of parentheseswith the Scratch conventions

In a final step, all class wrote a collective program taking into account several mathematical parameters. Some parameters wererequested from the user and others were modifiable directly in the program. Thus, pupils learn the notion of 'variable' in computer science, distinct from the mathematical notion (with the necessity, in programming, to initialize the variable in order to define it). After that, the math teacher checked this learning with several exercises in individual work.

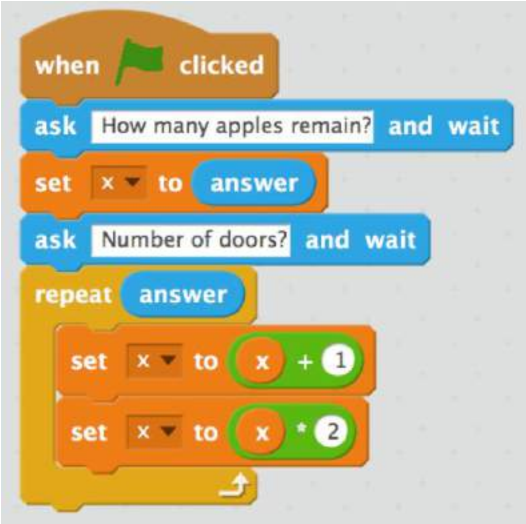


Figure 4.15: The collective coding written with the help of the math teacher

Finally, pupils were invited to think about the mathematical links between the 'apple orchard problem' and the following one.

A certain man went on business to Lucca to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money, and spent 12 denari. Then he returned to Pisa, doubled his money and it is proposed that he had nothing left. It is sought how much he had at the beginning.(Fibonacci, 1857, 258–259; Sigler 2002, 372–373; Moyon, 2016, 54–56)

They easily produced, individually, the following program taking into account the previous one. They understood the generality of the method of inversion proposed by Fibonacci and the category of problems of which these are examples.

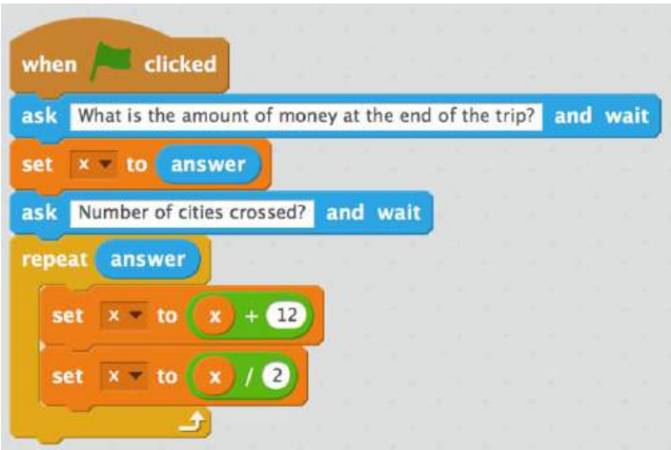


Figure 4.16: The Maxence's coding

5 Some extensions

5.1 For High-Schools curriculum

The mathematical reasoning of Fibonacci to solve the 'apple orchard problem' can be reworkedwith the notion of sequence for more advanced pupils (16 to 18 years old in France). Here, it is possible to consider the extract of the *Liber Abaci* as a starting point to write mathematics at a higher level than did Fibonacci himself.

5.1.1 First sequence working the algorithm backwards

Let be $(\boldsymbol{u_n})_{\boldsymbol{n\geq 0}}$ the sequence defined recursivelyrepresenting the number of apples remaining after the $\boldsymbol{n^{th}}$ door, $\boldsymbol{u_0}$ being the number of applesinside the first door. Thus, we have:

$$\forall n \in \mathbb{N}, \begin{cases} u_0 = ? \\ u_{n+1} = \frac{1}{2}u_n - 1 \end{cases}$$

It is an arithmetico-geometric sequence and its general term[23]is given by:

$$\forall n \in \mathbb{N}^*, u_n = \frac{1}{2^n}(u_0 + 2) - 2$$

Thus, in Fibonacci's conditions, we have:

$$u_7 = \frac{1}{128}(u_0 + 2) - 2 = 1$$
$$\frac{1}{128}(u_0 + 2) = 3$$
$$u_0 = 3 \times 128 - 2$$
$$u_0 = 382$$

5.1.2 Second sequence using algebra

If *x* is the number of applesinitially picked, a new sequence $(v_n)_{n\geq 1}$can be defined giving the number of applesremaining after the n^{th} door by:

$$\forall n \geq 1, \qquad v_n = \frac{1}{2^n}x - \left(1 + \frac{2^{n-1} - 1}{2^{n-1}}\right)$$

Or, after reduction,

$$\forall n \geq 1, \qquad v_n = \frac{1}{2^n}x - \frac{2^n - 1}{2^{n-1}}$$

An elementary mathematical induction can prove it. Thus, in Fibonacci's conditions, we have:

$$v_7 = \frac{1}{128}x - \frac{127}{64} = 1$$
$$\frac{1}{128}x - \frac{127}{64} = 1$$
$$\frac{1}{128}x = 1 + \frac{127}{64}$$
$$x = 128 \times \left(1 + \frac{127}{64}\right) = 382$$

5.2 Another solution method and Other problems from the *Liber augmenti et diminutionis*

The *Liber augmenti et diminutionis* [Book on increase and decrease] is an anonymous text from the twelfth century.[24] Hughes (2001, 107) described itas "a book that reads like a set of lectures on how to solve problemsinvolving numbers". The author's main purpose is to teach the method of double false position. One of the nine chapters of the book is about "apples stolen from an orchard". It containssix problemsdivided into three groups, whose

[23]In France, students do not know the formula for the general term. They do derive it from the recurrence relation or with help from the teacher.

[24]I am preparing a new critical edition of the text with a French translation of the whole text.

statements may besummarized as follows:

1. There are 3 doors; the man gives half and 2 fruit more; 1 fruit remainat the exit.
2. There are 3 doors; the man gives half and 4 (resp. 6, 8) fruit more to the first (resp. second, third) doorkeeper; no fruit remains at the exit.
3. There are 3 doors; the man gives half and the first (resp. second, third) doorkeeper gives back to the man 2 (resp. 4, 6) fruit; the man has 10 fruit at the exit.

In appendix 2, I give the entire textof the first problem (the mostsimilar to the ‘apple orchard problem’ of the *Liber Abaci*). I detailhere the method of double false position used in the *Liber augmenti et diminutionis*.

Let be $n_1 = 100$ a first number (first false position). After the third doorkeeper, exactly 9 apples remain. So, the differencebetween 9 and 1, the number of applesthat should remain, is (an excess of) 8. Let be $n_2 = 200$a second number (second false position). After the third doorkeeper, exactly 21.5 apples remain. So, the difference between 21.5 and 1 is (an excess of) 20.5. Thus, we have two errors[25]: $e_1 = +8$ and $e_2 = +20.5$ (both excesses), correspondingto the numbers initially chosen: $n_1 = 100$ and $n_2 = 200$. Here, the solution is given by the following relation:

$$\frac{n_1 \times e_2 - n_2 \times e_1}{e_2 - e_1} = \frac{100 \times 20.5 - 200 \times 8}{20.5 - 8} = 36$$

The method of double false position is adapted for linear problems that, in algebraic form, reduce to the solution of equations of the type: $ax + b = cx + d$, such as in this case (Chabert, 1999, 83–112). In high-school the problem of proving the correctness of the method may be given as an exercise.

6 Elements of conclusion

In this emprirical contribution, I showed one more time that the history of mathematics may support mathematics education using medieval texts (*Liber Abaci* or *Liber augmenti et diminutionis*) inthe classroom.And, I also highlighted that historycreatesa suitable context to think about mathematical concepts and procedures. This is the main purpose of my contribution.

When pupils engaged with the source, most of them had questions (on terminology or on mathematical procedures) similar to those a professional historian of mathematics would ask, especially when they compared different historical solutions with their own. It is, for me, a great opportunity to develop the critical thinkingof pupils.

Furthermore, I focused only on problems today characterized as ‘recreational’ because I consider them as important sources for problem solving (for different educational levels, even in the context of initial teacher education as I experiment it in the university of Limoges). It is really interesting toappreciatethe richness of these sources.

Finally, I easily integratedan importantaspect ofliteracy in today’s society, namelythe ability to code computer programs (here, using Scratch). In spite of the antiquity of the mathematical source, it was so easy –without hard efforts – for the pupils to engage with these problems, thanks to their algorithmic nature.

[25]That’s why this method is called *ḥisāb al-khata’ayn* [calculation of the two errors] in the Arabic tradition. Fibonacci transliterated the expression by *elchatayn* in his *Liber Abaci* (Fibonacci, 1857, 318).

Appendix 1: the 'Apple Orchard Problem' in the *Liber Abaci*

Here is the text of Fibonacci translatedin English by Sigler (2002, 397) from the edition made by Boncompagni (Fibonacci, 1857).

A certain man entered a certain pleasure garden through 7 doors, and he took from there a number of apples; when he wished to leave he had to give the first doorkeeper half of all the apples and one more; to the second doorkeeper he gave half of the remaining apples and one more. He gave to the other 5 doorkeepers similarly, and there was one apple left for him. It is sought how many apples there were that he collected; you do thus: for the one apple which remained for him you keep 1 to which you add the one apple that he gave to the last doorkeeper; there will be 2 that you double; there will be 4, and he had this many when he came to the last doorkeeper; to this you add the apple that he gave to the sixth doorkeeper; there will be 5 that you double; there will be 10, and this many remained after he left 5 doors; to this you add the one apple of the fifth doorkeeper; there will be 11 that you double; there will be 22 to which you add 1 for the apple that he gave the fourth doorkeeper; there will be 23 that you double; there will be 46 to which you add 1 for the apple that he gave to the third doorkeeper; there will be 47 that you double; there will 94; to his you add 1 for the apple that he gave the second doorkeeper; there will be 95 that you double; there will be 190 to which you add the 1 that he gave at the first door, and you double this amount; there will be 382, and this total is the number of apples; and thus reversing the order that was proposed you will be able to solve any similar problem.

In another way you put the number of collected apples to be the thing from which he gave at the first door $\frac{1}{2}$ of it and 1 apple. There remained therefore $\frac{1}{2}$ thing minus 1 from which he gave one half and one apple at the second door; therefore there remained for him one quarter thing minus $\frac{1}{2}1$[26] apples from which he gave at the third door one half and 1 apple. Therefore there remained for him $\frac{1}{8}$ thing minus $\frac{3}{4}1$ apples, half of which and one apple, he gave at the fourth door, and thus there remained for him $\frac{1}{16}$ thing minus $\frac{7}{8}1$ apples; of this half and one apple more he gave at the fifth door; there remained for him $\frac{1}{32}$ thing minus $\frac{15}{16}1$ apples of which half and one apple more he gave at the sixth door; there remained for him $\frac{1}{64}$ thing minus $\frac{31}{32}1$ apples; of this still he gave at the seventh door half and one apple more; there remained for him $\frac{1}{128}$ thing minus $\frac{63}{64}1$ apples which is equal to one apple; this is namely the one which remained after his passing the seven doors. If $\frac{63}{64}1$ apples are commonly added, then it will result that $\frac{1}{128}$ thing is equal to $\frac{63}{64}2$ apples. Therefore you multiply the $\frac{63}{64}2$ by the 128; there will be similarly 382 apples.

[26]Inspired by the Maghrebian mathematics' texts, the writing $\frac{a}{b}n$ represents the sum $n+\frac{a}{b}$.

Appendix 2: the 'Apple Orchard Problem'in the *Liber augmenti et diminutionis*

I give below an English translation of the Latin text edited by Libri (1838, 336–339) and Hughes (2001, 140).

A certain man went into an orchard and picked some apples. The orchard had three gates each guarded by a bailiff. So that man gave the first bailiff half of what he picked plus two apples more. He gave the second bailiff half [of what remained] and two more apples. He gave the third half [of what remained] and two apples more. The man was left with one [apple]. How many apples did he pick?

The procedure consists in taking a platter[27] with one hundred. You give half and two more to the first [bailiff]. You still have forty-eight [apples]. You give half and two more to the second. You still have twenty-two. You give half and two more to the third. You still have nine. So compare this with the one [apple] which was left. Thus now the error is eight by excess, that is the first error.

Then take a second platter which is two hundred. And give half and two more to the first [bailiff]. You still have ninety-eight [apples]. And give half and two more to the second. You still have forty-seven. And give half and two more to the third. You still have twenty-one and a half. So compare this with the one [apple] which was left. Thus now the error is twenty and a half by excess, that is the second error.

So multiply the first platter which is one hundred, by the error of the second platter which is twenty-one and a half. It results two thousands and fifty. Then multiply the second platter by the error of the first platter, that is to say multiply two hundred by eight and it results one thousand and six hundred. So take off the smaller of the two numbers from the larger, i.e. diminish one thousand and six hundred from two thousands and fifty. It remains four hundred and fifty. Then subtract one of both errors from the other, i.e. take off eight from twenty-one and a half. It remains twelve and a half. Then divide four hundred and fifty by it, and it results thirty-six. This is the number of apples picked.

REFERENCES

Caianiello, E. (2012). La vita e l'opera di Leonardo Pisano, Appendice 1. In G. Matino (Ed.), *Forme e modi delle linguedei testi tecnici antichi* (pp.54-60). Naples: D'Auria.

Caianiello, E. (2013). Leonardo of Pisa and the Liber Abaci. Biographical elements and the project of the work. In A. Bernard, & C. Proust (Eds.) *Scientific sources and teaching contexts throughout history: Problems and perspectives* (pp. 217-246). Dordrecht: Springer.

Chabert, J.-L. (Ed.) (1999). *A History of Algorithms. From the Pebble to the Microchip*. Paris: Belin.

Datta, B., & Singh, A.N. (2004). *History of Hindu Mathematics*. (vol. 1). Delhi: Bharatiya Kala Prakashan.

Fibonacci, L. (1854). *Tre scritti inediti di Leonardo Pisano*. B. Boncompagni (Ed.). Firenze: Tipografia Galileina di M. Cellini.

Fibonacci, L. (1857). *Il liber abaci*. B. Boncompagni (Ed.). Roma: Tipografia della scienze matematiche e fisiche.

Fibonacci, L. (1862). Practica geometriae. B. Boncompagni (Ed.). Roma: Tipografia della scienze matematiche e fisiche.

Folkerts, M. (1978). *Die älteste mathematische Aufgabensammlung in lateinischer Sprache: die Alkuin*

[27]The Latin word is *lanx*. It refers to the maghebian terminology of the 'method of the Scales' (Chabert & al., 1999, 101–103).

zugeschriebenen "Propositiones ad acuendos iuvenes". Vienna: Springer Verlag.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391–408.

Fried, M.N., Guillemette, D. & Jahnke, H.N., (2016). Theoretical and/or conceptual frameworks for integrating history in mathematics education. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (pp.211–230). Montpellier: IREM de Montpellier.

Hadley, J., & Singmaster, D. (1992). Problems to Sharpen the Young. The Mathematical Gazette, *76*(475), 102–126.

Hadley, J.& Singmaster, D. (1995). *Problems to Sharpen the Young*. London: South Bank University.

Hannah, J. (2011). Conventions for recreational problems in Fibonacci's Liber Abbaci. *Archive for History of Exact Sciences*, *65*(2), 155–180.

Heeffer, A. (2014). How algebra spoiled recreational problems: A case study in the cross-cultural dissemination of mathematics. *Historia Mathematica, 41,* 400–437.

Hermelink, H. (1978). Arabische Unterhaltungsmathematik als Spiegel Jahrtausendealter Kulturbeziehungen zwischen Ost und West. *Janus, 65*, 105–117.

Høyrup, J. (2009). Fibonacci's De Practica Geometrie, Edited by Barnabas Hughes. *Historia Mathematica*, *36*, 73–89.

Høyrup, J. (2008). Mathematics, Practical and Recreational. In H. Selin (Ed.), *Encyclopaedia of the History of Science, Technology, and Medicine in non-Western Cultures* (pp.1352–1359). Dordrecht: Springer.

Høyrup, J. (2016). Fibonacci, Leonardo. In *The Encyclopaedie of Islam* (vol.3, pp.58–61). Leiden-Boston: Brill.

Hughes, B. (2001). A Treatise on Problem Solving From Early Medieval Latin Europe. *Mediaeval Studies*, *63*, 107–141.

Hughes, B. (2004). Fibonacci, Teacher of Algebra: An Analysis of Chapter 15.3 of Liber Abbaci. *Mediaeval Studies*, *66*, 313–361.

Hughes, B. (2008). *Fibonacci's De Practica Geometrie*. New York: Springer.

Libri, G. (1838). *Histoire des sciences mathématiques en Italie, depuis la Renaissance des Lettres jusqu'à la fin du XVIIe siècle*. (vol. 1). Paris: J. Renouard.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*, 235–261.

Miura, N. (1981). The Algebra in the Liber Abaci of Leonardo Pisano. *Historia Scientiarum, 21*, 57–65.

Moyon, M. (2012). Algèbre & Practica geometriae en Occident médiéval latin: Abū Bakr, Fibonacci et Jean de Murs. In S. Rommevaux, M. Spiesser, & M.-R. Massa Esteve (Eds.), *Pluralité de l'algèbre à la Renaissance* (pp. 33–65). Paris: Honoré Champion.

Moyon, M. (2013). Diviser en multipliant les approches... Quand les mathématiques remontent aux sources. *Repères-IREM, 93*, 47–77.

Moyon, M. (2016). *Fibonacci: extraits du Liber Abaci*. Paris: ACL-Les éditions du Kangourou.

Moyon, M. (2017). *La géométrie de la mesure dans les traductions arabo-latines médiévales*. Turnhout: Brepols.

Moyon, M. (2019). Récréations mathématiques et algorithmiques dans le *Liber Abaci* de Fibonacci (XIII^{e} siècle): une mise en oeuvre pédagogique. In N. Chevalarias, M. Gandit, M. Morales, & D. Tournès (Eds.), *Mathématiques récréatives: éclairages historiques et épistémologiques* (pp. 225–251). Grenoble: UGA Éditions.

Moyon, M., & Spiesser, M. (2015). L'arithmétique des fractions dans l'œuvre de Fibonacci: fondements & usages. *Archive for History of Exact Sciences*, *69*(4), 391–427.

Moyon, M. (2017). Dividing a triangle in the Middle Ages: an example from Latin works on practical geometry. In É. Barbin (Ed.). *Let History into Mathematics Classroom* (pp.17–29). Cham: Springer.

Moyon, M., & Tournès, D. (Eds.). (2018). *Passerelles: enseigner les mathématiques par leur histoire au cycle 3*, Bouc-Bel-Air: ARPEME.

Picutti, E. (1983). Il Flos di Leonardo Pisano.Traduzioni e commenti. *Physis*, *25*. 293–387.

Rommevaux, S. (2008). Fibonacci's De Practica Geometrie, Edited by Barnabas Hughes. *Aestimatio*, *5*, 152–156.

Sesiano, J. (2009). *An introduction to the history of algebra: solving equations from Mesopotamian times to the Renaissance*. Providence: American Mathematical Society.

Sesiano, J. (2014a). *The* Liber mahameleth*. A 12th-century mathematical treatise*. Cham: Springer.

Sesiano, J. (2014b). *Récréations mathématiques au Moyen Âge*. Lausanne: Presses polytechniques et universitaires romandes.

Sigler, L. (2002). *Fibonacci's Liber Abaci*. New York: Springer.

Singmaster, D., (1988). Some early sources in recreational mathematics. In C. Hay (Ed.), *Mathematics from Manuscript to Print 1300–1600*, (pp.195–208). Oxford: Clarendon Press.

Singmaster, D., (2013). *Assorted Articles on Recreational Mathematics and the History of Mathematics*. Available at https://www.puzzlemuseum.com/singma/singma-index.htm Accessed: 22 August 2018.

Tropfke, J., Vogel, K., Reich, K., & Gericke, H. (1980). *Geschichte der Elementarmathematik* (4. Aufl.). Berlin: de Gruyter.

ENGAGING WITH PRIMARY SOURCES IN A MATHEMATICS FOR THE LIBERAL ARTS COURSE

Marcela CHIORESCU
Georgia College, 231 W Hancock St, Milledgeville, GA, USA
Marcela.chiorescu@gcsu.edu

ABSTRACT

There is evidence that integrating original sources in the mathematics classroom has significant pedagogical value, however, more empirical studies of implementing history of mathematics in teaching is needed. This paper describes a case study about replacing the textbook with primary historical sources projects for two topics: the Babylonian numeration system and the triangular numbers in a math for liberal arts course. Using history-as-a-tool in both projects, the author investigates whether the students can engage with primary sources to uncover math concepts and to develop their own understanding of these concepts, and obstacles and benefits of learning from primary sources.
Keywords: primary historical sources, student project, Babylonian numeration, triangular numbers.

1 Introduction

During the past four decades integrating the history of mathematics in the teaching and learning of mathematics attracted an increased amount of interest, see (Barbin, Jankvist and Kjeldsen, 2015). “Using history of mathematics in the classroom does not necessarily make students obtain higher scores in the subject overnight, but it can make learning mathematics a meaningful and lively experience, so that (hopefully) learning will come easier and will go deeper. The awareness of this evolutionary aspect of mathematics can make a teacher more patient, less dogmatic, more humane, less pedantic. It will urge a teacher to become more reflective, more eager to learn and to teach with an intellectual commitment.” (Siu, 1997/2000)

Tzanakis and Arcavi (2000) distinguished five main areas which can benefit from using the history of mathematics in mathematics teaching:

a) The learning of mathematics;
b) The development of views on the nature of mathematics and mathematical activity;
c) The didactical background of teachers and their pedagogical repertoire;
d) The affective predisposition towards mathematics; and
e) The appreciation of mathematics as a cultural-human endeavor.

Fauvel and van Maanen (2000) gave ideas and examples for thirteen possible ways to implement history of mathematics in the classroom:

1. Historical snippets;
2. Research projects on history texts;
3. Primary sources;
4. Worksheets;
5. Historical packages;

6. Taking advantage of errors, alternative conceptions, change of perspective, revision of implicit assumptions, intuitive arguments;
7. Historical problems;
8. Mechanical instruments;
9. Experiential mathematical activities;
10. Plays;
11. Films and other visual means;
12. Outdoors experience;
13. The world wide web.

The focus of this paper is on number three, namely on the use of primary sources. The use of primary sources in the classroom is “the most ambitious of ways in which history might be integrated into the teaching of mathematics” (Fauvel and van Maanen, 2000) but there is evidence that integrating primary historical sources in teaching and learning of mathematics has significant pedagogical value. (Furinghetti, Jahnke, & van Maanen, 2006; Pengelley, 2011; Jankvist, 2014, Barnett, Lodder and Pengelley, 2014). However, more empirical studies on incorporating primary sources in teaching and learning mathematics are needed as emphasized by Clark and Thoo (2014): “It is our hope that consumers of the contributions here (as well as from other publication outlets) will design empirical investigation of their own, analyze the results of those efforts, and critically reflect upon the experience in form of a conference paper, book chapter, or journal article.” or by Clark, Otero and Scoville (2017): “There was scant focus on the use of primary sources as a classroom tool in the early work in the field of history in mathematics education. However, more recent work on the use of primary sources has been done in countries such as Denmark and Brazil, while such research has not yet been conducted with student populations in the United States”

The purpose of this paper is to share about the integration of two mini projects with primary sources in a mathematics for liberal arts course, Quantitative Skills & Reason, at Georgia College during spring 2017. After presenting the context of the study I will discuss the results and conclusion.

2 Context of the study

Georgia College (GC), a college located in Milledgeville, GA, about a two-hour drive from Atlanta, is Georgia's designated public liberal arts university. It has approximately 5,900 undergraduate students and 300 graduate students in 37 undergraduate programs and 25 graduate programs. The Department of Mathematics offers a Bachelor of Science in mathematics with an optional teaching concentration. It has about 80 major students instructed by 18 full time faculty. Mathematics, a cornerstone of a liberal arts education, is required at GC for every student. As part of their core courses, Georgia College's students must complete three hours in the Area A2 Quantitative Skills. The courses that satisfy this requirement are: Quantitative Skills & Reason, Introduction to Mathematical Modeling, College Algebra, College Trigonometry, Precalculus, and Calculus I.

MATH 1001 Quantitative Skills & Reason is a mathematics for liberal arts course and is not intended to supply sufficient algebraic background for students who intend to take Precalculus or the calculus sequences for mathematics and science majors. This course places quantitative skills and reasoning in the context of experiences that students will be likely to encounter. It covers a variety of topics such as logic, counting methods and basic probability, data analysis, and modeling from data. For my two sections of MATH 1001 in spring 2017, I decided to explore a couple of these topics with primary sources, namely the Babylonian numeration system and triangular numbers.

"Many topics in the modern undergraduate mathematics curriculum are presented as a fast-paced newsreel of fact and formulas with little discussion of the motivating problems or intellectual struggle behind the textbook definitions, lemmata or algorithms. We deprive our students of knowledge about the origin of the subject and reveal little about the initial motivation or its study, leaving our students to believe that the subject emerged as a logically precise edifice with the present version of the textbook". (Jerry Lodder, 2014) With primary sources we can create authentic math inquiry activities, where students work like real scientists. Original sources expose students to "mathematics-in-the-making" as opposed to "mathematics-as-an-end-product" they found in their textbooks. (Siu and Siu, 1979). "Through observation, analysis, interpretation, synthesis and evaluation, students discover clues and integrate new information into their knowledge base" (Carlston, 2009).

Jankvist (2009) describes a way of organizing and structuring the discussion of why and how to use history in mathematics education. He is proposing two approaches for the 'whys' and different approaches for the 'hows'. He distinguishes two big categories for the 'whys': history-as-a tool and history-as-a-goal. In both my projects with primary sources I used history-as-a-tool, especially playing the role of a cognitive tool in supporting the learning of mathematics.

Jankvist (2009) also presents three major categories in which history can be used in the teaching and learning of mathematics: illumination approaches, the modules approaches and the history-based approaches. In my case study, I used the modules approaches which are similar to the "guided reinvention" activities of Freudenthal (2002) who strongly believes that "learners should be allowed to find their own levels and explore the paths leading there with as much and as little guidance as each particular case requires. There are sound pedagogical arguments in favor of this policy. First, knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others. Second, discovery can be enjoyable and so learning by reinvention may be motivating. Third, it fosters the attitude of experiencing mathematics as a human activity." I will refer to my projects with primary sources as *guided exploration modules* (GEMs), since my idea of using primary sources in the classroom resonates with Freudenthal's didactical principles in (2002).

2.1 Guided exploration modules

My GEMs are inspired from two primary sources projects (PSPs) of the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) project: Babylonian Numeration was developed by Dominic Klyve (2017) and is about the Babylonian numeration system. The second is about triangular numbers and is a part of the Construction of the Figurate Numbers project designed by Jerry Lodder (2017).I used these two GEMs in two of my sections of MATH 1001 in spring 2017. With one of them I met for 50 minutes three times per week (Monday, Wednesday and Friday) and with the other one for 75 minutes twice per week (Tuesday and Thursday).

2.2 Babylonian numeration

The GEM on the Babylonian numeration system was based only on a picture of a tablet discovered in

Figure 2.1: Cuneiform Texts, ed. Hilprecht, Vol. XX, Part 1. (1906). It's on Plate 16, No. 27.

the Sumerian city of Nippur (in modern-day Iraq) and dates to around 1500 BCE (figure 2.1). The purpose of this module was to introduce the Babylonian numeration to students. Students' task was to discover by themselves this numeration system. For my two sections I designed this GEM to be integrated in only one class meeting. I implemented it after I discussed with my students other numeration systems such as, the Hindu-Arabic numeration system and the Mayan numeration system but mentioned nothing about the Babylonian numeration system. For both classes, students were divided in groups of size 2-3. The groups were formed from the first day of class. At the beginning of the class students were asked to close their notebooks, textbook and turn off their electronic devices. After a very short introduction of the project I asked my students to study the image of the tablet and to answer the following questions written by Klyve (2017):

(1) How do Babylonian numerals work?
(2) Describe the mathematics on this tablet.
(3) Write the number 72 in Babylonian numerals.

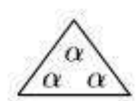
2.3 Triangular numbers

The main goal of the second GEM was for students to explore one category of figurate numbers, namely triangular numbers. This module was integrated after we discussed some elements of logic and some counting techniques as an example of inductive reasoning. This GEM is based on the Lodder's project (2017) and has as starting point excerpts from Book Two of Nicomachus' ancient Greek text *Introduction to Arithmetic* (1926), namely Chapter VIII. This module was also implemented in only one class meeting for both sections and,similar to the first one, students were not allowed to consult any other materials. After a short introduction of the project, students were asked to read the English translation of Nicomachus' text and to answer the following questions. Some were identical with the ones developed by Lodder (2017) and some were a slightly modified version of these:

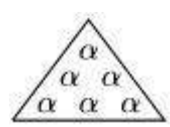
Exercise 1: In this exercise we introduce the **triangular numbers**, which count the number of dots in regularly shaped, equilateral triangles. Although Nicomachus is somewhat reluctant to state that the value of the first triangular number is 1, since the triangle drawn around one alpha, is only potentially the first triangle, 

we will begin the triangular numbers with the value of one. Let T_1 denote the first triangular number. Then $T_1 = 1$.

a) Let T_2 denote the second triangular number. Compute the numerical value of T_2 by counting the number of alphas (α) in the triangle:



b) Let T_3 denote the third triangular number. Compute the numerical value of T_3 by counting the number of alphas in the triangle:



c) Let T_4 denote the fourth triangular number, T_5 the fifth triangular number,…, and let T_n denote the nth triangular number for a natural number n. Fill in the following table for the first eight triangular numbers:

n	1	2	3	4	5	6	7	8
T_n								

d) Carefully explain how T8 is computed. Does Nicomachus include a picture of this eighth triangle in his writing?

Exercise 2: Nicomachus writes that "The triangular number is produced from the natural series of number…by the continued addition of successive terms, one by one, from the beginning…"

a) Write T_2, the second triangular number, as the sum of two successive whole numbers by using the number of alphas in each row of the second triangle.

Thus, $T_2 = \boxed{} + \boxed{.}$

b) Write T_3, the third triangular number, as the sum of three successive whole numbers by using the number of alphas in each row of the triangle.

Thus, $T_3 = \boxed{} + \boxed{} + \boxed{}$

c) Following the above format, write T_4 as the sum of four successive whole numbers so that

$T_4 = \boxed{} + \boxed{} + \boxed{} + \boxed{}$.

d) What ten successive whole numbers would be needed to be added together to produce T_{10}, the 10th triangular number?

Exercise 3 Arrange two copies of T_n to produce a rectangle with n rows and (n+1) columns. Use this to answer the following:

a) Find a simple formula for T_n in terms of n. What geometric idea does this formula represent?

b) Find a simple formula for 1+2+3+…+n using part (a).

3 Results

As a teacher my main goal has always been to create a rigorous, supportive learning environment, which fosters and encourages the ability to learn, to reason, and to communicate with proficiency. The teaching strategies that I implement to reach my goal focus on methods that allow actual participation in mathematical activities, such as: learning as discovery using group work, discussions among peers and between students and teacher, and presentations of problems at the blackboard either from their homework or from in-class material.

Using history-as-a-tool in my projects, especially playing the role of a cognitive tool in supporting the learning of mathematics, my research study was framed around the following questions:

- Are the students able to engage with primary sources to uncover math concepts and to develop their own understanding of these concepts?
- What do students identify as obstacles and benefits of learning from primary sources?

To answer these questions, I analyzed my students' work from their projects and their answers to a post-project reflection survey.

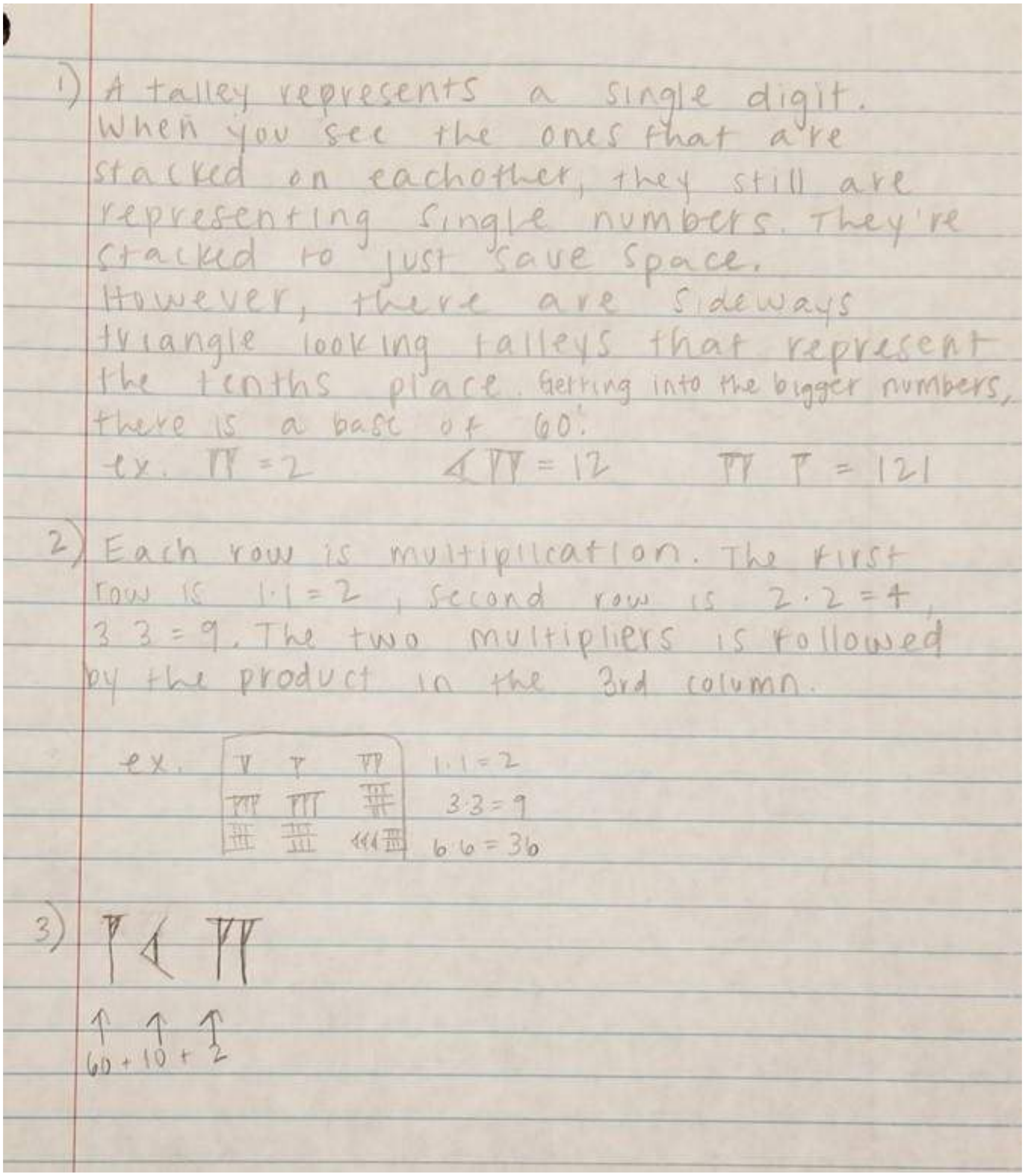
From both sections, I had in total 40 students who engaged in the two GEMs. They tackled these in groups of 2-3 with little or no help from me. My students were graded on these projects on participation only, since my goal for them was to leave them to explore math without the stress of a grade, which implicitly leads to the stress of making mistakes.

Table 3.1

Are the students able to engage with primary sources to uncover math concepts and to develop their own understanding of these concepts?

Babylonian numeration	Question 1:Describe the mathematics of this tablet	Question 2: How do Babylonian numerals work?	Question 3:Write the number 72 in Babylonian numerals
Percentage of students who answered correctly	83%	70%	70%

Most of the students (83%) discovered immediately that the first two columns of the tablet represent the numbers from 1-13 and the third one looks like the product of the first two. Also, 70 % of the students gave (a fairly) complete response with reasonable explanations to the second and third question, showing understanding of the concepts. Figures 3.1 and 3.2 are two samples of student work.



1) A talley represents a single digit. When you see the ones that are stacked on eachother, they still are representing single numbers. They're stacked to just save space. However, there are sideways triangle looking talleys that represent the tenths place. Getting into the bigger numbers, there is a base of 60.

ex. 𒐖 = 2 𒌋𒐖 = 12 𒐖 𒁹 = 121

2) Each row is multiplication. The first row is 1·1 = 2, second row is 2·2 = 4, 3 3 = 9. The two multipliers is followed by the product in the 3rd column.

ex.

1·1 = 2

3·3 = 9

6 6 = 36

3)

60 + 10 + 2

Figure 3.1

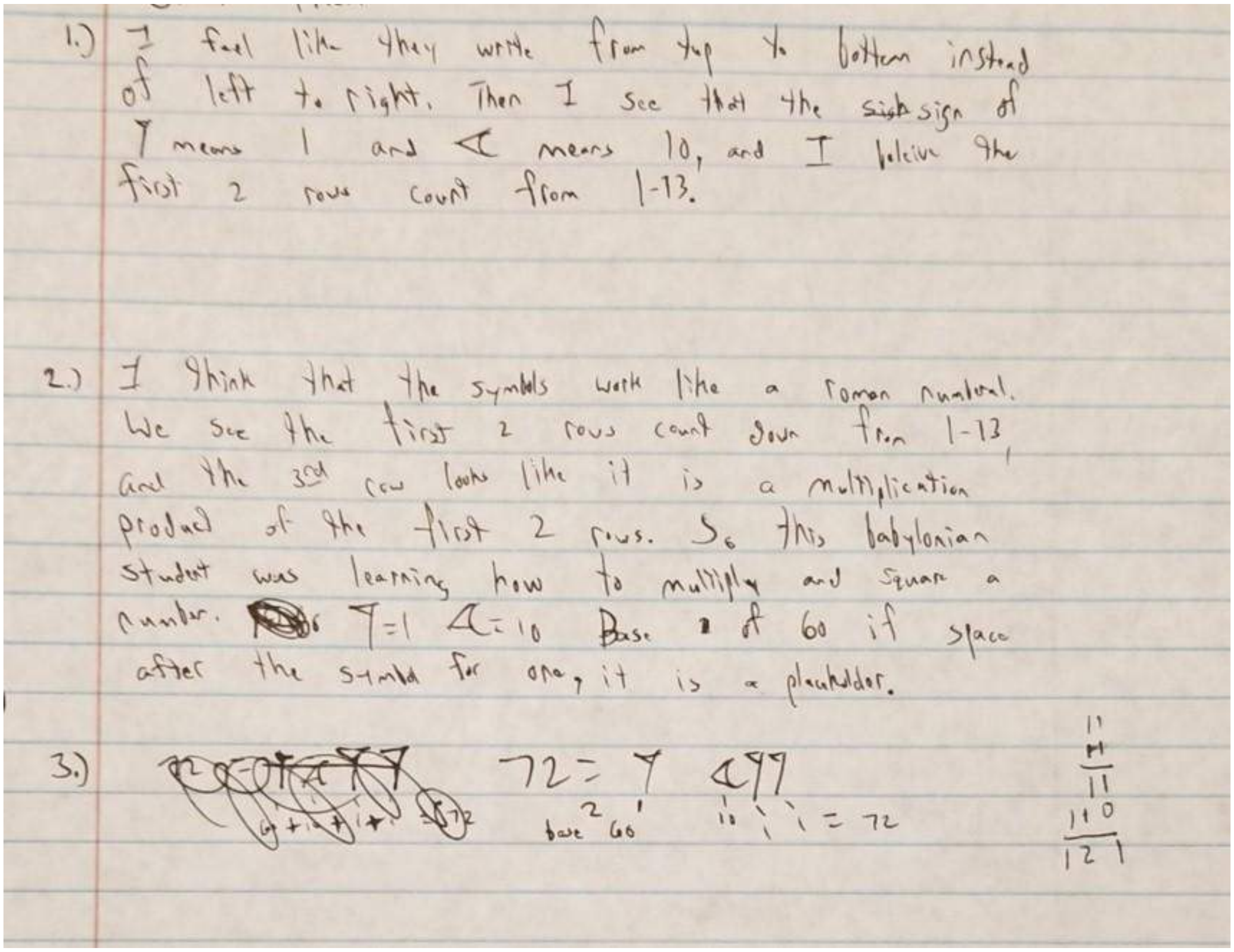


Figure 3.2

After students finished their first GEM they need to complete at home a reflective survey about their experience with this project: “Using original sources in the teaching of mathematics makes it possible to contextualize the mathematics in ways that many textbooks cannot afford. Describe your experience using the mini project Babylonian numeration as a way to explore the benefits and obstacles you identified”. For the first GEM, students identified as challenges:

- “difficult to read at first”
- “none”
- “difficult at first”

and as benefits:

- “made me use my critical thinking skills and apply my math knowledge to historical artifacts”
- “made me think outside the box, learning something new”
- “much more interesting than simply reading the textbook; allows to apply what we have learned previously about numerals and other ancient math system; I enjoyed it; using alternate methods like this help me learn, understand and retain the knowledge better”
- “forced you to think beyond what is in your notes and try to seek out your answer”.

For the second GEM, the Triangular numbers, when examining Table 2, we notice that most students answered correctly the first two exercises and they had some problems with the third one.

Table 3.2

Are the students able to engage with primary sources to uncover math concepts and to develop their own understanding of these concepts?

Triangular numbers	Exercise 1:				Exercise 2	Exercise 3		
	a)	b)	c)	d)		Drew apicture	a)	b)
Percentage of students who answered correctly	81%	81%	93%	81%	70%	69%	50%	50%

Figures 3.3 and 3.4 are samples of student work for exercise 3.

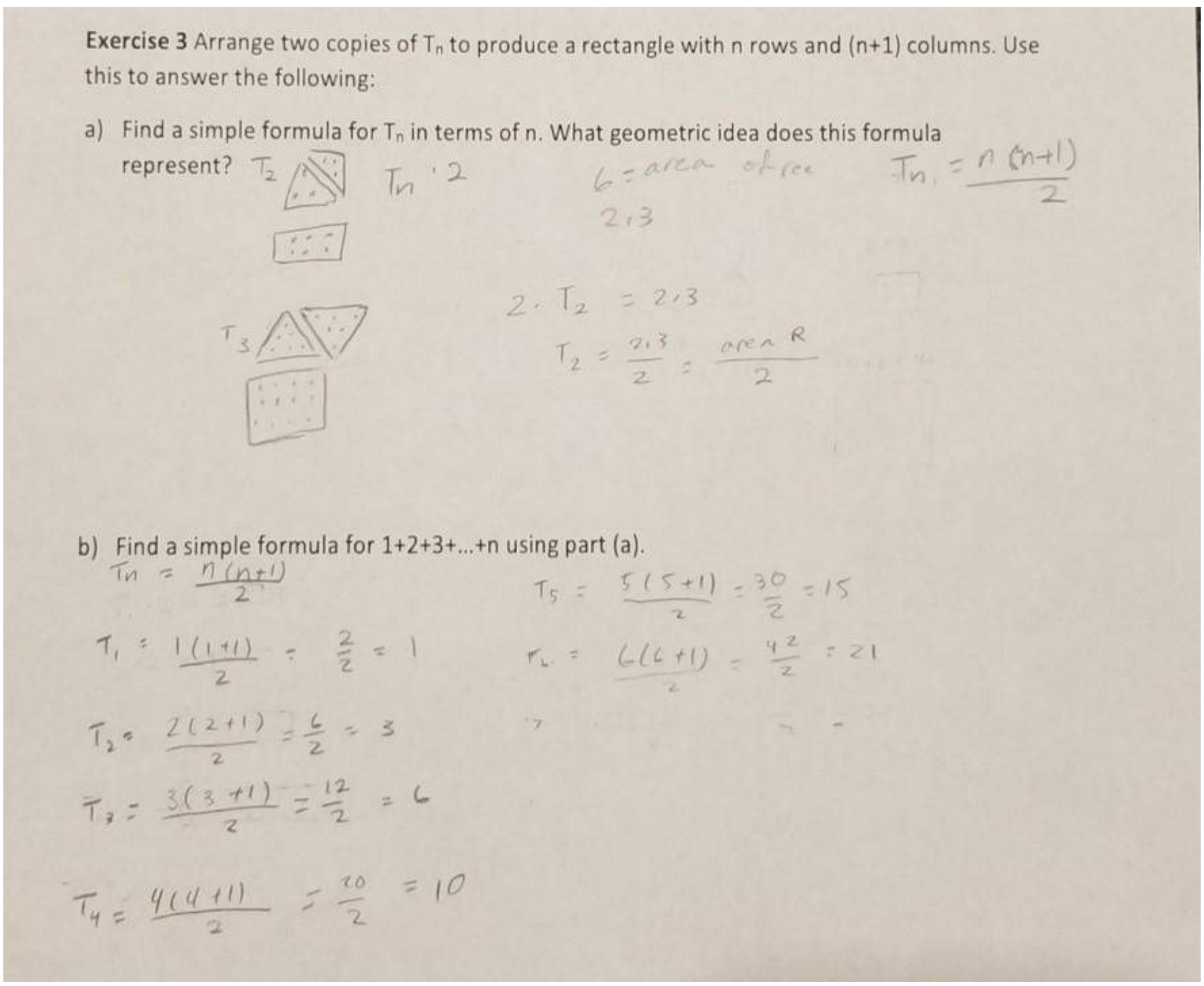


Figure 3.3

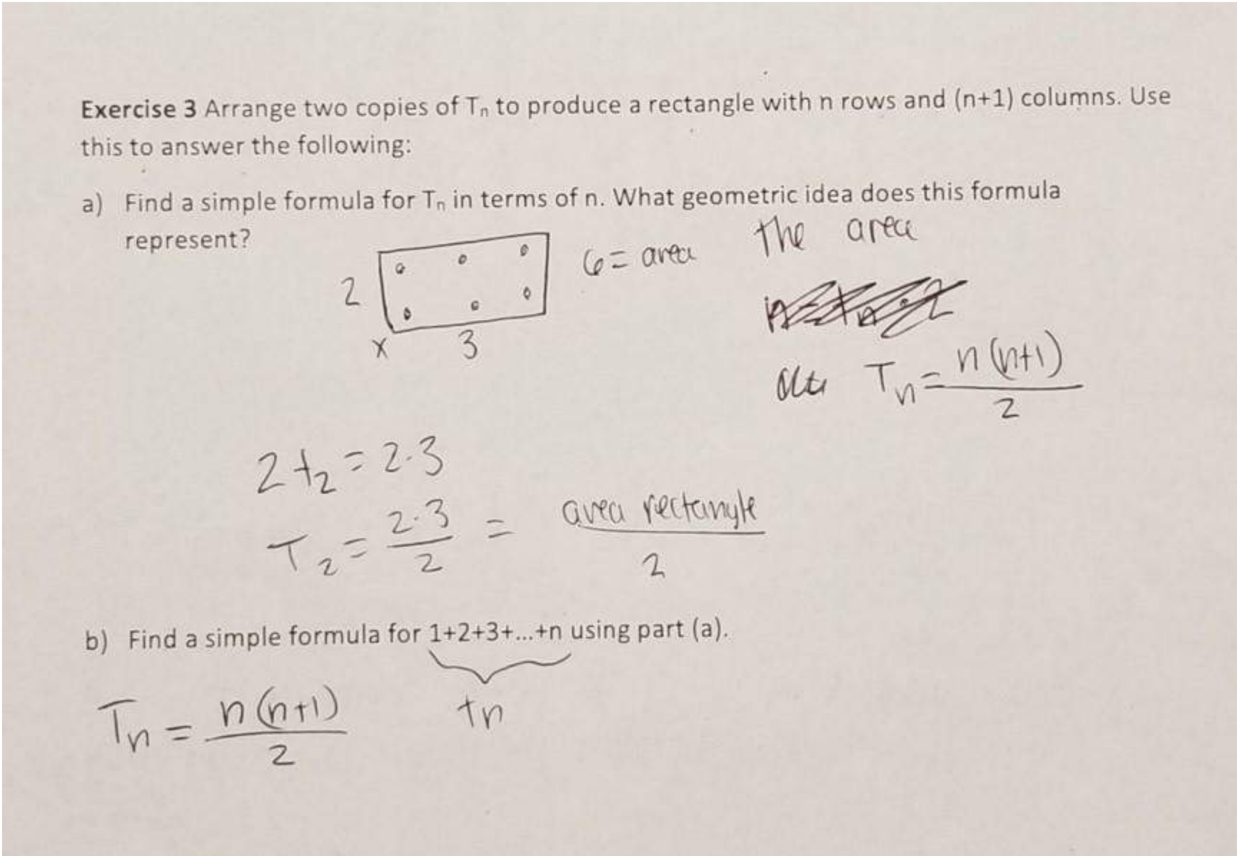


Figure 3.4

On their reflective survey, students identified for the second GEM the following challenges:

- "it does not provide as much explanation as a textbook might"
- "first I didn't understand but then it was fun"
- "challenging to create the formula"
- "last question was a bit difficult"
- "hard to understand what he said"

and the following benefits:

- "it was interesting to act as a mathematician and discuss the pattern"
- "work together to figure it out instead of just being told; better understand and retain the info"
- "allows you to see differences and patterns; beneficial in every day situation as bowling; bowling pins are set up in a triangular numeral of 5"
- "makes think more critically"
- "remember better; figure it out by myself".

4 Discussion and conclusion

Each of the two GEM's was designed to engage students in the discovery of mathematics. The goal of the first GEM was to allow students to discover by themselves the Babylonian numeration system. I was very thrilled to observe the excitement of most of the students when they were deciphering the tablet. Most of the students recognized with no help from me that the first two columns represent the numbers 1-13 and that the third column represents the multiplication of the first two columns. They also found immediately the symbols used for 1 and 10. Some students had a bit of trouble identifying the base for this numeration system, so I gave them a hint to look at the product of 11 and 11. For most of them this was enough. For the second GEM, my goal was to have students discover some interesting relations between numbers, especially triangular numbers. With this project students had most difficulty with the third question. One of the possible reasons students had difficulties with the last question, is that in their previous math courses, they were rarely engaged in creative mathematical activities,it was more about doing mathematics as a rigid process by following fixed and predeterminated procedures.

Overall, I was happy with the results of this study.I found that using primary historical sources in the classroom has the potential to enrich students learning experiences,prompting them to develop their own understanding of concepts, bringing them close to the experience of mathematics creation. I was also pleasantly surprised to read in their post-GEMs reflections such as:

> “The experience of using this activity to further my knowledge of how Babylonian mathematics and numerals work was much more interesting and captivating than simply reading the textbook. The mini project allowed us to apply what we have previously learned about numerals and other ancient mathematics systems to understand the Babylonians. The obstacle of having no prior knowledge about this particular civilization contributed because we had to use basic observations and math skills to understand their numerals. I enjoyed this activity and I feel that using alternate methods like this help me learn, understand, and retain the knowledge better than a lecture or reading form textbook would.”

> “We have used original sources in multiple mini projects now. It helps us as students understand where the math originated from. It blows my mind how we have evidence from thousands of years ago to show how math was formed like the triangle and rectangle method that we learned Tuesday. I enjoyed it even though it took a lot of thinking and coming up with ideas.”

REFERENCES

Barbin, É., Jankvist, U., & Kjeldsen, T. H. (Eds.) (2015). *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU*. Copenhagen: Danish School of Education, Aarhus University.

Barnett, J. H., Lodder, J. & Pengelley, D. (2014) The Pedagogy of Primary Historical Sources in Mathematics: Classroom Practice Meets Theoretical Frameworks. *Science & Education, 23*(1), 7-27.

Carlston, B. PatriciaandClenveson, B. Rhonda (2009). Engaging all learners with primary sources. *Teaching with Primary Sources Quarterly-Winter 2009*, Vol. 2, No 1.

Clark, K., Otero, D. & Scoville, N. (2017). Primary sources projects in an undergraduate mathematics classroom: A pilot case in a topology course. 10th Congress of European Research in Mathematics Education, Institute, Available here: https://keynote.conference-services.net/resources/444/5118/pdf/CERME10_0529.pdf

Clark, K., &Thoo, J. B. (2014) Introduction to the Special Issue on the use of History of Mathematics to Enhance

Undergraduate Mathematics Instruction, *PRIMUS*, *24*(8), 663-668.

Fauvel, J., & van Maanen, J. (Eds.) (2000). *History in mathematics education: the ICMI study* Dordrecht: Kluwer Academic Publisher.

Freudenthal, H. (2002). *Revisiting Mathematics Education: China Lectures*. Dordrecht: Kluwer.

Furinghetti, F., Jahnke, H. N., & van Maanen, J. (2006). Report No22/2006 on Mini-workshop on studying original sources in mathematics education. *Oberwolfach Reports 3*(2), 1285-1318.

Jankvist, U. T. (2009), A categorization of the 'Whys' and 'Hows' of using history in mathematics education. *Educational Studies in Mathematics, 3,* 235-261.

Jankvist, U. T. (2014). On the use of primary sources in the teaching and learning of mathematics. In: M.R. Matthews (Ed.), *International handbook of research in history, philosophy and science teaching* (Vol. 2, Chapter 27, pp. 873-908). Dordrecht: Springer.

Klyve, D. (2017), Babylonian Numeration. *Number Theory 2*. https://digitalcommons.ursinus.edu/triumphs_number/2

Lodder, J. (2014). Networks and Spanning Trees: The Juxtaposition of Prufer and Boruvka. *PRIMUS 24*(8), 737-752

Lodder, J. (2017). Construction of the Figurate Numbers. *Number Theory*, *4*. https://digitalcommons.ursinus.edu/triumphs_number/4

Nicomachus (1926). *Introduction to Arithmetic*, D'Ooge, M. L. (transl.) (pp. 241-242). New York: The Macmillan Company, 1926.

Pengelley, D. (2011). Teaching with primary historical sources: should it go mainstream? Can it? In: V. Katz and C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 1-8). MAA Notes 78. Washington D.C.: The Mathematical Association of America.

Siu, M. K. (1997/2000). The ABCD of using history of mathematics in the (undergraduate) classroom, *Bulletin of the Hong Kong Mathematical Society*, *1*, 143-154; reprinted in V. Katz (ed.) *Using History To Teach Mathematics: An International Perspective* (pp. 3-9). Washington D.C.: Mathematical Association of America.

Siu, F.-K., & M.-K. Siu: 1979, "History of mathematics and its relation to mathematical eduction'. *International Journal of Mathematical Education in Science and Technology* 10(4), 561-567.

Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) project available here: http://webpages.ursinus.edu/nscoville/TRIUMPHS.html

Tzanakis, C., Arcavi, A., de Sá, C. C., Isoda, M., Lit, C.-K., Niss, M., de Carvalho, J. P., Rodriguez, M., & Siu, M. K. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J.Fauvel&J. van Maanen (Eds.), *History in mathematics education: The ICMI Study* (pp. 201-240, Chapter 7). Dordrecht: Kluwer Academic Publishers.

REVERSED PROCEDURE AND KUTTAKA METHOD

The calculation of Indian Mathematics (*ganita*) in *Aryabhatiya* and *Brahma-sphuta-siddhanta*

Iolanda GUEVARA[1], Carles PUIG-PLA[2]

[1]Dept. d'Ensenyament de la Generalitat de Catalunya, Via Augusta 202, 08021, Barcelona, Facultat de ciències de l'Educació, UniversitatAutònoma de Barcelona, Barcelona, Spain

2Dept. de Matemàtiques, ETSEIB (Universitat Politècnica de Barcelona), Diagonal 647, 08028, Barcelona, Spain.

iguevara@xtec.cat, carles.puig@upc.edu

ABSTRACT

Between the 4th and 5th centuries, there was remarkable mathematical activity in India related to astronomy which produced the *siddhantas*, texts that included whole chapters devoted exclusively to mathematical calculus (*ganita*). Aryabhata (476-550) and Brahamagupta (598-668) wrote their siddhantic texts, *Aryabhatiya* and *Brahma-sphuta-siddhanta*, which were a reference to later astronomers and mathematicians. In this presentation, we will focus on some of the issues related to Aryabhata and Brahamagupta calculation. High school students can study them to establish bridges between the current methods of resolution and the old Indian methods (Reversed Procedure and *Kuttaka* Method). We will present two recent didactic experiences in the classroom: the first related with the Reversed Procedure of Aryabhata, presented during a pre-service teacher course at the University of Barcelona; the second with the *Kuttaka* method, made with students from an elective course of the History of Mathematics at the Polytechnic University of Catalonia.

Keywords: Indian Mathematics, Aryabhata, Brahmagupta, *Ganita*, Reversed procedure, *Kuttaka* method.

1 Introduction

Different members of the Catalan Historians of Science group had already presented several didactic works using original historical sources1. We had begun to try introducing some aspects of ancient India mathematics. When we talk about mathematics in ancient India, we are referring to the Indian subcontinent, i.e. India, Nepal, Pakistan, Bangladesh and Sri Lanka. In the Indian subcontinent, one of the oldest civilizations in the history of humanity, the civilization of the Indus was developed. The initial basic rudiments on geometry and metrology evolved over time to achieve remarkable astronomical and mathematical knowledge. There was a long development through different stages in which the transmission and learning of knowledge was made using singular mechanisms involving aspects related to commerce and technology (civilization of the Indus 2500-1800 BC), ritualistic religious and linguistic structures (Vedic age 1500-500 BC) or non-theist cosmological ideas (Jainism 500-200 BC) (Puig-Pla& Guevara, 2017).It was during the so-called classical era (400-1200) when such prominent personalities as Aryabhata (5th and 6th centuries), Brahmagupta (7th century) or Bhaskara II (12th century) appeared.
In this presentation, we will focus on some of the issues related to Aryabhata and

[1]These works are related with Greeks, medieval Muslim or ancient Chinese mathematics. See Romero *et al.*, 2015 (last ESU-7 in Copenhagen) or Massa *et al.*, 2011 (ESU-6 in Vienna).

Brahamagupta calculation (implemented in the 2017-2018academic year) in the classroom: the Reversed Procedure and the *Kuttaka* Method.

The Reversed Procedure from Aryabhata was presented during a pre-service teacher courseduring an Interuniversity Master in Teacher Training on Didactics of Mathematics - Universitat de Barcelona, Universitat Autònoma de Barcelona, Universitat Politècnica de Catalunya, Universitat Pompeu Fabra- (UB, UAB, UPC, UPF).

The *Kuttaka* Method from Aryabhata and Brahamagupta was offered in an elective History of Mathematics course for students of the Degree in Mathematics – Universitat Politècnica de Catalunya (UPC).

In general, if we want to implement a mathematical activity based on historical texts in the classroom it seems advisable, first, to locate the country and the historical time, then to identify the context in which it was developed and, finally, to raise the problem that it wanted to solve.

2 Mathematics in siddhanta texts

In the 4th and 5th centuries, there was remarkable mathematical activity in India related to astronomy which produces the s*iddhantas*, treatises on mathematical astronomy. They were texts with instructions to calculate positions of celestial bodies and solve questions related to the calendar, geography or astrology (Joseph, 1996:360).

The authors of the *siddhantas* wrote their works in Sanskrit verses following a structure established by the different astronomy schools (or *paksas).*In general, *siddhantas* pursued three goals: the first one consisted of predictive astronomy, the computation of times, locations and appearances of celestial phenomena –future or past– as seen from any given terrestrial location; the second was computational astronomy, explanations on computational procedures in terms of the geometry of the spherical models (presented in a separate section called *gola* (sphere)); and the third objective was instruction in general mathematical knowledge, basic arithmetic operations, calculation of interest on loans, rules for finding areas, volumes, sum of series, etc.

The *siddhanta* texts include one or some chapters dedicated exclusively to calculation in the proper mathematical sense(*ganita*).

2.1 Aryabhata & Brahamagupta

Two Indian astronomers and mathematicians, Aryabhata (476-550) and Brahamagupta (598-668), wrote two relevant *siddhanta* texts, *Aryabhatiya* and *Brahma-sphuta-siddhanta*, respectively. They were references for later astronomers and mathematicians. In Indian mathematics, *Aryabhatiya*(499) played, in some manner, the role of the *Elements* of Euclid in Greek mathematics (Moreno, 2011:46). *Brahma-sphuta-siddantha* (or *Corrected Teatrise of Brahma*), a work written in 628 when Brahmagupta was 30 years old, was in some way, a response to *Aryabhatiya*. He criticized Aryabhata, for example, for considering in a *kalpa* (the period between the creation and recreation of a world) 1008 *mahayugas* (a period of time) instead of 1000 according to Hinduism.

The last work of Brahmagupta was *Khanda-khadyaka*, written at the age of 67. It is a *karana* or manual of mathematical astronomy with simplified calculations.

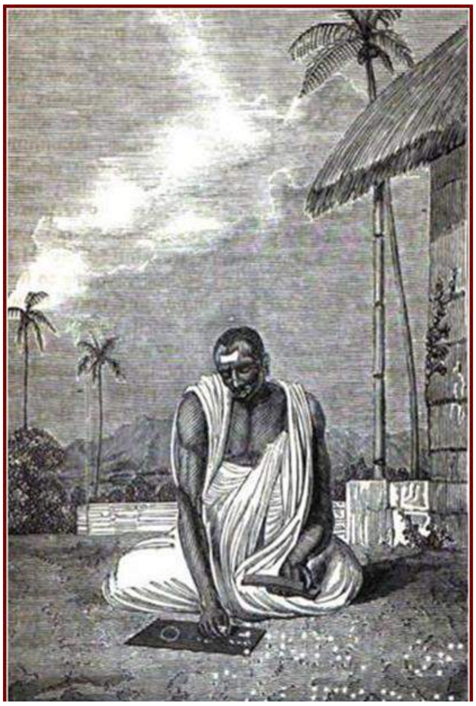


Figure 2.1: Imaginary representation of Brahmagupta (598-670) according to a 19th century engraving

2.2 *Aryabhatiya* and *Brahma-sphuta-siddantha*

The *Aryabhatiya*, written in 499 by Aryabhata, is the earliest completely preserved *siddhanta*. Chapter 2 is devoted to mathematics (*ganita*) and it consists of 33 verses[2] in a Sanskrit metric including methods for resolution of first degree, quadratic and first-degree indeterminate equations (Plofker, 2009, pp. 122-136). Bhaskara I wrote *Aryabhatiyabhasya* (629), a comment in Sanskrit prose about these 33 verses of chapter 2 of the *Aryabhatiya* (Keller, 2005, pp. 279-280).

From the *Aryabhatiya* of Aryabhata with commentary by Bhaskara I, it is known what Aryabhata said about how to teach the Reversed Procedure (Chapter 2, 28): "*In a reversed* [operation], *multipliers become divisors and divisors, multipliers, and an additive* [quantity], *is a subtractive* [quantity], *a subtractive* [quantity] *an additive* [quantity]".

In relation to the *Brahma-sphuṭa-siddhanta* (or BSS from now on), the book has 24 chapters. Chapters 1 to 10 deal with basic topics of astronomy: average lengths of planets; true lengths of the planets; the problems of daytime rotation; lunar eclipses; solar eclipses; rising and setting of the sun; phases of the moon; the shadow of the moon; conjunctions of the planets and conjunctions of the planets with the fixed stars. Chapter 11 is a criticism of *Aryabhatiya*. Chapter 12 deals calculation with numbers (*ganita*) and chapter 18 with calculation with unknown quantities and contents of *kuttaka* method. The construction of the sine appears in chapter 21. Chapters 13 to 17, 19 to 20 and 22 to 24 are dedicated to the topics related to astronomy.

[2]Verses 11 through 17 of *Aryabhatiya* contain calculations of half strings (which will evolve and become the sine) (Puig-Pla*et al*., 2011: 53).

3 The Reversed Procedure and the*Kuttaka*Method

Reversed Procedure and *Kuttaka* Method are both procedures of calculation to solve equations. In the first case it is about finding a number if we know the final result of several consecutive operations with the first number. The problem informs about the operations and the final result; the question is which is the initial number. We have to reverse the operations described in the problem, starting with the last one and finishing with the first one.

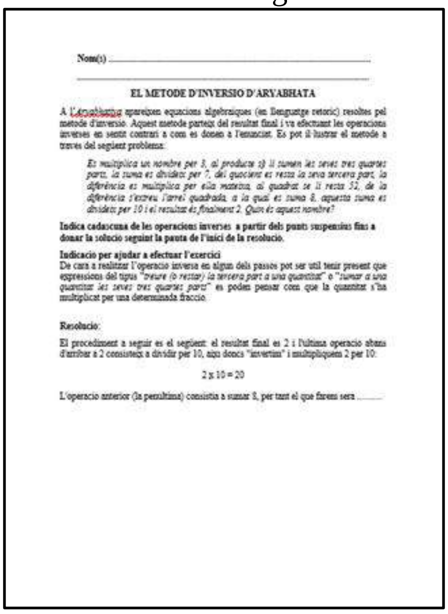
In the second case, we take successive steps to transform an equation with two unknowns into a simpler one repeating the process until reaching an equation that has one of the coefficients equal to 1, so that from thence the solutions for the original equation can be obtained by going backwards. This method can be considered as an application of the reversed procedure. Indeed, when the original equation $ax + c = by$ is transformed into $a'x' + c = y'$, clearly $x' = 0$ and $y' = c$ constitute one solution, from whence a solution (indeed all solutions) of the original equation $ax + c = by$ will be obtained, and the procedure agrees exactly with the algorithm described in the *Kuttaka* Method.

On the one hand, this explains the term *Kuttaka* ("Pulverizer" or reducing to powder) in that the coefficients are stepwise made smaller through the Euclidean algorithm (equivalent to the ancient Chinese algorithm in the *Geng Xiang Jian Sun* method) as well as tying up the two topics under discussion in this paper.

3. 1 The activity using the Reversed Procedure of Aryabhata

We presented this activity in a *pre-service teacher course* in an Inter-university Master in Teacher Training. After locating the country and the historical time (Ancient India at the end of 5th century), we gave a short explanation about the context (mathematics in *siddhanta* texts) and the procedure in the *Aryabhatiya* of Aryabhata. We used the*Aryabhatiya*commentary of Bhaskara I. After students were asked to solve an activity.

We prepared an initial exercise for students. It was based on verse 50 of the work *Lilavati* by Bhaskara II, a problem that had to be solved by the Aryabhata method (Plofker, 2007:454). The exercise had the following information and questions:



Nom(s) ....

EL METODE D'INVERSIO D'ARYABHATA

A L'Aryabhatiya apareixen equacions algebraiques (en llenguatge retoric) resoltes pel metode d'inversio. Aquest metode parteix del resultat final i va efectuant les operacions inverses en sentit contrari a com es donen a l'enunciat. Es pot il·lustrar el metode a traves del seguent problema:

Es multiplica un nombre per 3, al producte se li sumen les seves tres quartes parts, la suma es divideix per 7, del quocient es resta la seva tercera part, la diferència es multiplica per ella mateixa, al quadrat se li resta 52, de la diferència s'extreu l'arrel quadrada, a la qual es suma 8, aquesta suma es divideix per 10 i el resultat és finalment 2. Quin és aquest nombre?

Indica cadascuna de les operacions inverses a partir dels punts suspensius fins a donar la solucio seguint la pauta de l'inici de la resolucio.

Indicacio per ajudar a efectuar l'exercici
De cara a realitzar l'operacio inversa en algun dels passos pot ser util tenir present que expressions del tipus "*treure (o restar) la tercera part a una quantitat*" o "*sumar a una quantitat les seves tres quartes parts*" es poden pensar com que la quantitat s'ha multiplicat per una determinada fraccio.

Resolucio:

El procediment a seguir es el seguent: el resultat final es 2 i l'ultima operacio abans d'arribar a 2 consisteix a dividir per 10, aixi doncs "invertim" i multipliquem 2 per 10:

2 x 10 = 20

L'operacio anterior (la penultima) consistia a sumar 8, per tant el que farem sera ....

Figure 3.1: The Reversed Procedure Exercise

“In *Aryabhatiya* algebraic equations appear (in rhetorical language) solved by the Reversed procedure. This method starts from the final result and performs the reversed operations in the opposite direction as given in the statement. The procedure can be illustrated through the following problem:

> *A number is multiplied by 3, the product is added to its three quarters, the sum is divided by 7, the ratio is subtracted from its third part, the difference is multiplied by itself, the square is reduced to 52, from the difference is extracted from the square root, which is added 8, that sum is divided by 10 and the result is finally 2. What is this number?*

Indicate each of the inverse operations from the ellipsis in order to give the solution following the guideline of the beginning of the resolution.

Guidelines to solve the exercise.

In order to perform the reversed operation in one of the steps, it may be useful to bear in mind that expressions of the type "remove (or subtract) the third part from an amount" or "add up to three quarters" can be thought of as the amount that has been multiplied by a certain fraction.”

Resolution

The Reversed Procedure is as follows: the final result is 2 and the last operation before reaching 2 is to divide by 10, so we do the "reversed operation" by multiplying 2 by 10:

$$2 \times 10 = 20$$

The previous operation (the penultimate one) consisted of adding 8, therefore what we will do will be ....”

3.2 Students’ production susing the Reversed Procedure of Aryabhata

26 students carried out the activity. They formed freely 7 groups that had to be from 3 to 4 students (in fact there was also 1 group of 2 students and 1 group of 5).The group code is indicated thus, e.g. G2 (3) means group 2 with 3 students. In the statement the students had the pattern to follow (at the beginning of the resolution). They were asked to write a rhetorical explanation about the previous operation to carry out the reversed operation.

Although all the groups arrived at the correct solution, only one of them, G3(4), followed "exactly" this pattern. They wrote rhetorical explanations such as “[…] the previous operation was to extract the square root, therefore we will now square, $12^2 = 144$ […]”.

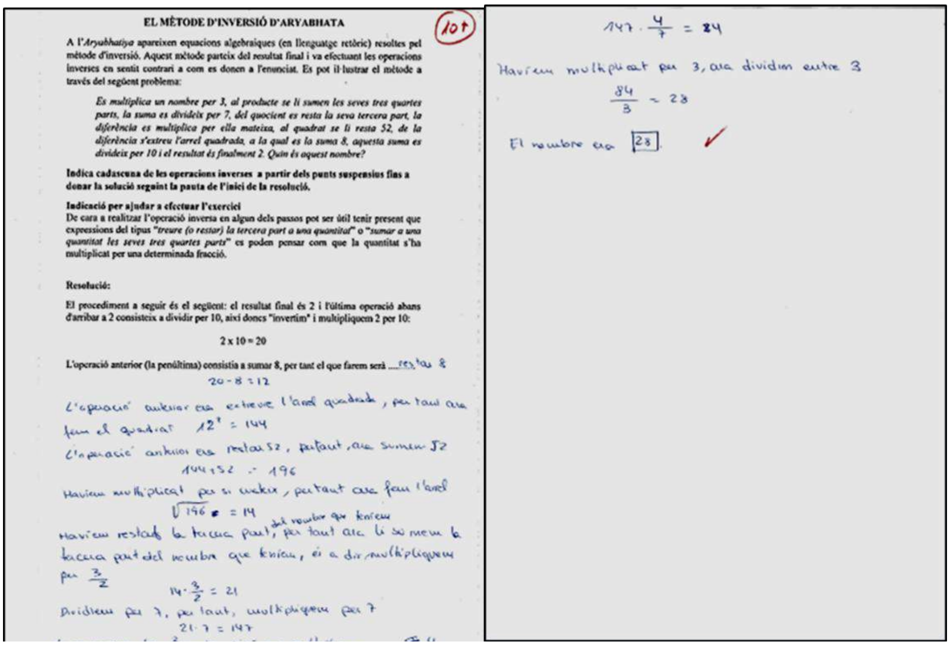


Figure 3.2: A student production from group G3(4) of the Reversed Procedure

All students correctly indicated the operations to be performed in each of the eight steps but "without" or "with few" rhetorical explanations.

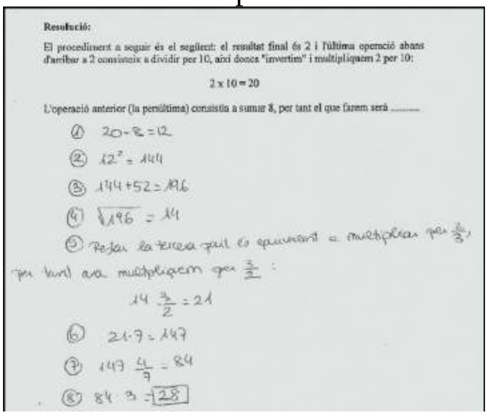


Figure 3.3: A student production from group G4(4) of the Reversed procedure

Two groups G5(4) and G6(2) introduced modern mathematical formalism by writing equations (using "x" for the unknown).

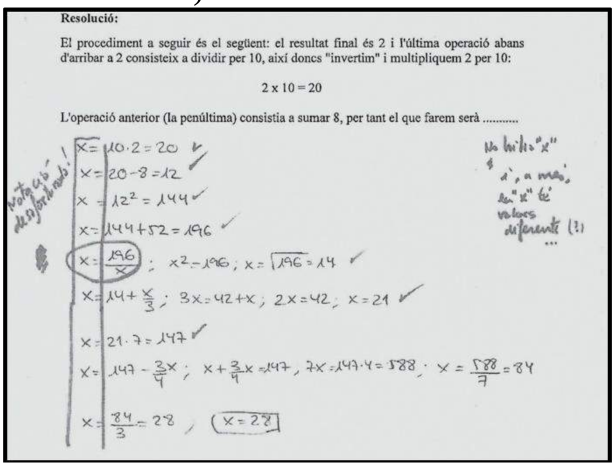


Figure 3.4: A student production from group G5(4) of the Reversed procedure

In general, we can conclude thatthe students of the pre-service teacher course easily understood the Reversed Procedure, although not all of them were able to solve the problem “in the manner of Aryabhata”, that is, in a rhetorical way and without formal symbolism (23% of students used the modern formalism of algebra).

3.3 The *Kuttaka* Method

We presented this activity in an *elective course of the History of Mathematics* of the Degree in Mathematics (Universitat Politècnica de Catalunya). After locating the country and the historical time (Ancient India at the end of 5^{th} century), we gave a short explanation about the context (mathematics in *siddhanta* texts) and the problem that Indian astronomers wanted to solve (the conjunction of the planets).

We continued with a short explanation of the method of Brahmagupta in *Brahma-Sphuta-Siddhanta* (chapter 18, 3-6) and finally students were asked to solve a problem, to assess the method and to answer a questionnaire about the history of math.

3.4 Chapter 18: Calculation with Unknowns

In this chapter Brahmagupta explains a calculation procedure with unknowns, first and second degree. But the innovation is the*Kuttaka*Method or “pulverizer" to treat equations or systems of indeterminate equations.The establishment of the calendar and astronomical calculations led to this type of procedureto solve equations or systems of indeterminate equations.

About the subjects that a master must know, he stated: “*A master* [acarya] *among those who know treatises* [is characterized] *by knowing the pulverizer, zero, negative* [and] *positive* [quantities], *unknowns, elimination of the middle* [term, that is, solution of quadratics], *single-color* [equations, or equations in one unknown], *and products of unknowns, as well as square nature* [problems, that is, second-degree indeterminate equations]” [BSS, Ch. 18, 2].

Brahmagupta gave the first known explanation of Indian mathematics about “the rules of signs and the arithmetic of zero”. Almost all the rules are identical to our modern algebra: “[The sum] *of two positives is positive, of two negatives negative; of a positive and a negative* [the sum] *is their difference; if they are equal it is zero. The sum of a negative and zero is negative,* [that] *of a positive and zero positive,* [and that] *of two zeros is zero* [. . .]. *The product of a negative and a positive is negative, of two negatives positive, and of positives positive; the product of zero and a negative, of zero and a positive, or of two zeros is zero.* [BSS, Ch. 18, 30 and 33, respectively]”.

Verses 43 through 59 refer to techniques and examples for solving equations with an unknown, both first and second degree. In the case of second degree, when a multiple “b” of the unknown added to a multiple “a” of the square of the unknown is equal to a number c. The calculation algorithm is focused on “removing the middle term” given the equation in the form: $ax^2 + bx = c$.

3.5 The *Kuttaka* or “pulverizer” method

Pulverizer (*Kuttaka*) is the successive steps to transform an equation with two unknowns into a simpler one. Given an equation $ax + c = by$ where a and b do not have common divisors, by means of a change of variable, the initial equation is transformed into another equivalent, until reaching an equation that has one of the coefficients equal to 1.From here the solutions are reconstructed until they reach the initial equation. But, for which part of astronomy were these calculations necessary? What was the problem to be solved?

We have to consider that we have a geocentric point of view, as the ancient Indians had. Then, it is a question of counting the days that have elapsed since the last time that two planets, A and B, were in conjunction at a certain point with respect to the fixed stars (that is since they had the same ecliptic longitudeλ_0).

For example, let us say we know - after long systematic observations - the following data:

1) Planet A takes an average time of 90 solar days to return to the same point on the stellar background (through the Zodiac) while planet B takes 33 days (see figure 3.5).

2) 19 days have elapsed since planet A completed an entire number of turns following the Zodiac (from λ_0).

3) 28 days have elapsed since planet B completed an entire number of turns following the Zodiac (from λ_0) in the same direction (see figure 3.6).

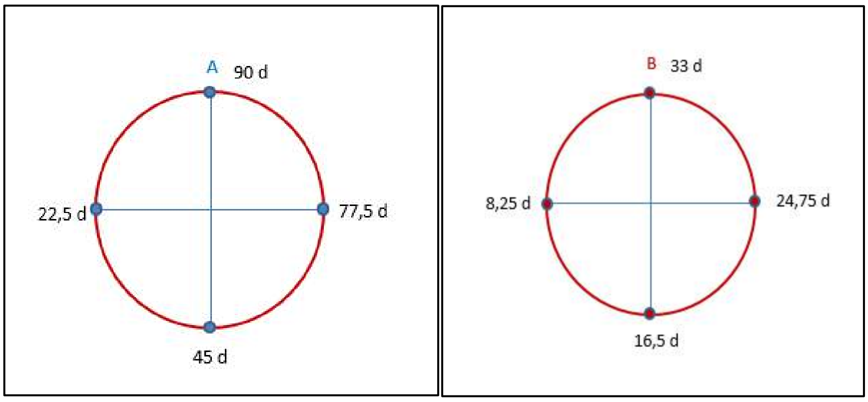


Figure 3.5: Planet A (to the left) takes 90 days to return to the same point. Planet B (on the right) takes 33 days

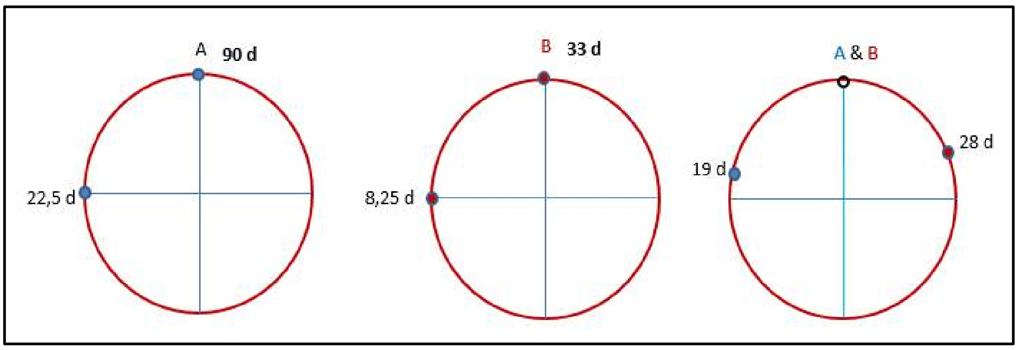


Figure 3.6: Planet A (on the left), Planet B (in the middle) and Planets A & B (on the right) after completed a different integer number of revolutions each one of them

The problem to solve is "how many days (N) have passed since the last conjunction?" Using the current notation, if x is the integer number of revolutions performed by A from the conjunction,

$$N = 19 + 90x$$

If y is the integer number of revolutions performed by B from the conjunction,

$$N = 28 + 33y$$

then, we have $N = 19 + 90x = 28 + 33y$

so, $90x - 33y = 28 - 19$

in other words that is,

$$90x - 33y = 9$$

a linear equation with two unknowns (with infinite integer solutions). We look for the positive integers such as $90x - 33y = 9$.

3.6 Brahmagupta in *Brahma-Sphuta-Siddhanta* (Chapter 18, 3-6)

Brahmagupta explains in verses 3-6 how to proceed with *Kuttaka* method:

1. *Divide the divisor having the greatest remainder* [*agra*] *by the divisor having the least remainder*

   [Indication: Euclid's algorithm].

2. *Once mutually divided, the last remainder will be multiplied by an arbitrary number* [integer] *such that if the product that we obtain is added* [if the number of quotients in the process is odd] *or we take it* [if it is even] *the difference of remainders* [the additive], *which results is divisible by the penultimate remainder.*

3. *Place the quotients of the mutual divisions one below the other in columns, until you reach the optional divisor and then the quotient you have obtained.*

4. *Multiply the penultimate by the previous one and the one that follows is added to it. Repeat the process* [the result is saved in the next column and occupies the penultimate position and then the penultimate number of the previous column].

5. *Divide the last number obtained* [*agranta*] *by the divisor having the least remainder. Then multiply the remainder by the divisor having the greatest remainder and add the largest remainder. The result will be the remainder of the product of the divisors* [to obtain a smaller solution].

In our activities, we explain the *Kuttaka* method Brahmagupta has created and we ask students to do the same. We also give them the translation of the method from Nolla (2006: 196-203).

3.7 The activity

As Brahmagupta explained in verses 3-6 of Chapter 18 in *Brahma-Sphuta-Siddhanta*, we try to solve the following problemof the *Aryabhatiya* by Aryabhata with comments from Bhaskara I (Chapter 2, 33):

"[A quantity when divided] *by twelve has a remainder which is five, and furthermore, it is seen by me* [having] *a remainder which is seven, when divided by thirty-one. What should one such quantity be?*"

$$\left.\begin{array}{l} 12y + 5 = N \\ 31x + 7 = N \end{array}\right\} \qquad 31x + 2 = 12y$$

3.8 Students' solutions of the activity

Although the answers of the students are not in English, we have collected their answers step by step and the connection they made with the Bézout method. We follow the assignments of two students, A and B.

Student A, translates the problem to modern notation

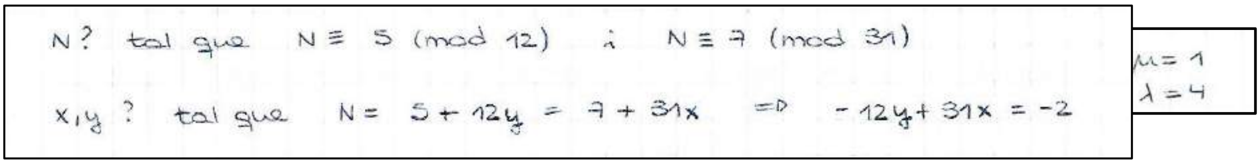


Figure 3.7: Student translation of the problem

1. Student A makes the first step, obtains the divisors and the residues and orders them in columns as established by the method

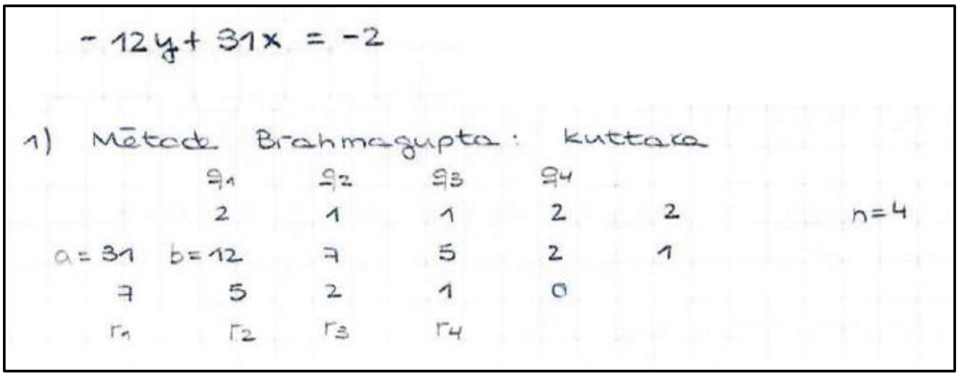


Figure 3.8: Student A's solution, step 1

2. Student A makes the second step, to find numbers λ and λ that meet certain conditions

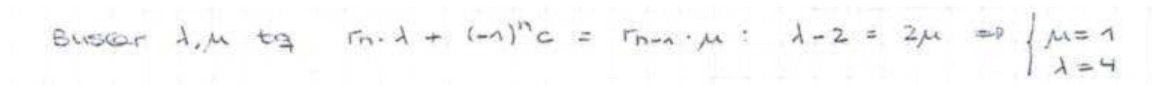


Figure 3.9: Student A's solution, step 2

3. Student A places the quotients of the mutual divisions one below the other in columns, until the optional divisor is reached and then the quotient is obtained.

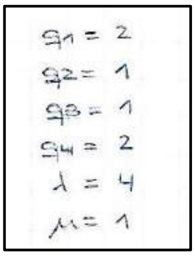


Figure 3.10: Student A's solution, step 3

4. Student A obtains a solution

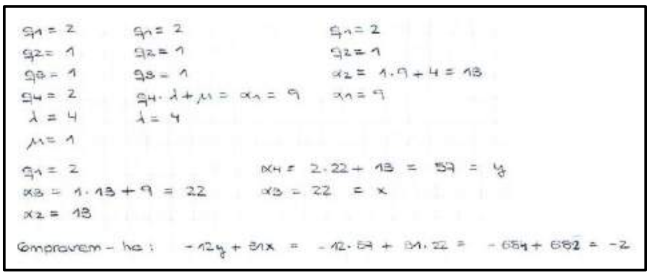


Figure 3.11: Student A's solution, step 4

5. Student A obtains a smaller solution

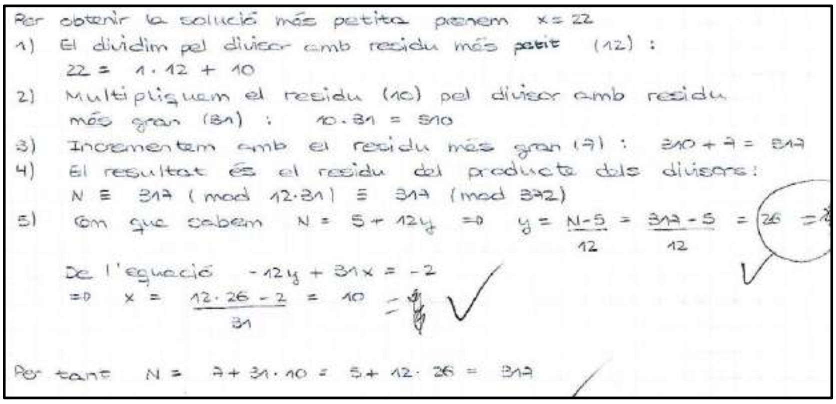


Figure 3.12: Student A’s solution, step 5

We can see another student’s solution (Student B) in figure 3.13. This student also used Bézout’s identity to solve the problem.

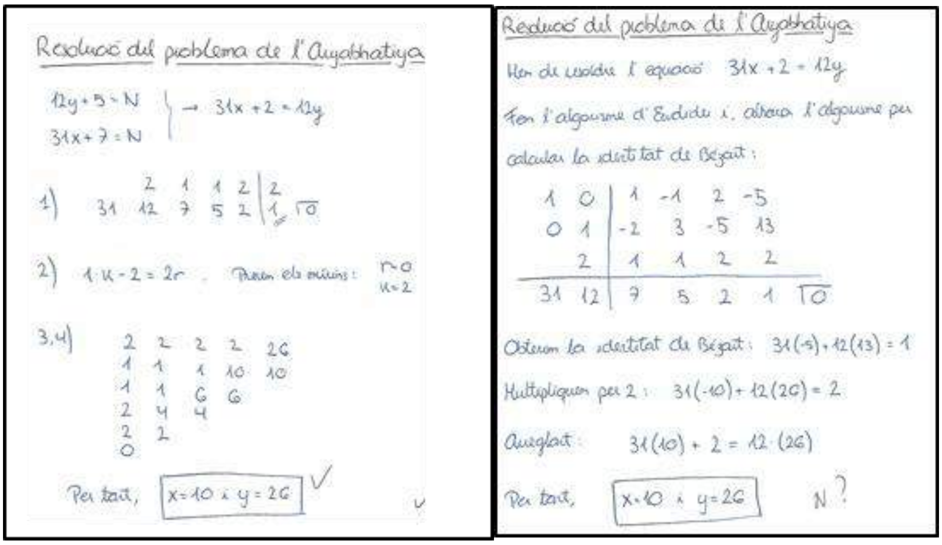


Figure 3.13: Student B’s solution of Aryabhata’s problem using *Kuttaka* method (left) and Bézout’s identity (right)

We did not explain Bézout’s identity in this course but some students connected the *Kuttaka* method with Bézout’s identity: *Let a and b be integers with greatest common divisor d. Then, there exist integers x and y such that ax + by = d*, and they solve the same problem with both methods, as we can see in the two pages of figure 3.13 done by student B. These students of the Degree in Mathematics are in the fourth year and had studied Bézout’s identity in the first year.

4 Questionnaire for Students

After the *Kuttaka* activity we asked the students some questions in order to assess the method but also some questions about their knowledge of math history. Again, we write the question followed by some students’ answers. The group had 8 students.

Question1. Did you have any knowledge about mathematics in ancient India?

1.1.- If yes, how did you get acquainted with it?
Explain briefly what you knew about it

Students' answers

Student P: positional notation and decimal base of numbers.
Student Q: research project in High School about Indian Math (only early civilizations). She remembered that the actual system of numbers came from Indian numbers.

Question 2. Before studying the *Kuttaka* method, did you know any other way to solve this kind of equations (linear with two unknowns)?

2.1. If yes, where did you learn this other method? Which one?
2.2. Evaluate the advantages and disadvantages of both methods.

Students' answers

All of them knew another method for solving this kind of equation. They learned it in the subject ofFoundations of Mathematics (1^{st} year).

Only two of them (students A and B) knew the name: Bézout Identity and only one of them (student B) said the full name: Étienne Bézout.

In general they preferred Bézout's method.

Students A & B like Bézout Identity to solve these equations because they think it is a proof but they thought "both methods" were similar.

Question 3.The *Kuttaka* method is a calculation procedure to solve indeterminate equations. Can you cite applications or contexts (current or historical) in which indeterminate equations are used?

Students' answers

Five of them relate this procedure to astronomy, the context used in the classroom to introduce *Kuttaka* method.

The student who knew Étienne Bézout (student B) said that linear equations with two unknowns are also used to calculate the intersection of varieties.

Student C referred to Fermat's theorem and Pythagorean Triples.

Question 4. Throughout the Degree in Mathematics or in high school, have you studied the solutions of a linear equation with two variables or unknowns?

Students' answers

All students said they had studied how to solve this equation with integer solutions in Foundations of Mathematics (1^{st} year).

One of them studied Diophantus's method in high school and did a research project in high school about Indian math.

Question 5.Is there any subject in the Degree in Mathematics in which the history of the involved conceptswas introduced?

Students' answers

Five students said: **no**

Three students said:

Student A: **yes** a little, sometimes teacher introduced the historical context and the biography of the mathematician who discovered a theorem.

Student B: **yes** in many subjects, the biography of the mathematician who discovered a theorem, but never in context.

Student C: **yes** but it depended on the teacher of the subject.

Question 6. Do you remember if any historical development of any mathematical concept was introduced in high school?

Students' answers

Six students said: **no**. One of them said: I would have liked to know something.

Two students said: **no**, only about Pythagoras and his theorem. One of them said: the goal of high school is for students to understand the concepts in order to pass the university entrance exam.

Question 7. Give your opinion about if History of Mathematics can help to understand better the topics studied in the Degree.

Students' answers

All students agree on the importance of knowing the History of Mathematics. In fact, they have chosen this elective course.

There are two points of view:

a) In order to understand modern concept sbetter.

b) Although it is not useful for a better understanding of modern mathematics, it could be interesting for understanding ancient mathematics.

Four students take position a) and four in b).

Question 8.For which mathematical topics would you recommend introducing the history of mathematics?

Students' answers

Some students associate history with specific topics: geometry, root of a polynomial, limit, calculus, and so on.

Others said the history of math must be taught at the beginning, in order to understand why math developed as well as in high school or in the first year of the degree.

Yet others said, in general in all the subjects and in both ways:

a) Related to concepts, to understand why other civilizations use other methods.

b) Related to problems, to compare motivations to solve problems and to have a broader view of the problem.

5 Final Remarks

We have been discussing throughout the paper various questions related to each topic exposed. As final remarks we can say that the activities based on the analysis of historical texts connected to the curriculum contribute to improving the students' integral training, giving them additional knowledge of the social and scientific context of the periods involved. They also get students to achieve a vision of mathematics, not as a final product but as a science that has been developed on the basis of trying to answer the questions that mankind has been asking throughout time about the world around us.

NOTE:

This research is included in the project: HAR2016-75871-R

REFERENCES

Guevara, I., & Puig, C.(2017). *Brahmagupta. El àlgebra de las estrelles*. Barcelona: RBA.

Joseph, G. G., (1996). *La cresta del pavo real. Las matemáticas y susraíces no europees*. Madrid: Editorial Pirámide.

Keller, A. (2005). Making diagram sspeak, in BhaskaraI'scommentary on the *Aryabhatiya*. *Historia Mathematica 32*, 275-302.

Massa-Esteve, M. R., Guevara, I., Romero, F., & Puig-Pla, C. (2011). Understanding Mathematics using original sources. Criteria and conditions. In E. Barbin, M. Kronfeller, & C. Tzanakis (Eds.) *History and Epistemology in Mathematics Education. Proceeding of the 6th European Summer University* (pp.415-427). Wien: Holzhausen Verlag.

Moreno, R. (2011). *Aryabhata, Brahmagupta y Bhaskara. Tres matemáticos de la India.* Madrid: Nivola.

Nolla, R. (2006). *Estudis i activitats sobre problemes clau de la història de la matemàtica*. Barcelona: Publicacions de la SCM, IEC.

Plofker, K., (2007). Mathematics in India. In V. Katz (ed.) *TheMathematics of Egypt, Mesopotamia, China, India, and Islam: a sourcebook* (pp. 385- 514). Princeton and Oxford: Princeton University Press.

Plofker, K., (2009). *Mathematics in India*. Princeton: Princeton and Oxford University Press.

Puig-Pla, C., Guevara, I., Romero, F., & Massa, M. R. (2011). La trigonometria a la matemàtica de l'Antiga Índia. Algunes idees per treballar a l'aula. In P. Grapí, & M. R. MASSA (Eds.). *Actes de la VI Jornada sobre la Història de la Ciència i l'Ensenyament "Antoni Quintana Marí"*(pp. 53-60). Barcelona: Societat Catalana d'Història de la Ciència i de la Tècnica – Institut d'Estudis Catalans,

Puig-Pla, C.,& Guevara, I. (2017). Maons, altars i versos: els singulars mètodes de transmissió dels coneixements matemàtics i astronòmics a l'antiga Índia. In P. Grapí, & M. R. Massa (Ed.), *Actes de la XV Jornada sobre la Història de la Ciència i l'Ensenyament* (pp.135-142). Barcelona: SCHCT-IEC. DOI: 10.2436/10.2006.03.17.

Romero, F., Massa, M.R., Guevara, I., & Puig-Pla, C. (2015). Teachertraining in theHistoryofMathematics. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (eds.) *History and Epistemology in Mathematics Education. Proceedings of the Seventh European Summer University* (pp. 113-128). Copenhagen: Aarhus University.

A CASE STUDY OF THE IMPLEMENTATION OF PRIMARY SOURCES IN UNDERGRADUATE MATHEMATICS

Matthew MAUNTEL
Florida State University, 1114 West Call Street, Tallahassee, Florida, USA
mmauntel@fsu.edu

ABSTRACT

In this paper I present a pilot case study of three different university mathematics instructors who implemented the same primary source project (PSP) as part of the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) project. I describe four implementations (two by the author, two by non-author instructors) using a combination of student and instructor interviews, instructor implementation reports, and instructor and student surveys. Survey data revealed that most students reported perceived academic gains from their work with the PSP, with the author's students reporting some of the greatest gains. I conclude that there are differences in implementation based upon observed lines of communication and how instructors view distinctive features of the implementation, including the importance of group work and understanding language use within sources.The author stressed the importance of group work and the productive struggle associated with language, resulting in a different implementation than both non-authors.

1 Introduction

Mathematics faculty and educational researchers increasingly recognize the value of the history of mathematics as a support to student learning. The expanding body of literature in this area includes recent special issues of Science & Education (Jankvist, Fried, Katz, & Rowlands, 2014) and Problems, Resources and Issues in Undergraduate Mathematics Education (PRIMUS; Clark & Thoo, 2014), both of which include direct calls for the use of primary historical sources in teaching mathematics. For many instructors, the current lack of classroom-ready materials poses an obstacle to the incorporation of history into the mathematics classroom. As noted by Jankvist (2009), "the 'urgent task' of developing critical implements for using history in the teaching and learning of mathematics" (p. 256) is also essential to further research on the benefits and effectiveness of using history of mathematics to teach.

For decades much of the research literature in the United States on the impact of the history of mathematics on students, particularly at the secondary level or tertiary level, was focused on students' attitudes (e.g., Marshall, 2000; McBride & Rollins, 1977). There has been little focus on the use of primary sources as a classroom tool in the early work in the field of history in mathematics education. While more recent work on the use of primary sources has been done in countries such as Brazil (e.g., Bernardes & Roque, 2018), Denmark (e.g., Kjeldsen & Blomhøj, 2012), and Turkey (e.g., Alpaslan & Haser, 2015), similar research has not yet been conducted with student populations in the United States. Thus, the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (or, TRIUMPHS) project is committed to investigating the ways in which mathematics students respond to being taught concepts within the undergraduate curriculum via primary historical sources.

In this paper we share initial findings on the research questions we identified for this pilot study:

1. What are the differences between the author's implementation of a PSP and two non-author implementations of the PSP?
2. How do students' perceived gains in understanding content-related material differ between implementations?
3. What were the reported benefits and obstacles of each implementation of PSP from a student point of view and from an instructor point of view? How do these views align or differ?

Thus, the aim of this particular analysis was to determine whether differences exist when the instructional materials –in this case, the primary source project of interest – are implemented by the author of the project when compared to other site-testing instructors. In what follows, we explore the data sources that informed our analysis, and the ways in which we are attempting to make sense of students' experiences with PSPs. We will discuss the differences we identified from each implementation of the primary source project, *Solving a System of Linear Equations using Ancient Chinese Methods*, the ways in which multiple course populations reported similar student gains, and the ways in which students' reported benefits and obstacles for learning with primary source materials can inform future implementations in the TRIUMPHS project.

2 Brief Literature Review

Numerous examples from the literature describe outcomes of purposeful use of history of mathematics in undergraduate mathematics instruction (see, for example, Barnett, 2014; Jahnke et al., 2002; Liu, 2014; Ruch, 2014; Tamulis, 2014). However, these examples often describe the instructional approach that was undertaken (including important details of the primary sources used), or only offer anecdotal evidence observed by the authors, regarding their students' experience with the historical content or instructional materials. In other, more empirical examples, the primary focus of the intervention is on pre-service mathematics teachers (e.g., Charalambous, Panaoura, & Philippou, 2009; Clark, 2012; Huntley & Flores, 2010).An open question for the field of history in mathematics education is: What might be possible to investigate regarding the implementation of particular instructional materials (in this case, the PSPs being developed)?

Teaching from primary sources has been widespread in the other fields such as social sciences (De Guzmán, 2007; Klyve, Stemkoski, & Tou, 2013). The use of original sources enables students – as well as teachers – to enrich their understanding of subject being taught (Laubenbacher, Pengelley, & Siddoway, 1994). Due to a lack of appropriate classroom materials, the integration of history into mathematics classrooms remains a difficult approach for many.

Precisely because the use of primary sources provides students the opportunity to interpret results as they were originally presented and then reformulate them in modern terms, original (primary) source readings enable instructors to present a different view of mathematics to students. That is, instead of a classroom scenario in which "definitions and theorems are usually presented first, and then, motivation and application follow afterwards" (Jankvist, 2014, p. 879), primary source materials provide instructors with the opportunity to motivate instruction first with a problem or application, "the solution of which leads to theorems, proofs, and in the end definitions" (Jankvist, p. 879). In this way, primary historical sources have the potential to transform students' images of or views about mathematics.

3 The TRIUMPHS project

3.1 Overview

In 2015, the National Science Foundation in the United States funded a five-year, seven-institution collaborative project to design, test, and evaluate curricular materials for teaching standard topics in the university mathematics curriculum via the use of primary historical sources. The team of principal investigators (PIs) consists of six mathematicians and one mathematics education researcher. The primary source projects (PSPs) are developed by TRIUMPHS PIs, as well as external authors, and once PSPs have gone through an extensive review and testing in the author's (or authors') classroom, they are site-tested by project PIs (when appropriate) and classroom instructors (mathematicians and mathematics teacher educators) who were recruited either through workshops or through other recruitment means. There is an extensive evaluation-with-research component of the TRIUMPHS project, which addresses aspects of faculty expertise and student change. Three TRIUMPHS PIs (two mathematicians and the mathematics education researcher) and several graduate students facilitate the evaluation of classroom site testing of the PSPs.

The goal of the TRIUMPHS project is to promote students' learning and their development of a deeper interest in and appreciation of mathematical concepts by creating educational materials in the form of PSPs based on original historical sources written by mathematicians involved in the discovery and development of the topics being studied. In TRIUMPHS, PSPs contain (1) excerpts from one or several historical sources, (2) a discussion of the mathematical significance of each selection, and (3) student tasks designed to illuminate the mathematical concepts that form the focus of the sources. PSPs are designed to guide students in their explorations of these original texts in order to promote their own understanding of those ideas.

The numerous PSPs are the life force of the TRIUMPHS project. During the grant-funded effort, the PIs promised that some 50 PSPs (which span the undergraduate mathematics curriculum, from basic statistics and trigonometry, to real analysis, abstract algebra, and topology) will be developed, tested, and evaluated. Of the 50 PSPs, 20 are planned to be "full-length" and 30 are what we refer to as "mini-PSPs." Full-length PSPs are designed to typically encompass at least two to four class sessions, which represents the same amount of time that it normally takes to teach the mathematical topic of focus within the PSP. However, among the full-length PSPS there are also longer ones that could be used by instructors to comprise an entire course's content[1]. Alternatively, "mini-PSPs" can be completed in one to two class sessions and each of the mini-PSPs have been developed to teach a particular topic or concept in mathematics that would normally be addressed in a single class session, but which will be done via a primary historical source. To date, 28 full-length PSPs and 21 mini-PSPs have been developed. Though we have exceeded our commitment to develop 20 full-length PSPs, there are additional full-length PSPs in development, as well as the remaining, promised mini-PSPs.

[1] In fact, this was done recently (Spring 2018) by Janet H. Barnett, in an Abstract Algebra course.

In Fall 2015 the first PSPs were tested in two undergraduate mathematics classrooms in the United States; in Year 3 (academic year 2017–18), 46 distinct site testers tested one or more PSPs in undergraduate mathematics classrooms. In total, by the end of Year 3, 53 instructors have site tested PSPs as part of the TRIUMPHS project, with some one-third of those serving as repeat testers. In the first semester of Year 4, we have 25 site testers; again, of these, we have several repeat site testers, where 14 are new to site testing TRIUMPHS PSPs.

3.2 *Solving a System of Linear Equations using Ancient Chinese Methods*

The *Solving a System of Linear Equations using Ancient Chinese Methods* PSP (Flagg, 2017) was created to introduced students to row reduction in an introductory linear algebra course. The PSP is based upon the text *The Nine Chapters on the Mathematical Art* (Shen et al., 1999) and covers basic arithmetic using counting rods and solving systems of equations involving the Fancheng rule (see Appendix for a section of the PSP relating to the Fancheng rule). When first learning the Fancheng rule, students perform operations using columns operations in grids as opposed to row operations. Modern matrix notation is introduced in the latter half of the PSP. The PSP also introduces modern terminology, including echelon form, and requires students to complete several problems from *The Nine Chapters on the Mathematical Art* (Shen et al., 1999) using modern notation. Flagg (2017) highlights the value of the Fancheng rule in avoiding complex fractions until the very end of the row reduction process.

4 Context and setting for the study

The *Solving a System of Linear Equations using Ancient Chinese Methods* PSP (Flagg, 2017) was implemented in Fall 2017 and Spring 2018[2]. Students completed initial and final course surveys, in which they shared their beliefs about mathematics, prior experience with primary source materials in undergraduate mathematics courses, views about mathematics learning, and general demographic information. Upon completion of the PSP, students provided responses to post-PSP survey items, which captured students' perceived gains in skills relating to linear algebra content, general mathematical skills including reading and writing about mathematics, and attitudes and confidence in mathematics. Additional questions asked about the interaction of students with peers, the instructor, and the primary source material inside and outside of class. Finally, several open-ended questions asked students to reflect upon their experience with the PSP, including their perception of benefits and obstacles of learning mathematics using primary sources, and their attitude towards using primary sources in a linear algebra course. Implementation reports and surveys were collected from instructors and an instructional guide ("Notes to Instructors") was provided by the author.

4.1 Instructors and students

There were three course instructors of interest, across four implementations of the *Solving a System of Linear Equations using Ancient Chinese Methods* in 2017–18. The four student populations, as well as the survey data collected for each, are briefly described in Table 4.1 1.

[2] Flagg is a mathematician and university instructor and not a mathematics education researcher.

Table 4.1: Student populations in Linear Algebra courses (2017-18)

Instructor	Number of consenting students (estimated students enrolled)	Students completing all surveys (I = Pre-Course; P = Post-PSP; F = Post-Course)
Author Flagg (Fall 2017)	14 (15)	10 (I: 14, P: 11, F: 10)
Professor Monty (Fall 2017)	10 (25)	5 (I: 8, P: 7, F: 5)
Author Flagg (Spring 2018)	17 (21)	7 (I: 13, P: 11, F: 9)
Professor Carl (Spring 2018)	29 (32)	23[3]

In addition to the surveys and implementations reports, Professors Flagg and Monty were interviewed about their fall implementations. Three students from Professor Carl's class and two students from Professor Monty's class were also interviewed. The interviews were reviewed and relevant excerpts were transcribed for analysis and triangulation with data collected from the surveys and implementation reports.

In the following sections we describe each implementation using either the instructor's perspective, student's perspective, or both, when available, to describe the ways in which the four course populations reported similar perceived learning gains, and the ways in which students' reported benefits and obstacles for learning with primary source materials can inform future implementations in the TRIUMPHS project. We begin with our conceptualization of implementation of the PSP of interest.

5 Modeling communication utilized during PSP implementation

As part of this pilot study, we analyzed the interactions (i.e., lines of communication) between students, the instructor, and the author of the PSP in question, *Solving a System of Linear Equations using Ancient Chinese Methods* (Flagg, 2017). First, we considered the role of the student and how they supplied information to the instructor and to the author. For example, each student could supply information to the instructor (who was also implementing the PSP) by completing the assigned tasks and asking questions in class. Each student also completed a survey (e.g., post-PSP survey) about the project which we also analyzed. Students could communicate with each other via group work taking place during class.

[3] Due to the implementation of the PSP at the start of the semester, students in Professor Carl's course only completed one survey, which included items from each of the three surveys, and was completed immediately after the PSP was completed.

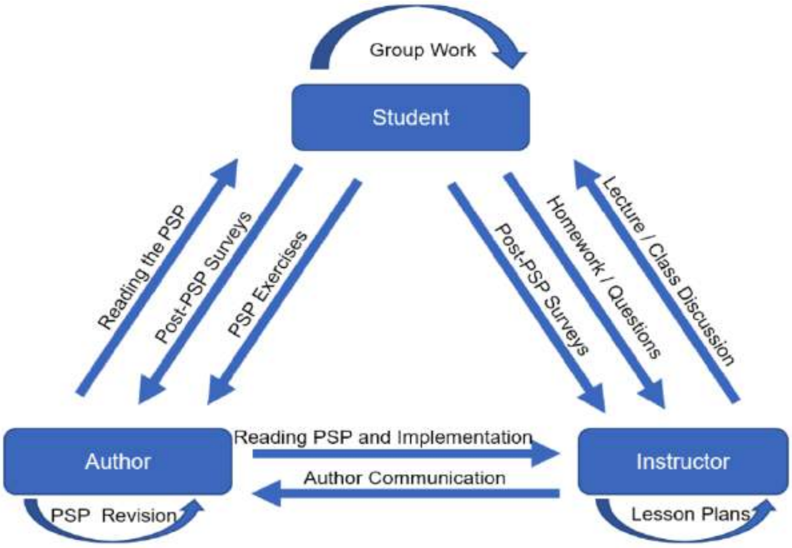


Figure 5.1: Lines of communication in a PSP implementation

Another line of communication was in the form of instructors with the students or author, through instructors lecturing or leading in-class discussions with students or via the instructor communicating directly with the author by email or phone if they had questions about implementation. The author was able to communicate both with students and the instructor through the written text of the PSP. The author also included "Notes to Instructors" with the PSP, which provided suggestions for implementation, including a timeline for implementation. Finally, it was possible for the author to further process the lines of communication by revising their PSP. The various forms of communication that we conjectured taking place are illustrated in Figure 5.1.

5.1 Case 1: The author as instructor (Flagg)

The first case we analyze is one in which the author is the instructor. According to the author's implementation report, a typical class consisted of reviewing the reading and homework from the earlier session, doing some group work in class relating to task(s) in the PSP, and having a concluding whole-class discussion. Students were expected to read the PSP outside of class as well as work on several tasks in the PSP as homework. The author was excited to share her passion for the material demonstrated by the following passage:

> I benefited from being able to share my passion for where ideas originate with my students and give them the opportunity to discover "new" old ideas. The students benefited from the struggle of trying to understand the unfamiliar language and notation. (Flagg's Implementation Report, Fall 2017)

5.2 Case 2: Non-author as "guide on the side" (Monty)

During this implementation, the instructor provided the students with the PSP and asked them to read portions of it outside of class. Class time consisted of students working together in small groups, which entailed completing assigned tasks during class. Students could ask the professor questions during class, but there was often not a concluding whole-class discussion. This is described in the instructor's implementation report in the following passage which described three days of implementation:

> Students worked in groups (3-4students each) on Tasks 2-9.Students worked on a white board (rather than their own paper) so that they had to interact (as I limited it to one white board pen for each group). (Monty, Implementation Report)

This indicates that the instructor saw part of the value and struggle of the PSP as coming from the students working as a group. The instructor also noted in his commentary on the implementation that "I probably should have stepped in and brought the entire class together. I think some of them were starting to veer off track and re-orienting, them may have been helpful" (Implementation Report). This indicates the instructor's reluctance to interfere with student group work during implementation (even when, upon later reflection, the instructor indicated it might have been useful). This implementation is characterized by its focus on student interactions as opposed to instructor-student interactions (Figure 5.2).

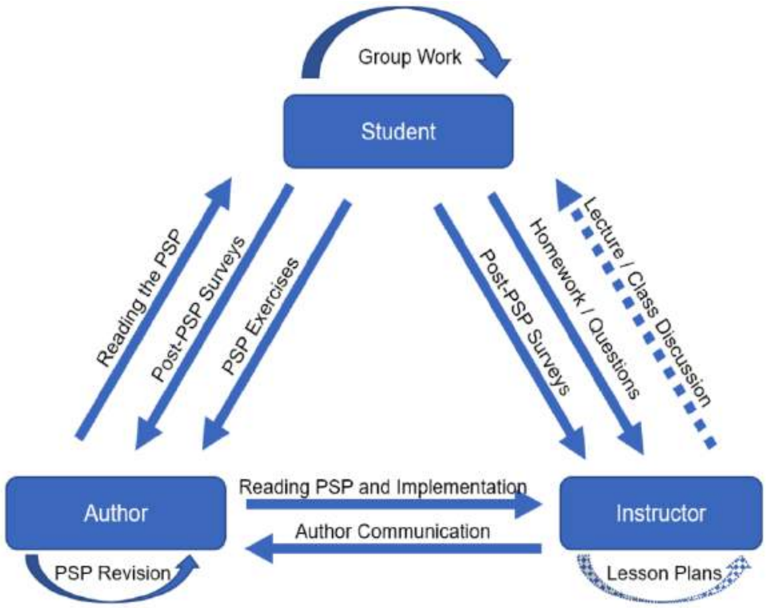


Figure 5.2: Non-author "guide on the side" approach

5.3 Case 3: Non-author without group work emphasis (Carl)

During this version of implementation, students were asked to read the material outside of class and complete some of the PSP tasks. The course consisted mostly of lecture with some in-class discussion but no group work. This implementation is depicted in Figure 5.3. During this implementation it was possible that the importance of reading was undermined by the in-class discussion as represented by the following comment offered by a student during a post-course interview:

> We would spend a couple of hours outside of class trying to go through five questions and a small section of the reading, and then we would come to class and he would answer all of our questions in like 10 minutes. (Student Interview)

Here the student noted that instructor would cover the material quickly the next day, undermining the value of struggling with the material on their own. To highlight this point, the arrow connecting author to the student is dotted to represent that the students did not struggle through the PSP.

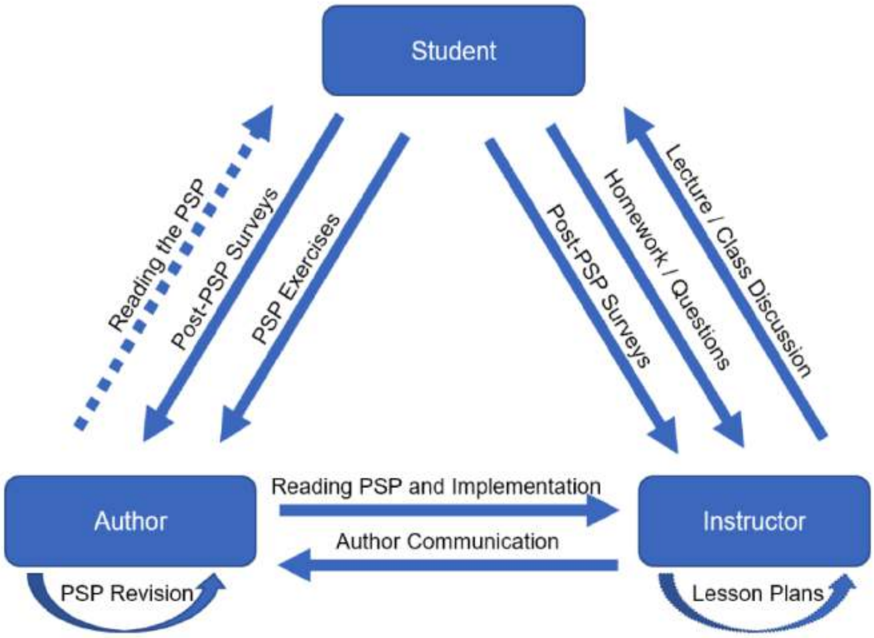


Figure 5.3: Non-author, without group work approach

6 Analysis and findings

6.1 Perceived student gains

As Figures 6.1, 6.2, and 6.3 show, students reported perceived gains in three major topics related to the PSP. We note several trends from the data collected. First, there is a trend toward students' reported perceived gains as "great gain" and "good gain," indicating that students perceived that they experienced progress in key course topics. Second, for the author's (Flagg) implementations, there was a high percentage of students reporting "good" and "great" gains, which might have resulted in the difference in implementation between the author and non-authors. Third, Figure 6.3 indicates a trend toward students' reported perceived gains as "no gain" or "small gain." A possible reason for this trend is that students in the United States are often taught back substitution prior to taking a Linear Algebra course, thus resulting in students perceiving smaller gains.

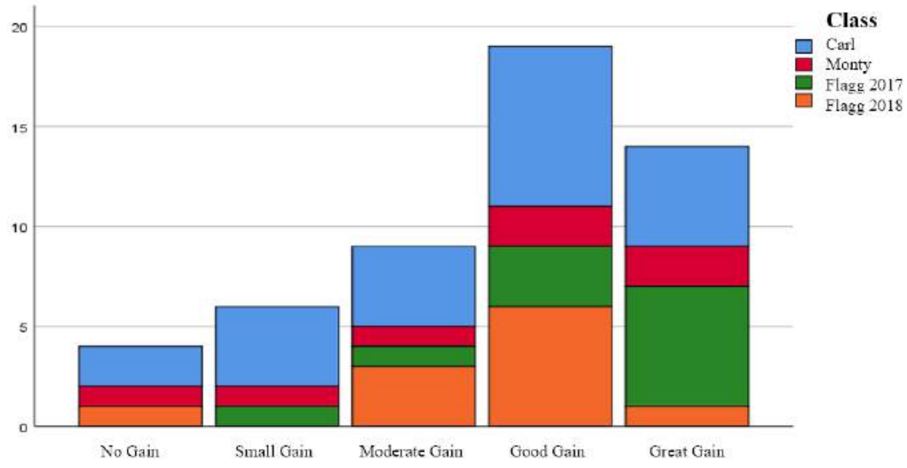


Figure 6.1: Perceived learning gains:Students' ability to set-up an augmented matrix

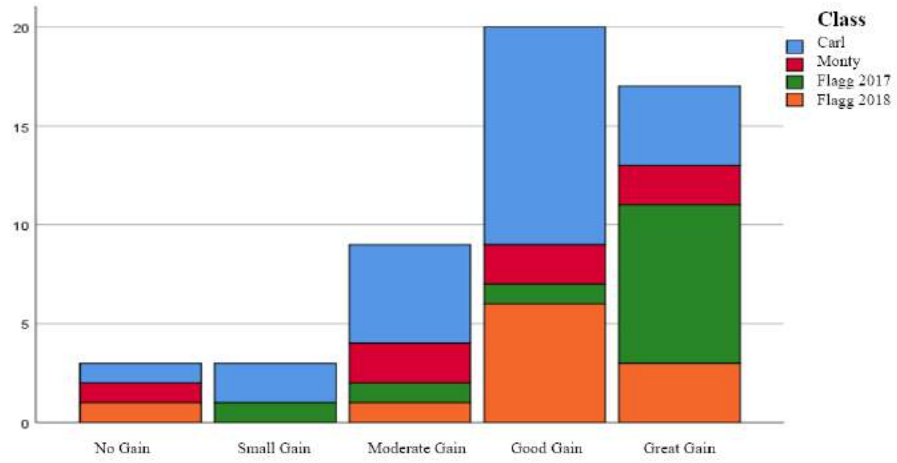


Figure 6.2: Perceived learning gains: Students' use of elementary row operation on an augmented matrix to find the unique solution of a non-singular matrix.

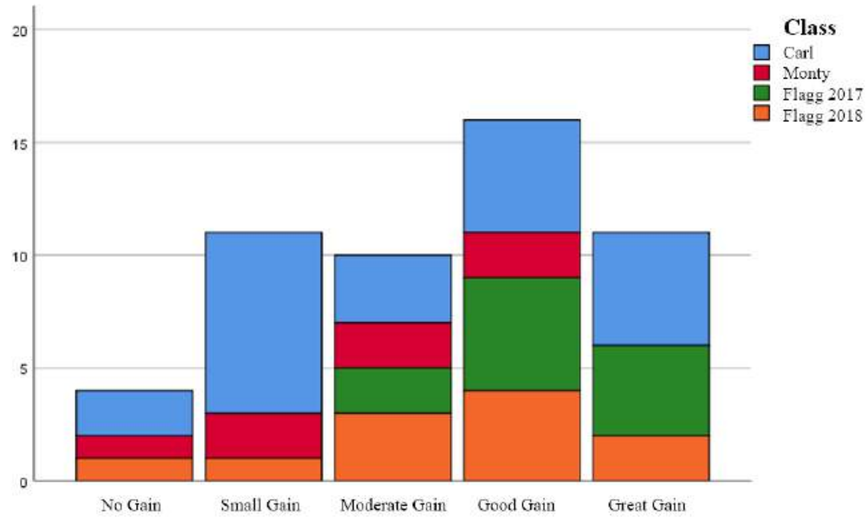


Figure 6.3: Perceived learning gains: Students' use of back substitution method for finding the solution of a system of linear equations represented by an upper triangular matrix.

6.2 Students' reported benefits and obstacles to using PSPs

On the post-PSP survey, students were asked what they believed the benefits and drawbacks were from learning mathematics by reading primary historical sources used in the *Solving a System of Linear Equations using Ancient Chinese Methods* project. Students identified multiple benefits, but one of the most prevalent was that students noted that seeing the development of mathematics or mathematical context was important and could instill confidence in the material. This is exemplified in the following student response:

> I think the benefit is knowing where methods originated, and know that what is studied and used today is completely valid and has history. It creates confidence in the subject because it has backing. (Flagg Student, Spring 2018)

A second benefit the students noted was that they experienced a change in perspective during PSP implementation. This is exemplified by the following student's response:

> One of the benefits of learning mathematics by reading historical sources is for students to have a perspective [on] how mathematics developed throughout history. This also gives [an] example how different societies can have multiple ways to solve a mathematical problem that differs from contemporary mathematics. (Flagg Student, Fall 2017)

Students also reported obstacles they experienced during this PSP implementation. Two general themes that arose were difficulty with language and general confusion. The first obstacle noted was a difficulty with the text itself, in which students described the language as archaic or that translations did not provide the clarity they needed. This is exemplified by the following comment:

> The main drawback was the archaic language used to describe the methods. I didn't have a clue what the author was asking me to do until the professor explained in more fluid language. (Monty Student, Fall 2017)

It is possible that students used the term "language" to represent general frustration with the material. There is some indication that the order in which topics were presented in the first implementation was less than optimal. This was acknowledged by students (in surveys) and in the instructor's implementation reports in Fall 2017. The PSP was revised for the 2018 implementation by the author (Flagg). Despite these revisions, there were still objections to the difficulty of understanding the language contained within the PSP, indicating that students were challenged with interpreting some "foreign" and unfamiliar terms.

The second obstacle that students reported was that the material was confusing or frustrating. This was often mentioned with reference to difficulty in understanding language, but it was not exclusively paired with this difficulty. For example, one student reported the following:

> It's more confusing than learning in a more traditional manner and I struggled with even understanding some of the primary source material. (CarlStudent1, Spring 2018)

Here it is important to note that the student was comparing the material to traditional material. This was another common theme among reported obstacles. This might indicate that students had problems learning from what was not a traditional, didactical text and the shift from such a format to something quite different presented a possible difficulty. In the United States, it is entirely possible that students might not have encountered a non-didactical mathematical text prior to this experience; thus, students may not know how to approach the text. As well, this is indicated by several students reporting that they had to read more than in a standard mathematics class.

Students were also asked what they would tell a friend about their PSP experience. Here, a number of students reported that potential peers should expect confusion and frustration, but the confusion and frustration had a payoff in terms of knowledge gains:

> They should expect to have to put a lot of time and effort into the project, but it will be worth it in the end. It will also be frustrating, but then interesting. (Carl Student 2, Spring 2018)

6.3 Instructors' consideration of language

The fact that students experienced difficulty interpreting language was also reported by instructors in their implementation reports. In general, there were two views relating to this obstacle reported by instructors. This first viewpoint was the perspective of Monty. He indicated that language was an unnecessary bug in the PSP, and perhaps the PSP could be modified to provide a clearer explanation in terms of modern terminology. In this instance, Monty wanted to "make the unfamiliar familiar" for the students. That is, Monty proposed to do this cognitive work for the students, rather than students potentially engaging in productive struggle to do this work.

The other perspective was that the difficulty interpreting language provided a great opportunity to discuss the use of language in communicating mathematical ideas for example the need for clarity and precision. This is best exemplified by Carl describing his process of "turning it [language] into a feature not a bug." This viewpoint was endorsed by the author, who stated that

> …some struggle is important to understand the power and challenge of mathematical language as [students] go deeper into the subject. *A little exercise in having trouble reading the PSP can be leveraged into a lesson on why it is important to be clear.* (Flagg, Fall 2017, emphasis added)

For both Carl and Flagg, "making the familiar unfamiliar" through the PSP was a meaningful and significant aspect of the use of the PSP with regard to overall mathematics instruction, and was a shared cognitive activity among students and instructors alike.

7 Conclusion and Discussion

7.1 Limitations and future research

This study was a pilot used as a preliminary analysis of the implementation of the PSP. The researcher's viewpoints into the implementation are a result of self-reported data from the students and instructors. These viewpoints may have omitted vital information regarding classroom interactions. Any results regarding how the students and instructors recalled these interactions are limited. Also, the number of participants included in this study is small; therefore, any quantitative results should be considered limited in scope (in particular the learning gains).Future research should include data collection (and analysis) of implementation. Observing student teacher interactions may result in increasing or decreasing the cognitive demand of the tasks contained within the PSP. This might also have an impact on how students perceive the PSP and their own learning gains, and could be grounds for further research.

It is also unclear if the student remarks about the language in the PSP resulted from frustration with the mathematics of the problem (which were communicated as language issues), issues with translating particular words within the text, the didactical style of the text which was different than most traditional text books in the United States, or something else. A careful analysis of each of these conditions in future PSP implementations might shed some light on what students mean by the claim that the language (in the PSP) is frustrating.

7.2 Discussion

In this pilot study, the data indicated that there were discernible differences among different implementations of the *Solving a System of Linear Equations using Ancient Chinese Methods* PSP. First, there were different lines of communication that were open among students, the instructor, and the author. The author's implementations contained all possible lines of communication while some lines of communication were not present in non-author implementations of the PSP. The presence of these lines of communication might be a contributing reason for why students in the author's implementations of the PSP reported high perceived learning gains.

While many students viewed language as a difficulty in the PSP, there was a significant difference in how instructors viewed language. Carl and Flagg both viewed language as an obstacle for students, but one that students should embrace as part of the learning experience, as opposed to a bug that needs to be corrected in future implementations. This is in line with the conceptions of Barnett, Lodder, and Pengelley (2014) who stated that

> In short, the primary source is now being used not just to introduce the mathematics in an authentically motivated context, but also as a text which the student is explicitly challenged to actively "interpret" as part of their personal process of making modern mathematics their own. In alignment with this shift, the tasks we now write for students increasingly adopt a more active "read, reflect, respond" approach to these sources. (p. 10)

Monty and Flagg viewed group work as productive struggle for students and prioritized it in their implementation. Furthermore, Flagg noted in her implementation that

> I learned a great deal from writing and implementing my project. I think that I learned the most from incorporating the readers' suggestions in framing tasks as more open-ended questions. I will continue to use that lens as I create course material for all my classes. The implementation of the project also takes me one step closer to creating a more interactive classroom. (Flagg, Implementation Report).

7.3 Implications for Instruction

Many participants viewed using the PSP as positive experience even if it was frustrating at times. The results indicate that some students had a frustrating time reading and interpreting excerpts from a primary source. Some students were able to overcome this difficulty by the time they submitted their post-PSP surveys, others were not. This indicates that it is important for instructors to carefully monitor student progress through the PSP to ensure that while some student struggle is productive, excessive student struggle might make for an unpleasant and unproductive experience. In particular, it is important that students receive feedback early and often at the beginning of the PSP when they are first encountering a new didactical style, new mathematics, and/or issues with translation that might introduce complexity to the problem.

7.4 Conclusion

In conclusion, we found that instructors can implement PSPs in vastly different ways, even when they are provided with supporting “Notes to Instructors” that are included as part of the PSP. Monty viewed the PSP as a chance to change the social dynamic of his classroom into a more active learning environment. Carl viewed the PSP as a chance to change the academic material presented to engage students in a cognitively different way, especially through overcoming the challenge of reading and interpreting the PSP. And, the author considered both of these struggles as productive and sought to incorporate them into her classroom which resulted in some of the greatest perceived learning gains on the part of students. This is only a preliminary report and this pilot study provides some evidence that the author might be better positioned to implement their own PSP (in terms of perceived student gains). This relationship could be due to the fact that authorship and preparation of “Notes to Instructors” require the author to think deeply about not only the material, but its implementation in the classroom, which includes the consideration of student engagement and learning. We believe that the relationship between authorship and quality of implementation is grounds for interesting future research, especially with regard to consideration of professional learning experiences for non-author instructors prior to PSP implementation.

REFERENCES

Alpaslan, M., &Haser, Ç. (2015). Selecting and preparing original sources for pre-service mathematics teacher education in Turkey: The preliminary report of a dissertation. In E. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds), *History and Epistemology in Mathematics Education: Proceedings of the Seventh European Summer University*. Copenhagen, Denmark: Danish School of Education, Aarhus University.

Barnett, J. H. (2014). Learning mathematics via primary historical sources: Straight from the source’s mouth, *PRIMUS, 24*(8), 722–736.

Barnett, J. H., Lodder, J., &Pengelley, D. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education, 23*(1), 7–27.

Bernardes, A., & Roque, T. (2018). History of matrices: Promoting commognitive conflicts and encouraging reflection on meta-discursive rules in prospective teachers. In K. M. Clark, T. H. Kjeldsen, S. Schorcht, & C. Tzanakis (Eds.), *Mathematics, Education and History: Towards a harmonious partnership*, ICME-13 Monographs(pp. 209–227). Cham, Switzerland: Springer International Publishing.

Charalambous, C. Y., Panaoura, A., &Philippou, G. (2009). Using the history of mathematics to induce changes in preservice teachers’ beliefs and attitudes: insights from evaluating a teacher education program. *Educational Studies in Mathematics*, *71*(2), 161–180.

Clark, K. M. (2012). History of mathematics: illuminating understanding of school mathematics concepts for prospective mathematics teachers. *Educational Studies in Mathematics*, *81*(1), 67–84.

Clark, K., &Thoo, J. B. (Eds.). (2014). Special issue on the use of history of mathematics to enhance undergraduate mathematics instruction. *PRIMUS, 24*(8), 663–773.

De Guzmán, M. (2007). Enseñanza de las Ciencias y la Matemática. *RevistaIberoamericana de Educación*, *43*, 19–58.

Flagg, M. (2017). *Solving a system of linear equations using ancient Chinese methods.* Retrieved from https://digitalcommons.ursinus.edu/triumphs_linear/1/

Huntley, M. A., & Flores, A. (2010). A history of mathematics course to develop prospective secondary mathematics teachers' knowledge for teaching, *PRIMUS, 20*(7), 603–616.

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., da Silva, C. M. S., & Weeks, C. (2002). The use of original sources in the mathematics classroom. In J. Fauvel& J. van Maanen (Eds.), *History in mathematics education: The ICMI Study* (pp. 291–328). Dordrecht, The Netherlands: Kluwer.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Jankvist, U. T. (2014). On the use of primary sources in the teaching and learning of mathematics. In M. R. Matthews (Ed.), *International handbook of research in history, philosophy and science teaching* (pp. 873–907). Dordrecht, The Netherlands: Springer Science+Business Media.

Katz, V., Jankvist, U. T., Fried, M. N., & Rowlands, S. (Eds.). (2014). Thematic issue: History, philosophy and mathematics education. *Science & Education, 23*(1), 1–250.

Kjeldsen, T. H., &Blomhøj, M. (2012). Beyond motivation: history as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics, 80*(3), 327–349.

Klyve, D., Stemkoski, L., &Tou, E. (2013). *Teaching and research with original sources from the Euler Archive*. Retrieved from http://www.maa.org/press/periodicals/convergence/teaching-and-research-with-original-sources-from-the-euler-archive

Laubenbacher, R., Pengelley, D. &Siddoway, M. (1994). Recovering motivation in mathematics: Teaching with original sources. *Undergraduate Mathematics Education Trends.6*, 1–7.

Liu, P.-H. (2014). When Liu Hui meets Archimedes: Students' epistemological and cultural interpretations of mathematics. *PRIMUS, 24*(8), 710–721.

Marshall, G. (2000). *Using history of mathematics to improve secondary students' attitudes towards mathematics.* Unpublished doctoral dissertation, Illinois State University, Bloomington-Normal, IL.

McBride, C. C., & Rollins, J. H. (1977). The effects of history of mathematics on attitudes toward mathematics of college algebra students. *Journal for Research in Mathematics Education*, *8*(1), 57–61.

Ruch, D. (2014). Creating a project on difference equations with primary sources: Challenges and opportunities. *PRIMUS, 24*(8), 764–773.

Tamulis, A. (2014). Using Albrecht Dürer's*Instruction on Measurement* for projects in the mathematics classroom. *PRIMUS, 24*(8), 753–763.

Appendix

Excerpt from *Solving a System of Equations by Ancient Chinese Methods* (Flagg, 2017)

3 Solving by the Fangcheng Rule

The Fangcheng Rule gives step-by-step instructions for solving problems on a counting board that are equivalent to systems of linear equations. The Rule can be broken into two separate steps. The first step is sometimes called **forward elimination** in modern mathematics, since the procedure uses an equation, starting with the first, to remove variables from the equations that follow. As we will see, the ancient Chinese version of forward elimination will result in the array resembling the shape of a triangle, and this form of the array will be referred to as **triangular form** or **lower triangular form**.

The algorithm for solving the triangular array will be referred to as **substitution** since the simplest method of solution is to start by solving the equation with only one variable, and then substitute the known values of variables into the remaining equation to solve for one variable at a time.

The ancient Chinese method of forward elimination is equivalent to the modern method, yet the procedure for substitution presented in the Fangcheng Rule is different from modern methods. To highlight the similarities and differences in the procedure and the resulting arithmetic, we will separate the elimination and substitution steps.

3.1 The Rule

Before we begin a careful explanation of each step of the Chinese method, read the English translation of the Fangcheng Rule. The term shi in the Fangcheng Rule is the yield of grain or fruit. It refers to the seeds of rice as it comes off the plant and before it is husked. It also means the constant in the equation [Shen et al., 1999, p. 400].

> The Fangcheng Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. Similarly for the middle and left column. Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge. Again multiply the next [and] follow the pivoting [11]. Then use the remainder of the medium grade paddy in the middle column to multiply the left column and pivot. The remainder of the low grade paddy in the left column is the divisor, the entry below is the dividend. The quotient is the yield of low grade paddy. To solve for the medium grade paddy, use the divisor [of the left column] to multiply the shi in the middle column then subtract the value of the low grade paddy. To solve for the top grade paddy also take the divisor to multiply the shi of the right column then subtract the values of the low grade and the medium grade paddy. Divide by the number of bundles of top grade paddy. This is the number of bundles of top grade paddy. The constants are divided by the divisors. Each gives the dou of yield [of one bundle].

[11] The word 'pivot' used here as a modern term the translator chose to use to describe the process, it is not the original Chinese word.

Task 10 Try solving your array for Problem 1 of the *Nine Chapters* using the Fangcheng Rule before reading further. At what step were the instructions unclear?

Did you follow the whole rule? Probably not! Don't be discouraged. The fact that Liu Hui gave detailed commentary to the Fangcheng Rule in his edition from 263 CE indicates that other ancient Chinese mathematicians needed help as well! The original students would have likely read the rule in conjunction with a visual demonstration of the procedure on a counting board. Since we no longer have any record of the visual part of the lesson, it will take a little more effort to translate the verbal description.

In this section we will use Liu Hui's commentary to help us understand how to perform the Fangcheng Rule. We will use modern numerals to aid in understanding, but the layout will correspond to the ancient Chinese format in columns. Read Liu's introduction to the Fangcheng Rule.

The character *cheng* means comparing quantities. Given several different kinds of item, display [the number for] each as a number in an array with the sums (shi) [at the bottom]. Consider [the entries in] each column as rates, 2 items corresponds to a quantity twice, 3 items corresponds to a quantity 3 times, so the number of items is equal to the corresponding [number]. They are laid out in columns [from right to left], [and] therefore called a rectangular array (*fangcheng*). [Entries in each] column are distinct from one another and [these entries] are based on practical examples.

Task 11 What is the significance of Liu's statement that 'Entries in each column are distinct from one another'? Why is it important that the numbers are based on practical examples?

3.2 Forward Elimination in the *Nine Chapters*

We will now work through the Fangcheng Rule one step at a time, using Liu's commentary to help us understand the procedure. Our goal in this section is to reduce the array to triangular form.[12]

Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. Similarly for the middle and left column.

Liu's Commentary: This is the general rule [for arrays]. It is difficult to comprehend in mere words, so we simply use paddy to clarify. Lay down the middle and left column like the right column.

[12]In this section the source text will be labeled as part of the original (Fangcheng) Rule or from Liu's commentary to make the distinction clear.

Task 12 Why is the Fangcheng Rule given using the numbers in a specific example instead of as a general procedure?

The array for Problem 1 is the following:

1	2	3
2	3	2
3	1	1
26	34	39

(9)

The first step of the solution procedure is the following:

Rule: Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge.

Liu's Commentary: The meaning of this rule is: subtract the column with smallest [top entry] repeatedly from the columns with larger [top entries], then the top entry must vanish. With the top entry gone, the column has one item absent. However, if the rates in one column are subtracted [from another column], this does not affect the proportions of the remainders. Eliminating the top entry means omitting one item from the sum (shi). In this way, subtract adjacent columns from one another. Determine whether [the sum is] positive or negative. Then one can obtain the answer. First take top grade paddy in the right column to multiply the middle column. This means homogenizing and uniformizing. To homogenize and uniformize means top grade paddy in the middle column also multiplies the right column. For the sake of simplicity, one omits saying homogenize and uniformize. From the point of view of homogenizing and uniformizing this reasoning is natural.

Lui first uses the terms 'homogenize and uniformize' in Chapter 1 of the *Nine Chapters* after Problem 9 when he is explaining the rules for adding fractions. The term refers to the process of multiplying the numerator and denominator of each fraction by a specific factor in order to create equivalent fractions over a common denominator [Shen et al., 1999, pp. 70-72]. In the case of the Fangcheng Rule, 'homogenizing and uniformizing' refers to multiplying a column by the specified number in order for an entry to cancel when one column is subtracted by another.

We are instructed to first multiply the middle column by 3, the number for the top grade paddy in the right column. To multiply a column by a number means to multiply each entry in that column by the given number. After multiplying the middle column by 3, the middle column now has a larger top number than the right column. Liu tells us to subtract the right column repeatedly from the middle column until the top number vanishes (is zero). Subtracting the right column from the middle column means to replace each entry in the middle column by the difference between that number and the number on the same row of the right column. Modern mathematicians would record a zero if that was the result of subtraction. However, the ancient Chinese did not use a symbol for zero in counting board arithmetic, so we will follow the traditional procedure and leave the space blank.

Task 13 Use the instructions in this piece of the Fangcheng Rule to eliminate the number for the top grade paddy in the middle column of the array for Problem 1.

Liu next explains how to continue the process of elimination.

Liu's Commentary: Again eliminate the first entry in the left column. Again, use the two adjacent columns to eliminate the medium grade paddy.

Task 14 Follow the same procedure Liu outlines.

(a.) Eliminate the 1 in the top row of the left column.

(b.) Follow the elimination procedure for the middle and left column with the medium grade paddy.

At this point your array should resemble a triangle. Today this is called *lower triangular form* or simply *triangular form*.

The array for Problem 1 should now be in lower triangular form. Practice this procedure with the other Problems from the *Nine Chapters* and the modern problem by completing the following tasks.

Task 15 Reduce the array you created in Task 6 for Problem 3 from the *Nine Chapters* to lower triangular form using the elimination procedure the Fangcheng Rule. Note that this problem involves using negative numbers in the elimination process.

Task 16 Reduce the array you created in Task 7 for Problem 7 from the *Nine Chapters* to lower triangular form using the elimination procedure the Fangcheng Rule.

Task 17 Reduce the array for the modern problem created in Task 9 to lower triangular form using the Fangcheng Rule.

The Fangcheng Rule, together with the Sign Rule, were used to solve problems that involved excess and deficit in the form of both positive and negative numbers. However, problems arose from practical examples, so the final solutions were always positive. In triangular form, the column containing only one unknown and a yield should be positive. Consider again Problem 8 in the *Nine Chapters*.

"WHAT IS MATHS WITHOUT A CHALLENGE!"

Reporting on how undergraduate mathematics students in an Irish university worked with original sources in a novel context

Maurice O'REILLY
CASTeL & School of STEM Education, Innovation & Global Studies, Institute of Education, Dublin City University, Ireland
maurice.oreilly@dcu.ie

ABSTRACT

In 2016, the author curated an online exhibition of the mathematical works in the Edward Worth Library in Dublin. Of about 4400 works, dating from 1475 to 1733, in the library, about one hundred are mathematical, the earliest of these printed in 1538. The process by which the exhibition was created involved working with twenty undergraduate students (on a three-year BA programme) who were taking a module in the History of Mathematics. As part of this module, students were required to contribute to the exhibition. Specifically, they were asked to choose one mathematical book from the collection and explore it in whatever way they could.

The background to the exhibition, students' reaction to their initial visit to the library, an overview of how students were supported in their work, and a summary of the eventual shape of the exhibition are all reported previously (at CERME10). In this paper the author reports on how the students coped with unfamiliar and challenging original material (mainly in Latin, but with some books in French and English).

In particular, three stages in the project are highlighted: students' initial work on their author and book, their revised work (an 800-1200 word draft entry for the exhibition) and the necessary editing to ensure coherence of the exhibition itself. For the revision (phase 2) of their work, students were provided with feedback on phase 1 and asked to check out resources on the web and in the (university) library relating to their chosen book, its author and the context of his life.

Inevitably students' approaches to the project varied greatly, as did the level of engagement. It cannot be claimed that the insights reported here might apply to another 'similar' cohort of (mainly) Irish students, and, even less, to students elsewhere. Nonetheless, it is hoped that a detailed presentation of the experience of working with these particular students will contribute to the discourse of working with original sources in an undergraduate programme where the students have some mathematical background but no prior exposure to the history of the subject.

HISTÓRIAS COM CIÊNCIA NA BIBLIOTECA ESCOLAR [(HI)STORIES WITH SCIENCE IN THE SCHOOL LIBRARY]

A project to bring topics of History of Science to secondary schools in Aveiro (Portugal)

Hélder PINTO[1,2,4], Teresa Costa CLAIN[1,3,5]
[1]CIDMA – University of Aveiro, Aveiro, Portugal
[2]ESE – Instituto Piaget, V. N. Gaia, Portugal,
[3]Escola Secundária D. Maria II, Braga, Portugal
[4]hbmpinto1981@gmail.com, [5]tcostacaracol@gmail.com

O binómio de Newton é tão belo como a Vénus de Milo.
O que há é pouca gente para dar por isso.

Newton's Binomial theorem is as beautiful as the Venus de Milo.
The problem is that very few people are able to realize it.

Álvaro de Campos [heteronymous of Fernando Pessoa]

ABSTRACT

In this paper, we intend to present the project *Histórias com Ciência na Biblioteca Escolar* [(Hi)stories with Science in the School Library], a project which brings History of Science topics to high schools in the city of Aveiro (Portugal). This project is a joint project of the University of Aveiro and the *Rede de Bibliotecas Escolares* [School Libraries Network, a program of the Ministry of Education]. The goal of this project is to join the University and High Schools in promoting scientific dissemination, namely the research carried out in the University, as well as extending the scientific culture to a pre-university audience. This project is coordinated by Professor António Andrade (Department of Languages and Cultures) and consists of a cycle of nine conferences in high school libraries by several investigators of the University. The themes of these conferences are very diverse and include areas such as astronomy, medicine, botany, literature, physics and mathematics. These conferences are always centered on episodes of the history of science, for instance, botany in the work of the epic Portuguese poet Camões, the importance of amateur astronomers in the past, the history of syphilis, the importance of the phonograph in Portugal, among others. In this paper, we will present the three mathematical conferences in detail.

1 Introduction

The History of Mathematics Group of the University of Aveiro participates in the project *Histórias com Ciência na Biblioteca Escolar* [(Hi)stories with Science in the School Library] with three conferences: Portuguese Arithmetic Books in the Portuguese Discoveries (Teresa Costa Clain), Real Problems – Historical Mathematical Solutions (Hélder Pinto) and Amateur Mathematicians – passions with limits? Simple problems, big challenges… (Helmuth Malonek).

In Clain's conference, the practical arithmetic treatises written in Portugal during the 16th century are presented. According to the traditional model, these treatises are mathematics texts with a practical vocation and with the objective of answering to the needs of professional training in the commercial world. Commercial arithmetic also became a source and a vector for the dissemination of an important set of problems that would mark the history of knowledge for centuries. In this session, the *Praticad'*

Arismetica (1540) by Ruy Mendes is briefly presented and some problems proposed by Bento Fernandes are analyzed, which illustrate the ludic side of mathematical knowledge at the time.

Pinto's conference shows several historical examples of real problems that were solved using mathematics, for instance, how to measure the distance to a ship in the sea (Thales), how to determine the size of the earth (Eratosthenes), how to measure the height of a mountain (China), how to measure the sun´s altitude (Portuguese instrument) and how to improve calculation with rudimentary calculators. In this conference, as an introduction, several examples that everyone who knows some mathematics are presented, such as percentages (e.g.: taxes and store promotions), measuring areas, "reading" schedules and tables, and so on (for instance, as an example to students, even when using a clock, several mathematical notions are being used: why is 17:15 the same as "a quarter past 5 pm"?).

Malonek's conference was not presented in the ESU-8 meeting because it is the most recent one in this project, only being implemented this year. However, in this paper the situation of the project is updated and this conference is also presented.

Finally, note that this project is not only about the contents presented in the conferences, the major goal is to enhance the scientific culture of high school students. This is a small step to increase, in the future, the audience that can understand the importance of science and mathematics through the history of mankind. Mathematics and science are very useful and beautiful; the major problem is that very few people realize this.

2 The project *Histórias com Ciênciana Biblioteca Escolar*

The University of Aveiro and the Network of School Libraries, a program of the Ministry of Education, met to jointly develop a project of scientific dissemination among high school students, centered on the School Library.

Figure 2.1: The project *logo*.

This project, entitled *[Hi]stories with Science in the School Library*, is made up of nine conferences, stimulated by a group of teachers and researchers with research developed in their areas of knowledge and composed by the following sessions:

- Plants in the Lyric and Epic of Camões. (Jorge Paiva / Functional Ecology Center of the University of Coimbra);

- Amateur astronomers - passions without limits? (Vitor Bonifácio / UA-DF / CIDTFF);
- The unicorn and the bezoar: between myth and reality. (António Andrade / UA-DLC / CLLC);
- The Phonograph, which presented on Sunday constipated and hoarse, presented on Monday clear and clear as never before - The phonograph and its presence in the teaching and popularization of science (19th century). (Isabel Malaquias / UA-DF / CIDTFF);
- Portuguese books of arithmetic in the Discoveries. (Teresa Clain, History of Mathematics Group, CIDMA – UA; D. Maria II High School);
- Real Problems - Historical Mathematical Solutions. (Hélder Pinto, History of Mathematics Group, CIDMA – UA; ESE – Instituto Piaget);
- Changes in history seen from a Chemistry perspective: some examples of molecules that have changed the world. (João Oliveira / UA-DQ / CESAM);
- Madness, medicine and literature (from the Archipathology of Filipe Montalto). (Joana Mestre Costa, UA-ISCA / CLLC);
- Amateur Mathematicians - passions with limits? Simple problems, big challenges… (Helmuth Malonek, History of Mathematics Group, CIDMA - UA).

The coordination of the conference cycle links schools and researchers. In this project, the School Library represents an aggregator of diverse knowledge and resources that could be implicated in the change of educational practices, by supporting learning methods and the curriculum.

The project *[Hi]stories with Science in the School Library* began with high schools in the city center of Aveiro and quickly spread to other schools in the region of Aveiro (and beyond). In these sessions, we try to show that Science is a “passion“, the result of human curiosity in an attempt to discover the world and everything that is part of it, including Mankind. Without knowledge of the origins, we hardly understand the present, hence the importance of this cycle of lectures.

In the following text, the contributions of the History of Mathematics as part of the history of science will be presented, namely the sessions facilitated by the History of Mathematics Group (GHM) of CIDMA of the Department of Mathematics of the University of Aveiro.

2.1 Portuguese Arithmetic Books in the Portuguese Discoveries (Teresa Costa Clain)

In the following subsection, the practical arithmetic treatises written in Portugal in the 16th century will be presented: *Tratado da Pratica d'arismetica* (Nicolas, 1519), *Pratica d'Arismetica* (Mendes, 1540) and *Tratado da Arte de Arismetica* (Fernandes, 1555). The main goal of this presentation is: to show that mathematical themes are present in the arithmetic treatises, as well as to disclose the authors' performance in the face of challenges of the surrounding commercial world and their contributions to the mathematical knowledge of the time, giving them a place in the historiography of Mathematics in Portugal. For this purpose, some axes on which the session will be developed are presented, that is, an approach around a socio-economic framework addressing the following issues: What was the economic, social and geographical context? Who were the authors? What do they tell us about themselves and their motivations?

At this point, the Portuguese empire and commercial expansion is explained. The large deals associated with expansion were not "going the right way". The arithmetic treatises are testimony of a preparation for the daily life of the national merchant in order to open up to the international markets. Territorial expansion was also synonymous with commercial expansion, and new social and natural realities. The arithmetic treatises addressed some business issues and dealt with problems related to the *New World*. One example was the spices business, which often had a loss of merchandise due to poor travelling conditions. Further, the "weight" of the fourth and twentieth tax on goods was added on top of this, as Gaspar Nicolas claims. Two important institutions of the Portuguese commercial network are also referred to: the House of India and Flanders Factory.

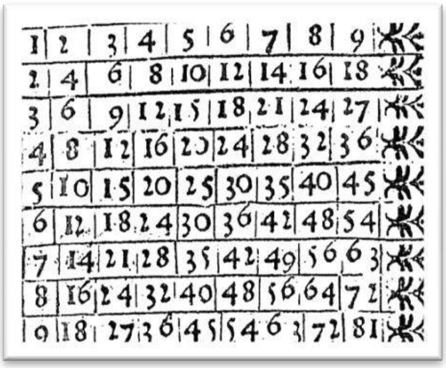
Biographical data on the authors is scarce; however, their motivations are explicit. A reading of the three treatises leads us to believe that the three authors presented similar themes through different methodologies to serve the same objectives. Gaspar Nicolas frequently mentions the questions that were asked in the House of India when he arrived in the city of Lisbon. Ruy Mendes, dealing with the same subjects, does so in a "more academic" manner, by organizing the subjects, considering his taste in problems with numbers and through the method in which he introduces the themes. Bento Fernandes transmitted through his work his experience as a merchant. He wrote a treatise for merchants and gave special emphasis to commercial rules, focusing on the importance of training merchants.

A global approach to these works will be: What is the organization of the treatises? What are the topics covered? New commercial routes, with an enormous amount and variety of products in circulation, as well as the emergence of increasingly complex commercial techniques required that the actors act wisely. Merchants needed to record, calculate earnings and predict risks. For these purposes, arithmetic knowledge was necessary.

In the session, the structure of the work by Ruy Mendesis presented. This is the only one of the three authors that separates the knowledge of basic rules from their applications. In this treatise, there are "raw" mathematics and commercial mathematics.

An approach around the mathematical language of the works is used: What are the concepts covered? What is the scientific vocabulary used? What arithmetic was it? Also, notice that the use and vulgarization of the Indo-Arab numbers transform the notion of arithmetic and allowed for the dissemination of written calculation.

To begin, basic calculations and basic operations are presented, such as the Lattice method of multiplication according to the Table of Multiplication.



1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

Figure 2.2: Multiplication Table by Ruy Mendes (f. 13)

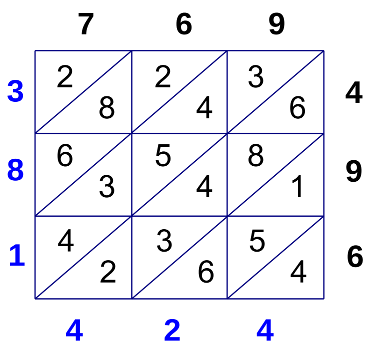


Figure 2.3: Lattice method of multiplication by Gaspar Nicolas (769 x 496 = 381424)

The entire division by *galera* method (Galley division) is also introduced.

	0	2					
	3	5	6				
0	4	7	8				
1	2	3	2	9			
9	**8**	**7**	**6**	**5**	(2	2	8
4	**3**	**2**	2	2			
	4	3	3				
		4					

Figure 2.4: An example of Galley division.

An example is presented by choosing **98765** as the dividend and **432** as the divider. The Galley division algorithm is considered difficult; however, this activity is proposed to the students who attended the session. The algorithm is outlined in the following intermediate steps by interpreting figure 2.4:

$$987 - 2 \times 432 = 123$$
$$1236 - 2 \times 432 = 1236 - 2 \times (400 + 30 + 2) = 1236 - (800 + 60 + 4) = (1236 - 800) - (60 + 4) =$$
$$376 - 4 = 372$$
$$3725 - 8 \times 432 = 3725 - 8 \times (400 + 30 + 2) = (3725 - 3200) - (240 + 16) =$$
$$(525 - 240) + 16 = 285 - 16 = 269$$

The students know the theme of "sequences" and in this context, a problem of "walks" is proposed, where several steps need to be fulfilled:

-Understanding the problem in the original text;

-Studying and comprehending the author's solution;

-Solving the same problem using knowledge acquired in school.

Here, the problem at hand is outlined: consider two men walking, which can be described by two arithmetic progressions. One of the men walks 9 miles each day and the other walks 1 mile more than the previous day. The author wants to know when they meet for the first time. The author's solution: duplicate 9 and take 1 away, we have 17.

In current language, the equation can be solved with a second degree $9n = \frac{n(n+1)}{2}$. To "avoid" the resolution of the two-degree equation, Nicolas proposed the equivalent expression $9 = \frac{n+1}{2}$, where $n = 9 \times 2 - 1 = 17$(Nicolas, 1963, f. 52 v).

Classical themes, such as progressions, and square and cubic roots, though disconnected from the mercantile world, have taken a prominent place in the Portuguese treatises. These themes are associated with calculating practices. Moreover, they present rough sketches of processes similar to those associated with mathematical thinking.

In the lecture, local characteristics are linked to the treatises: How have the themes presented been adapted to Portuguese commerce? What are the specific rules in Portuguese commerce?

The main goal of the treatises was essentially a commercial application of the developed methodologies. Among the classic rules considered, such as companies and leagues, there are particular Portuguese rules of commerce, such as the fourth and twentieth tax rule, and the rule of the account of Flanders. The fourth and twentieth tax rule established a model for calculating the payable tax for goods from the East by applying the rule of threes (80---23---a (to the tax) and 80---57---b (to the merchant)). This tax was an important source of revenue for the kingdom and corresponded to 28.75% of earnings. The account of Flanders was a conversion rule. Through these models, a mathematical understanding of the issues can be reached, as referred in Almeida (Almeida, 1994).

Nicolas, Mendes and Fernandes present the problems of the fourth and twentieth rule by addressing three steps to calculate the tax, see (Clain, 2016, p.352): a quarter of the merchandise, the twentieth of three-quarters of the remainder and finally what the merchant will keep after the taxes have been paid. The presence of this trichotomy in the statements for the practice of the fourth and twentieth rule can be observed.

On the rule of companies, the students are presented with problems similar to(Mendes, 1540, f. 71):

> Three merchants, Pedro, Luys, and Andre created a company in which Pedro invested 56 *cruzados*, Luys 78 and Andre 85. They earned a hundred *tostões*, how should they distribute the gains between the three of them?

With the students, the relation of the gains is established through the model

$$\frac{g_1}{i_1} = \frac{g_2}{i_2} = \frac{g_3}{i_3} = \frac{100}{219}$$

and we have, for example, Pedro's gains in this company

$$\frac{g_1}{56} = \frac{100}{219} \Leftrightarrow g_1 = \frac{100 \times 56}{219}$$

In the scope of proportional division, some problems of a playful nature, such as the classic inheritance problem, is presented:

> A man was about to die and his wife was pregnant, and he made a will in this way: he said that he left 600 *cruzados* in money, which he ordered to be distributed in this way: in case the woman had a male child, her son had 2/3 of the 600 *cruzados* and his wife 1/3. If his wife had a daughter, the daughter received 1/3 and the wife received 2/3. After the man died, the wife had a daughter and a son. How are the 600 *cruzados* to be distributed among the family? (Fernandes, 1555, f. 101v)

In the session, the problem is solved in current mathematical language, through a system of three equations and three unknowns (*x* – son; *y* – mother and *z* – daughter):

$$\begin{cases} x + y + z = 600 \\ x = 2y \\ y = 2z \end{cases}$$

These treatises have a large number of problems. The mathematical contents are found in the solutions presented by the authors. Through some stories told in the library and the examples presented, the students perceive that they are currently studying subjects that are found in the 16th century treatises, although the mathematical language is different. Symbolic language was not part of the 16th century, but mathematical thinking was. We leave the testimony of the students of the LH2 11th grade class:

> The class considered that the lecture was very enriching, constructive and appropriate to our course since it displayed mathematics in the historical context, more specifically, in the Discoveries. It allowed us to acquire a perception of how calculations were performed at that time and how much it has since evolved. The fact that we put into practice some of the processes used at the time of the Discoveries, made the speech more dynamic, interesting and useful. Given the vastness of the theme, we felt it was a short time since there was much more to be addressed. In short, the class considered that this kind of activity should be implemented in the classes more often.

2.2 Real Problems – Historical Mathematical Solutions (Hélder Pinto)

In this session, Pinto begins to sensitize the students to everyday life mathematics. The speaker intends to express the following message: mathematics are more present in our life than we imagine and, sometimes, those mathematics are very simple. The topics covered are diverse and offer different perspectives. Some examples are described:

- Hours and clocks: the sexagesimal system that everyone knows, fractions $\frac{1}{2}$, $\frac{1}{4}$ (half an hour, a quarter), the multiplication table of the number 5 (that "appear" in the minutes clock pointer) and Modular arithmetic (12h / 24h) are presented. In the conference, the high number of divisors of 60 (1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60) are referred to, which constitutes a possible reason for the appearance of this numbering system. This session is an occasion to review some aspects of number theory that are outside the curriculum of high school education in Portugal.
- Percentages (bank loans, sales, taxes, etc.): in Portugal, the Languages and Literature students take the Applied Mathematics to Social Sciences course, where financial models, among others, are studied. In this course, students are faced with financial calculations necessary in the everyday life of a taxpayer (state taxes on supermarket purchases, property tax, etc.). The students attending the Science and Technology course are not familiar with these topics since they arenot in the course´s curriculum. However, it is important for students to have an education in finance. A special focus on the use of percentages is addressed to calculate bank loans, sales and taxes. This knowledge fosters a practice for citizenship, avoiding unrealistic spending situations, and promoting consistent financial education.

After the first part of the conference, we explain that the utility of mathematics can be possibly found in many topics in the history of mathematics. Specifically, remark that the geometry of measurement is present in the mathematics curricula in Portugal and students are familiar with the basics of trigonometry. So, to introduce topics of the history of Mathematics, it is pertinent, above all, to show that mathematics has always been present, over time, in solving everyday life problems, such as construction, navigation, among others (see, for instance, Swetz (1994) and Katz (2000) for activities and to understand the importance of using History in teaching). In this context, several themes are presented:

- Measure the distance to a ship at sea (*Thales of Miletus*): in this part, different possibilities on how this measurement could have been done at the time are shown using only similarity or equality of triangles.

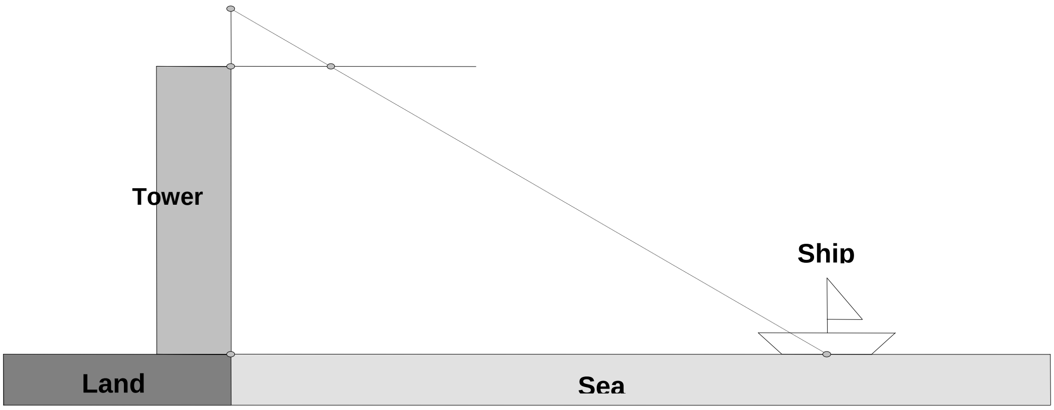


Figure 2.5: One possibility using similarity of triangles.

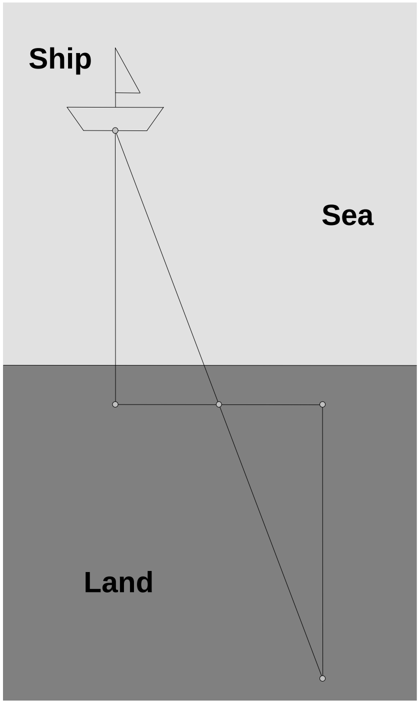


Figure 2.6: Another possibility using equality of triangles

- The height of a pyramid (*Thales of Miletus*): the situation with proportions is presented but it is also pointed out that the easier solution is to wait for the time of the day when all objects have shadows “perfectly equal” to themselves…

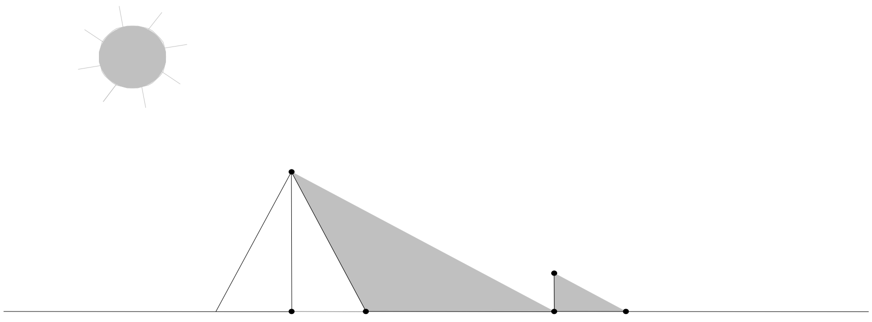


Figure 2.7: A representation of a pyramid and a vertical stick with their shadows

- Determination of the earth´s meridian length (*Eratosthenes)*: how amazing that it was possible to do this measurement with absolutely no modern technology just using mathematics!

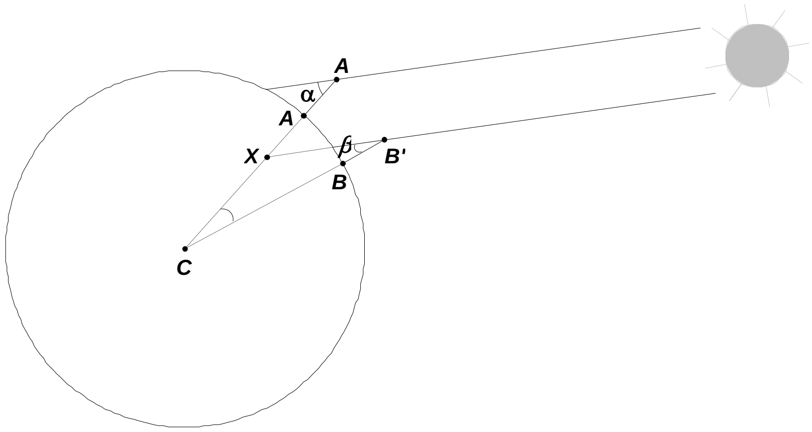


Figure 2.8: Picture shown to students to explain the Eratosthenes method

In another part of the conference, Napier rods and Genaille-Lucas rulers are presented in detail. We explain how these ancient instruments of calculation work (suitable only for multiplications) and highlight that the second instrument is an ingenuous improvement of the previous one (the second rulers are easier and quicker to work with). The aim in this part is to show that mathematics, like other sciences, is the consequence of the effort of several people and has evolved overtime. For a description of these instruments and how to use them in the classroom, see Pinto (2009) and (2010); for more information about these calculation methods see Seaquist, Seshaiyer & Crowley (2005).

Figure 2.9: Examples of these rulers used in classroom

Finally, the shadow instrument of Pedro Nunes is shown in this conference. Pedro Nunes (1502-1578) was a mathematician, cosmographer and professor at the University (the only one in Portugal in that time). Nunes lived in the period of the Portuguese Discoveries and had many contributions to the introduction of rigor in geometry and mathematics in 16th century Portuguese culture.

In his works, Pedro Nunes suggested several instruments that he imagined were useful for astronomical navigation used during the Discoveries. At stake was the measurement of the height of the Sun and other stars, namely the Polar Star. Using this measure, pilots could calculate latitude, which was very important in astronomical navigation; as stated in the conference, it was like the “GPS” of thetime. Among his creations, there are three that have survived and heavily contributed to the progress of scientific instrumentation: the *Nónio*, the Nautical Ring and the Instrument of Shadows, of which several examples are known. Among these, the instrument of the shadows is highlighted, which was a modest instrument, similar to a solar clock, but with a very ingenious innovation that made it possible to directly measure height through the shadows projected by the Sun.

The Instrument of Shadows was constructed like this: the base was a plate, usually square, where a circle was inscribed and a tangent to that circle was drawn. Mounted on the circle was a board similar to a style or gnomon, as in solar clocks. This plate had the shape of a rectangular isosceles triangle, with the legs´ length equal to the radius of the circle. The triangle had a hut resting on the radius of the circle that touched the tangent line, as shown in figure 2.10. A diameter parallel to the tangent was marked in the circle and the circumference from 0° to 90° was drawn in the directions from the diameter to the point of the tangent. To measure the height of the sun, one started by placing the base of the instrument horizontally. This base was then rotated until the edge of the shadow of the triangle coincided with the tangent line. The height of the star was directly “read” in the graduated circle. However, remark that this instrument was not used in real life because it was very difficult to have a completely horizontal surface athigh sea…

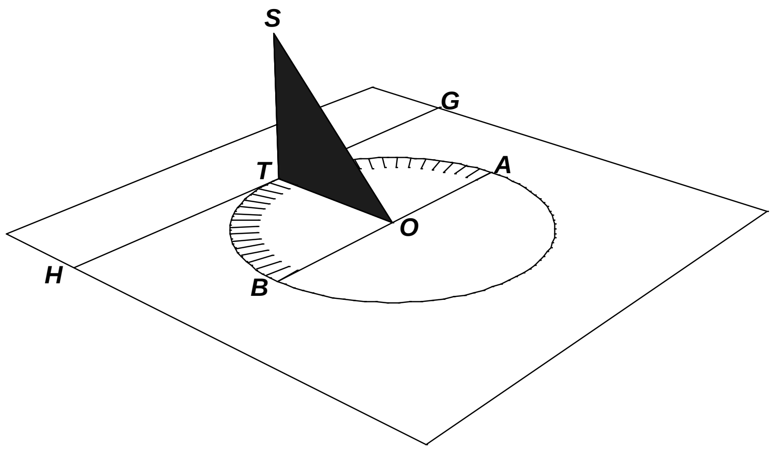


Figure 2.10: Representation of the Instrument of Shadows

The geometric principle of this instrument is very easy to understand: the sun’s rays make an angle with the horizontal plane that corresponds to the angular height of the star. When placing the plate to match the shadow with the tangent line, this angle is transferred to the horizontal plate, where it is directly “read”. The geometric demonstration is very basic and only use two mathematical results that students already know (see figure 2.11): the angles SS’T and TS’O are equal because the triangles [SS’T] and [TS’O] are equal (they have one common side as [TS’], [ST] and [TO] have the same length because of the instrument’s construction and both triangles are rectangle; the LAL criteria is used); the angles TS’O and XOA are equal because they are alternate angles between parallel lines. For more detailed information about this instrument, see Crato (2003).

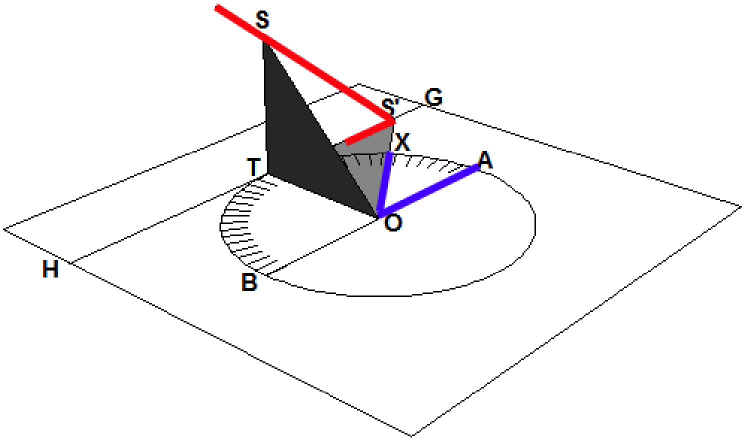


Figure 2.11: The correct position of the shadow in the instrument

The Instrument of Shadows is easily introduced in the classroom because it is very easy to construct replicas using only paper and cardboard (figure 2.12). For detailed explanations about how to construct these replicas and how to create activities with this instrument in the classroom, see Pinto (2009) and (2010).

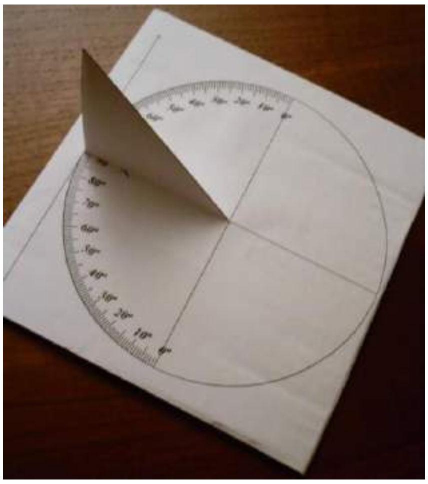


Figure 2.12: A replica of the instrument made with paper and cardboard

Pinto's lecture aims to motivate the learner to the practices based on the construction and use of simple instruments. In addition, Socio-Economic Sciences and Humanities course students are familiar with the theme of the Discoveries through the History of Portugal classes. Pedro Nunes is a heavily mentioned figure in the literature and his important contributions to the scientific knowledge are presented.

The students of the SE1 10th grade class left their opinion about this session:

> Professor Hélder Pinto showed that Mathematics is present in everything, from a simple clock to the most advanced spacecraft. In times when knowledge and technology were more rudimentary, mathematics was used to solve real problems, which today are considered substantially simpler. The class was surprised by the fact that certain measures of interest, such as the earth's radius,

> were very similar to the values obtained using more modern and rigorous processes. The lecture was very enriching and allowed us to realize that since the earliest times Man has always tried to perceive the world around him and to interact with it. We would like to participate in more activities of this nature, since it gave us a different perception of mathematical knowledge.

2.3 Amateur Mathematicians - passions with limits? Simple problems, big challenges… (Helmuth Malonek)

In this lecture, Malonek always starts by talking about his taste for mathematics, emphasizing the difference between amateur mathematicians and professional mathematicians. He introduces and explores several examples of problems developed by amateur mathematicians. Among others, the "friendly numbers"and the famous Goldbach conjecture, which is one of the oldest unresolved problems in mathematics, which says "any even number greater than 2 can be represented by the sum of two prime numbers". It is known that computer-based techniques have already confirmed this conjecture; however, the actual mathematical demonstration has not yet been presented. During the lecture, Malonek interacts with the students by telling some stories about his experience as a mathematician, raising their interest. Furthermore, the school library, where the lecture occurs, fosters a taste for reading. Malonek presents the book of Apostolos Doxiadis *The Uncle Petros and the Goldbach Conjecture*, a bestseller of the year 2001 (Doxiadis, 2001). During this session some chapters of this book are addressed, the main topics of the book are highlighted, i.e., the passion for studies, the discovery of mathematics, learning persistence, and the importance of a pedagogical practice based on trial and error. At the end of the conference, the students were invited to give their opinion on the session (The session was held in the library of the High School of Caldas das Taipas with the students of the SE2 10th grade class). We quote a student:

> It was a very enjoyable session, where we had the privilege of being able to listen to an experienced and well-known mathematician and above all, to learn. We all became aware of the reason for this theme and also had the opportunity to learn a little more about various mathematicians who, while amateurs, made a valuable contribution to the development of mathematics.

3 Final Remarks

The project is now in a consolidation process. The number of conferences available to schools is increasing and each speaker goes, on average, to three or four schools each year. The majority of the students that attend to conferences are, typically, 16-18 years old but, in some cases, the attendants were younger (13-15 years old). Overall, this is a very low cost project (each school only provides, sometimes, transportation to the lecturers) and each school chooses the best way of implementing this project: in some schools, the conferences were given along the school year (usually on the same week day, typically two each month); in others, the conferences were given in special weeks, for instance, during "science weeks", integrated in other initiatives of the schools. In the figure below, we show that the three mathematical conferences were given on the same dayin the High School of Caldas de Taipas, integrated in the Math day of the *Semana de Ciência e Tecnologia* [Science and Technology Week].

Figure 3.1: Photo in upper left corner: from left to right, Hélder Pinto, Teresa Clain, Helmuth Malonek and Fernanda Carvalho (library coordinator of the highschool, Escola Secundária de Caldas das Taipas). Photo in the lower right corner: from left to right, Teresa Clain, Gorete Branco (coordinator of the math department of the highschool, Escola Secundária de Caldas das Taipas), Helmuth Malonek and Hélder Pinto. The lectures are announced on posters. In this case, by Americo Costa of the same highschool (the first three images of the bottom line).

With these mathematical conferences, we intend to change the students' perspective about mathematics and humanize the discipline. In the end, we hope that the history of mathematics can help students understand the following:

> Mathematics contains some very subtle devices that serve not only to satisfy those who are intrigued by mathematical problems but also to help with all practical and mechanical endeavors and to lessen men's labors. (Descartes, 1637, p.7).

Acknowledgments. This work was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project UID/MAT/04106/2019.

REFERENCES

Almeida, A. (1994). *Aritmética como descrição do real (1519-1679).* Lisboa: Imprensa Nacional, Casa da Moeda.

Clain, T. C. (2016). Les règles d'un quarte et un vingtième et des comptes de Flandre comme modélisation du réel. In L. Radford, F. Furinghetti & T. Hausberg (Eds.), *HPM 2016: Proceedings of the 2016 ICME Satellite Meeting* (pp. 351-358). Montpellier: IREM de Montpellier.

Crato, N. (2003). O Instrumento de Sombras. In *Centro Virtual Camões – Ciência em Portugal, Personagens e*

Episódios. Retrieved from http://cvc.instituto-camoes.pt/ciencia/e32.html, accessed November 8, 2018).

Descartes, R. (1637). *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences*. Leyden.

Doxiadis, A. (2001). *Uncle Petros and Goldbach's Conjecture*. New York: Bloomsbury.

Fernandes, B. (1555). *Tratado da Arte de Arismetica*. Porto: Germão Galharde.

Katz, V. (Ed.) (2000). *Using History to Teach Mathematics: An International Perspective*. MAA Notes, vol. 51.Washington, DC: The Mathematical Association of America.

Mendes, R. (1540). *Pratica darismetica nouamente agora composta pelo licenciado Ruy Mendez: na qual se descraram por boa ordem e craro estilo as quatorze especias darte darismetica .&. as sete dellas por numeros inteyros e as outras sete por numeros quebrados: e assi mesmo trinta e cinco regras da dita arte muito sortil e breue e craramente decraradas*. Lisboa: Germão Galharde.

Nicolas, G. (1519 [1963]). *Tratado da Pratica d'Arismetica* [*facsmile*]. Portugal: Livraria Civilização.

Pinto, H. (2009). *História da Matemática na Sala de Aula*. Lisboa: Associação Ludus.

Pinto, H. (2010). The History of Mathematics in the classroom – some activities. In É. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the Sixth European Summer University – ESU 6* (pp. 245-257). Wien: Holzhausen Verlag.

Seaquist, C., Seshaiyer, P., Crowley, D. (2005). Calculation Across Cultures and History. *Texas College Mathematics Journal*, *1*, 15-31.

Swetz, F. (1994). *Learning Activities from the History of Mathematics*. Portland: J. Weston Walch Publisher.

JORDAN’S ISOMORPHISM CONCEPT IN THE WORK “TRAITÉ DES SUBSTITUTIONS ET DES ÉQUATIONS ALGÉBRIQUES”

Erika ZUBILLAGA-GUERRERO[1,3], María Teresa GONZÁLEZ-ASTUDILLO[2,4], Flor Monserrat RODRÍGUEZ-VÁSQUEZ[1,5]

[1]GuerreroAutonomousUniversity, Av. Lázaro Cárdenas s/n, Col. Haciendita, 39087, Chilpancingo de los Bravo, Guerrero, Mexico

[2]Universityof Salamanca, Paseo de Canalejas, 169, 37008, Salamanca, Spain

[3]eguerrero@uagro.mx, [4]maite@usal.es, [5]flor.rodriguez@uagro.mx

ABSTRACT

This paper discusses the concept of isomorphism from the analysis of Camille Jordan’s book “Traité des substitutions et des equations algébriques”. Methodologically, we resort to qualitative text analysis (Kuckartz, 2014), considering three levels of analysis: reference card of the work, context and purpose of the work and of the author, and presentation and treatment of the group isomorphism concept. The results show that Jordan explicitly introduced the concepts of holoedric (total) and meriedric (partial) isomorphism in the context of substitutions groups. Thus, Jordan used the term isomorphism to designate what is currently known as bijective homomorphism (isomorphism) and surjective homomorphism (epimorphism), distinguishing them with the terms holoédrique and mériédrique, respectively. This distinction suggests the possibility of a correspondence or transmission of some properties from one group to another, recognizing the usefulness of this concept due to the similarity of the properties presented by groups that are isomorphic.

1 Introduction

Over the past four decades there has been considerable interest and growth in research in Mathematics Education towards the line of historical research (Fauvel & van Maanen, 2000; Matthews, 2014; Clark, Kjeldsen, Schorcht, Tzanakis, & Wang, 2016). Two areas of study can currently be distinguished in relation to the contributions of history in this field (differentiated themes in the International Congress on Mathematical Education [ICME]): the history of the teaching and the learning of mathematics; and the role of the history of mathematics in mathematics education. The first focuses on the history of Mathematics Education due to its development as a scientific discipline, while the second emphasizes the importance of integrating historical and epistemological issues in the teaching and learning of mathematics.

Clark et al. (2016) point out that “mathematical knowledge is determined not only by the circumstances in which it becomes a deductively structured theory, but also by the procedures that originally led or may lead to it” (p. 136). However, the teaching of mathematics usually presents the final products of years of mathematicians’ work and does not consider the process of constructing those products, which would otherwise contribute to the appropriation of the meaning of mathematical concepts. In this regard, in the teaching and learning of mathematics, the integration of historical and epistemological aspects can favor the understanding of specific parts (e.g., concepts, theorems) of mathematics, and also lead to a deep awareness of the discipline itself (Clark et al., 2016).

Various researchers (e.g., Fauvel & van Maanen, 2000; Jankvist, 2009; Furinghetti, 2019) have provided arguments in favor of the use of history indicating the ways in which it can be used in the teaching and learning of mathematics. For example, Jankvist (2009) proposes two categories of arguments for the use of history, as a *tool* and as a *goal*. In the first category, history plays an important role as an auxiliary or supportive means of teaching and learning the in-issues of mathematics: mathematical concepts, theories, methods, algorithms, among others. In the second category, history as a *goal*, it is stated that although history can have a positive side effect of supporting the learning of mathematics, the main purpose of its use is not this one; it concerns the teaching of meta-issues of mathematics as a scientific discipline and its history.

We consider the use of history as a *tool* that can enhance a deep understanding of mathematical concepts, seeking to make more explicit the mathematical construction of a specific mathematical concept and other underlying concepts (in-issues) (Jankvist, 2009). Thus, we could see mathematics as something that develops and can be constructed, that is, highlighting the creative side of mathematics rather than the cultural side (Menghini, 2000). Furthermore, "to be interested in the process of knowledge construction requires a history of mathematical ideas and, therefore, an epistemological reflection" (Barbin, 1997, p. 21). Epistemology here refers to the question of the meaning of mathematical concepts and theories, what Barbin refers to as *épistémologie historique* (historical epistemology), considering that what gives meaning to concepts and theories are the problems they solve. In this respect, the historical and epistemological of concepts takes on special relevance.

On the other hand, a way in which historyhas been integrated into teaching and learning (as a tool and as a goal) of mathematics has been using primary (original) sources, that according to the categorization of Jankvist (2009) on how history may/should be used, the information from these materials is usually used as: (1) small extracts in historical epilogues (*illumination* approaches) and (2) readings of original sources or student projects (*modules* approaches). In this paper we consider the use of original sources for a historical and epistemological study in the (3) *history-based* approaches, where the main concern lies in the problems of mathematics, whose interest lies in their learning, and "do not deal with the study of the history of mathematics in a direct manner, but rather in an indirect fashion" (pp. 246-247), that is, the historical information is not directly discussed with the students.In particular, the analysis of an original source is intended to highlight the characteristics of a mathematical concept itself (e.g., its meaning and applications), and the social and cultural aspects that influenced its construction, and from the available information, rescuing episodes or enabling elements that can help students to understand mathematical concepts (Anacona, 2003).

Thus, we situate this work within the second area of study in relation to the contributions of history to mathematical education, namely, the role of the history of mathematics in mathematical education. We propose that a historical and epistemological analysis of concepts based on original sources, reveals the different approaches that mathematicians made around a concept, the difficulties that they had to overcome, the genesis of the concept and how other aspects determined the evolution of this concept until its consideration as we know it today. In this respect, some investigations justify that a historical approach to the concept may favor its understanding (e.g., Anacona, 2003; Furinghetti, 2019).

Specifically, we have carried out a historical and epistemological analysis of the group isomorphism concept that despite the importance of this concept in abstract algebra (Lajoie, 2000), research carried out around their teaching and learning in a first course of abstract algebra has shown difficulties presented by undergraduate students to understand this concept. For example, Larsen (2009) identified difficulties in determining when two groups are *essentially the same* (the same structure whose elements are not necessarily labeled differently), and in the formulating of the formal definition of isomorphism, that usually is given in terms of a bijective function, whereas the operation preservation property does not emerge easily. Leron, Hazzan, & Zazkis (1995), Lajoie (2000, 2001), Weber (2001), and Weber & Alcock (2004) showed difficulties in proving that two groups are isomorphic and in constructing an isomorphism between specific groups. In addition, Lajoie (2000, 2001) identified difficulties: in giving the interpretation experts give, to the idea that isomorphic groups are similar, in considering more than one possible isomorphism between two isomorphic groups, in seeing isomorphism both as an equivalence relation on groups and as a particular correspondence between two isomorphic groups, and in recognizing a usefulness to the isomorphism concept in algebra.

We specifically investigated the following questions:

- What kind of problems were the mathematicians of the 19th century trying to solve, which favored the emergence of the concept of group isomorphism?
- What meanings were attributed to the concept of group isomorphism by 19th century mathematicians?

This document presents the results of the analysis of the book by Camille Jordan (1838-1922): "Traité des substitutions et des equations algébriques", published in 1870 in the sections that refer to the concept of group isomorphism. The presentation of a didactic intervention based on the results of historical and epistemological analysis of this concept in Jordan's work is beyond the scope of this paper.

2 Method

This document studies the work of Jordan (1870), which from a literature review of secondary sources (Wussing, 1984; Kleiner, 2007), was identified as the work where the definition of isomorphism appears for the first time explicitly in the context of substitution groups (what today we would call a permutation of a finite number of letters).

For the analysis of the work we used the qualitative text analysis method (Kuckartz, 2014). Three levels of analysis (González, 2002) were considered, each one of which deepens in the information obtained in the previous one: *reference card of the work*, *context and purpose of the work and of the author*, and *presentation and treatment of the group isomorphism concept* (Table 2.1). The second and third levels have been subdivided into categories, which were constructed in a deductive-inductive way (Kuckartz, 2014).

The first level (*reference card of the work*) corresponds to the information specific to the work, which includes the name of the author, dates of birth and death, title of the work, year, publisher and place of publication analyzed and location of the work.

The second level (*context and purpose of the work and of the author*) provides an overview of the work, contextualizing it so that it can be properly analyzed in terms of the facts that influenced its writing and publication. The resulting categories were the following: (1) *Contextualization of nineteenth century mathematics*, which informs about the historical-cultural moment of the mathematics in which the work was written; (2) *Contextualization of the work,* which provides information about the historical (scientific) moment in which it was produced, the goals or intentions of the author about the work, as well as the innovations introduced in the work; (3) *Characterization of the structure of the work*, to show the extent and distribution of the content and the works considered as reference at the time of writing; (4) *Professional information of the author*, to inform about the institutions where he carried out his main studies, as well as the influences that he received from other mathematicians of the time, to highlight the most important works published by the author and to recognize his link with the mathematical object of interest.

The third level (*presentation and treatment of the group isomorphism concept*) considers the context of application of the concept by the author because it unravels its meaning and scope. In addition, a concept is only one element of a conceptual field that conveys its meaning to it, so a concept will never appear isolated, which was considered in determining how the concept of study was justified and applied (Schubring, 2005). The resulting categories were the following: (1) *Definition*, which are descriptions related to the mathematical concept group isomorphism, in which they are characterized from a theoretical mathematical point of view and which will depend on the context in which it was introduced: classical algebra, number theory, geometry or analysis; (2) *Other concepts involved*, which are the concepts underlying the group isomorphism concept and which are associated with its development; (3) *Types of examples* that the author uses to present the mathematical object group isomorphism; (4) *Applications of the concept*, to identify the importance of the group isomorphism concept from the problems that are solved by this, as well as to identify the scopes and the limitations of the definition.

Table 2.1: Methodology for the analysis of historical mathematical works

Levels of analysis	Categories	Units of analysis
Reference card of the work	Name of the author Date of birth and death of the author Title of the work Year, publisher and place of publication analyzed Location of the work	
Context and purpose of the work and of the author	Contextualization of nineteenth century mathematics	Historical-cultural moment of the mathematics
	Contextualization of the work	Historical moment and place where the work was written General objectives of the work Innovations introduced in the work

	Characterization of the structure of the work	Extent and distribution of the content References
	Professional information of the author	Author’s education Other published works
Presentation and treatment of the group isomorphism concept	Definition	
	Other concepts involved	
	Types of examples	
	Applications of the concept	

3 Analysis and interpretation of Jordan’s work (1870)

With respect to the presentation of the work, the proposed levels of analysis have been considered in an orderly manner: *reference card of the work*, where the data of the analyzed material are established; *context and purpose of the work and of the author*, for which in addition to the information that the analyzed material has been able to provide, it has been necessary to consider secondary sources such as Brechenmacher (2012), Kleiner (2007), Lebesgue (1926), Neumann (1999), Schlimm (2008), Timmermans (2012) and, Wussing (1984), these sources were selected according to two criteria: (1) review of mathematics history books where elements of Jordan’s life and academic work where identified, and (2) review of articles in mathematics history that provided specific information about life, Jordan’s academic work, and where specific content of the Traité was discussed; and finally, *presentation and treatment of the group isomorphism concept*. The development of the exposition of the work is organized according to the levels of analysis and is based on the categories and units of analysis previously established (Table 2.1).

Reference card of the work

Author: Marie-Ennemond- Camille Jordan.
Author’s date of birth and death: 1838-1922.
Title: Traité des substitutions et des équations algébriques.
Year, publisher and place of the edition analyzed:1870, Paris: Gauthier-Villars.
Location of work: Library of Sciences. Mathematics. University of Zaragoza. Spain.
Symbols: MAT-Ant 102; ZGG 404.

Context and purpose of the work and of the author

The French-Italian mathematician Joseph-Louis de Lagrange’s (1736-1813) work in 1770-71 had initiated the study of permutations in connection with the study of the solving of algebraic equations and had influenced, in the first third of the 19th century, the work of other mathematicians such as the Italian Paolo Ruffini (1765-1822) and the Norwegian Niels Henrik Abel (1802-1829), developing elements of the theory of groups of permutations (Kleiner, 2007). It was the French Évariste Galois (1811-1832) who first used the term *group* and distinguished between the general principles of what is now known as Galois Theory and an application of this, namely the solvability of algebraic equations by radicals. Galois’ ideas transmitted from his writings were not understood and assimilated by his contemporaries. The main works of Galois were published after his death for the first time in 1846 by Joseph Liouville (1809-1882), who like his French compatriots Charles Hermite (1822-1901), Victor Puiseux (1820-1883) and Joseph-Alfred Serret (1819-1885) studied and continued the works of Galois (Wussing, 1984); until they received a complete treatment by Jordan in his “Traité des substitutions et des equations algébriques” (Neumann, 1999).

In the first half of the nineteenth century, the French mathematician Augustin-Louis Cauchy (1789-1857) also made important contributions to the development of theory of groups of permutations, and it is because of Cauchy that this theory has been developed autonomously. In this regard, Kleiner (2007) points out that “before Cauchy, permutations were not an object of independent study but rather a useful device for the investigation of solutions of polynomial equations” (p. 24).

In France, a deep and rapid advance that revealed the wide range of the concept of permutation group through mathematics was achieved by establishing a connection between the Galois and Cauchy approaches (Wussing, 1984). The work of Serret of 1866, the third edition of the“Cours d’Algèbre Supérieure” and two Comments of Jordan on Galois, the first “Commentaire sur le Mémoire de Galois” of 1865, and its continuation “Commentaire sur Galois” of 1869 are examples of this interest of unification; but it would be Jordan’s work “Traité des substitutions et des equations algébriques”, in which the ideas of Galois and Cauchy were finally unified, which had a great influence on the evolution of group theory. Jordan published more than 30 articles on groups in the period of 1860-1880, and “the *Treatise* embodied the substance of most of Jordan’s publications on groups up to that time” (Kleiner, 2007, p. 25).

Camille Jordan was born on January 5, 1838 in Lyon, France. Jordan was admitted to the École Polytechnique in 1855. In 1873 he was appointed *examinateur* at the École and professor of Analysis, replacing Hermite in 1876. Jordan was elected member of the Académie des Sciences after the death of the French Michel Chasles (1793-1880) in 1881. Furthermore, in 1883 he was appointed professor at the Collège de France as Liouville’s successor and in 1885 he assumed the position as director of the *Journal de Mathématiques Pures et Appliquées*, one of the main mathematical research journals of the time (Lebesgue, 1926).

Jordan’s book “Traité des substitutions et des équations algébriques” of 667 pages was published in 1870 by Gauthier-Villars. In the preface the author states that the main goal of the work is “to develop the methods of Galois and compile them into a body of doctrine, showing how easily they solve all the major problems of equation theory” (Jordan, 1870, p. VII, our translation; see note 1). In addition to mentioning the influence of Serret’s work, “Cours d’Algèbre supérieure”, whose study inspired Jordan to contribute to the progress of Algebra, he recognized the contributions of Galois, for the invention of the principles of the Galois theory and the Italian Enrico Betti (1823-1892), for writing an important Memoir, where it was established for the first time the complete series of the theorems of Galois in a rigorous way. Jordan also mentioned the contributions of Germans Leopold Kronecker (1823-1891) and Abel, Hermite, and the Italian Francesco Brioschi (1824-1897) on the Galois groups of certain division problems of elliptic and Abelian functions, as well as the investigations of the German geometers Otto Hesse (1811-1874), Alfred Clebsch (1833-1872), and Ernst Eduard Kummer (1810-1893), the British Arthur Cayley (1821-1895), the Irish George Salmon (1819-1904), and the Swiss Jakob Steiner (1796-1863), who studied various geometrical problems to which Galois' methods can be applied.

The concept of substitutions group and its application not only to the theory of equations but in other areas of contemporary mathematics allowed Jordan to unify in his Traité the results of Galois, Cauchy and other mathematicians (Wussing, 1984; Kleiner, 2007). In general, Jordan's Traité contains sections dedicated to the study of substitution groups, to the Galois theory itself and to the applications of the Galois theory to equations that arise in several areas of mathematics.

Jordan's work (1870) is not a book with pedagogical intent. In fact, as Wussing (1984) indicates: "it was an expression of Jordan's deep desire to bring about a conceptual synthesis of the mathematics of his time. That he tried to achieve such a synthesis by relying on the concept of a permutation group" (p. 160). Also, Kleiner (2007) points out that "his aim was a survey of all of mathematics by areas in which the theory of permutation groups had been applied or seemed likely to be applicable" (p. 25).On the Traité of Camille Jordan one may consult Brechenmacher (2012).

The Traité consists of four books, each divided into chapters. Book I deals with the main notions related to congruences. Book II is divided into two chapters, the first dedicated to the study of substitutions in general and the second to linear substitutions. Book III is made up of four chapters. The first presents the principles of the general theory of equations and the other three contain applications of the Galois theory in algebra, geometry and problems concerning transcendental functions. Finally, in Book IV, divided into seven chapters, Jordan determines the general types of equations solvable by radicals and for each of them he obtains a complete classification system.

It is in Book II of Jordan's Traité entitled "Des substitutions", where the isomorphism concept can be found, and in which Jordan proves a part of the Jordan-Hölder theorem, "one of the fundamental theorems in the theory of groups" (Schlimm, 2008, p. 409).The first chapter deals with generalizations about substitutions and presents a synthesis of previous results from French mathematicians such as Cauchy, Serret, Joseph Bertrand (1822-1900), and Émile Mathieu (1835-1890). Jordan presents concepts (which had previously appeared in his "Commentaire sur Galois" in 1869) such as a substitution, an unit substitution, the product of two substitutions, a group (Jordan uses the term *faisceau* as a synonym for group), the derived group (le groupe dérivé), the order of a group, the degree of a group, the simple and composite groups, and alternating group. About Jordan's work on groups, he considers them as groups acting on sets and uses a generator and relations approach to groups. In Jordan's group definition, closure under multiplication is the sole property required.

> **23**. One gives the *substitution* name to the operation by which a number of things are exchanged than can be assumed to be represented by letters *a*, *b*, …. […]
>
> **27**. One says that a system of substitutions forms a *group* (or a *bundle*) if the product of any two of the substitutions of the system is again a member of that system.

> The various substitutions obtained operating successively whenever we want and, in any order, certain substitutions given A, B, C ... obviously form a group: we will call it the derived group of A, B, C, ..., and we will designate it by the symbol (A, B, C, ...). (Jordan, 1870, pp. 21-22, our translation; see note 2)

Jordan also proves the Lagrange theorem and the Cauchy theorem and introduces the isomorphism concept. In addition, he conducts an important study on transitivity and primitivity for substitution groups. In the second chapter of Book II, Jordan is devoted to the study of the properties of the General and Special linear groups.

Presentation and treatment of the group isomorphism concept

Jordan's Traité is an extensive and complex work. In this paper we do not present a general study of Jordan's group concept, rather, we consider a fragmented reading of the Traité, extracting some interesting elements about the isomorphism concept of this monumental work.

In Book II: Des substitutions, in §V.- Symmétrie des fonctions rationnelles, Jordan addresses the problem of the number of values taken on by a function of *n* variables as a result of their permutation, that it had been the subject of previous study by the same author in his 1860 thesis. Also, Jordan treats the concept of isomorphism.

Jordan imported the terms *holoédrique* (total) and *mériédrique* (partial) from Auguste Bravais' "Études cristallographiques" to substitution groups. "Similarly any two groups of operations applying to any object (movements of solid objects, substitutions of the roots of an equation, etc.) can transmit, in whole or in part, some of their properties" (Timmermans, 2012, p. 45).

Jordan used the word *isomorphism* to refer to what is now known as bijective homomorphism and surjective homomorphism, but distinguished between the two using the terms *l'isomorphisme holoédrique* and *l'isomorphisme mériédrique*, respectively. However, we must keep in mind that Jordan's isomorphism is not seen as maps.

> **67**. A group Γ is said to be *isomorphic* to a group *G* if it is possible to establish between their substitutions a correspondence such that: 1° to each substitution of *G* there corresponds a unique substitution of Γ and to each substitution of Γ one or more substitutions of *G*; 2° to the product of any two substitutions of *G* there corresponds the product of their respective corresponding substitutions.
>
> An isomorphism is said to be *meriedric* if many substitutions of *G* correspond to the same substitution of Γ, and *holoedric* in the opposite case. (Jordan, 1870, p. 56, our translation; see note 3)

Jordan does not present concrete examples of isomorphic groups. However, he explicitly shows the application of the definition of meriedric isomorphism by deducing that if a group has isomorphic groups of this type then it is composite (see note 4), as shown in the following quotation:

> Suppose, to fix the ideas, that G contains *m* substitutions $g_1, ..., g_m$ corresponding to the same substitution γ of group Γ. Let γ' another substitution of Γ, g' a substitution of G

corresponding to it: $g_1^{-1}g'$ will correspond to $\gamma^{-1}\gamma'$, and consequently each of the *m* substitutions $g', g_2 g_1^{-1}g', ..., g_m g_1^{-1}g'$ will correspond to $\gamma\gamma^{-1}\gamma' = \gamma'$. Each substitution of Γ having thus *m* corresponding in G, the order of Γ will be *m*-fold smaller than that of G.

Group Γ contains substitution I. Let $h_1, ..., h_m$ be the corresponding substitutions of G: they form a group to which all the substitutions ofGare permutable. Because if g is one of these, γ is its corresponding: $g^{-1}h_1 g$ has the corresponding $\gamma^{-1}I\gamma = I$: therefore, it belongs to the sequence $h_1, ..., h_m$.

If $m >$ I, the group $(h_1,...,h_m)$ cannot be reduced to the only substitution I: it will be less than G, if we assume that Γ is not reduced to the only substitution I; therefore G will be composite. Hence this conclusion: *The composite groups have only meriedric isomorphs* (not formed exclusivelyby substitution I). (Jordan, 1870, p. 56, our translation; see note 5)

Jordan then poses as a problem the determination of isomorphic groups to a given G group. In the development, Jordan treats the problem by constructing a new group through the values taken by a function of *n* variables as a result of their permutations, which turns out to be transitive (see note 6) and isomorphic to G:

68. PROBLEM. — *Determine the isomorphic groups to a given group* G.

The problem is reduced to determining those groups that are transitive. […]

69. So let's search isomorphic groups at G and transitive. We will see that their determination is reduced to that of the various groups contained in G.

Let $x, x_1, ...$ be the letters that G exchanges between them; H=$(h_1,...,h_n)$ any group contained in G. Substitutions of G can all be put in the form of $h_\alpha g_\beta, g_1,...,g_\beta,...,g_m$ which are appropriately chosen substitutions, the first of which is reduced to unit and whose number *m* equals the proportion of the orders of G and H.

Now let F_1 be any rational function of $x, x_1, ...$, invariable by substitutions H; F_s becomes substitution *s*. The *m* functions $F_1,...,F_{g_m}$ will be transformed into each other by any substitution of G. Indeed, the substitution $h_{\alpha'}g_{\beta'}$ transforms F_{g_β}, for example, into $F_{g_\beta h_{\alpha'} g_{\beta'}}$ .Moreover $g_\beta h_{\alpha'} g_{\beta'}$, belonging to G, can be put in the form $h_{\alpha''}g_{\beta''}$ and $h_{\alpha''}$ does not alter the function F_1 : so F_{g_β} will be transformed into $F_{h_{\alpha''}g_{\beta''}} = F_{g_{\beta''}}$.

Each substitution of G, performed in the functions $F_1, \dots$, is thus equivalent to a certain substitution performed *between* these functions. The latter substitutions obviously form a group Γ, isomorphic to G. This new group will be transitive, the substitutions $g_1, ..., g_m$ make it possible to transform F_1 into any of the functions $F_1,...,F_{g_m}$.

70. We will show reciprocally that any transitive group, isomorphic to G, is identical to one of those we have just formed. […]

Our proposition is thus established. (Jordan, 1870, pp. 56-59, our translation; see note 7)

Finally, in Jordan's treatment of the concepts of isomorphism and isomorphic groups, some theorems involving them can be identified, for example, in section §XI. Groupes isomorphes aux groups linéaires of Book II; as well as in Book III: Des irrationnelles, in Chapter IV: Applications a la théorie des transcendantes in the section § III. Fonctions hyperelliptiques; and in Book IV: De la résolution par radicaux, in the first chapter, Conditions de résolubilité. Below is a theorem taken from Book III.

THEOREM. — *Any group of degree q isisomorphicnot meriedricto a group of the degree* $2^{2k}-1$, *with abelian linear substitutions, where k is the largest integer contained in* $\frac{q-1}{2}$.(Jordan, 1870, pp. 364-365, our translation; see note 8)

4 Discussion

Implicitly, the groups became the object of study in algebra when mathematicians like Lagrange and Ruffini used permutations when dealing with one of the main problems of the eighteenth and nineteenth century in this mathematical domain, the solvabilityof algebraic equations of degree higher than the fourth by radicals (Kleiner, 2007).Abelproved that it was impossible solving the general equation of the fifth degree by radicals and posed the following problems: (1) Constructing all algebraic equations of a given degree that are solvable by radicals. (2) Given an equation, recognizing whether it is soluble by radicals and make this resolution when possible (Lebesgue, 1926).

On the other hand, the term group was usedfor the first time, without being defined, by Galois, who created a theory (Galois theory) considered one of the great achievements of the nineteenth century, and its application to the solvability of equations by radicals. After the publications of Galois' writings by Liouville in 1846, the interaction between the theory of equations and the theory of permutations favored the emergence and consolidation of the concept of a permutation group (Wussing, 1984). Thus, initially, group theory was considered under the aspect of finite group theory of permutations.

The independence of the theory of permutations resulted in the application of the concept of a permutation group outside the theory of algebraic equations, for example, in geometry, analysis, number theory, and mechanics in the works of mathematicians such as Jordan, Serret and Hermite, just to mention a few (Wussing, 1984).In fact, Schlimm (2008) points out that with the publication of Jordan (1870), "the theory of substitution groups was established as an independent tool for the study of algebraic equations" (p. 410).

Jordan (1870) explicitly introduced the concepts of *isomorphism holoédrique* (total) and *mériédrique* (partial), terms which he imported from crystallography, particularly from Études de Auguste Bravais, to his "Traité des substitutions et des equations algébriques" in the context of substitution groups. The distinction between meriedric and holoedric isomorphism presented by Jordan proves that isomorphism was not always bijective (holoedric). In his

Traité, Jordan uses the term *similitude* (similarity) to refer to the properties of isomorphic groups *identiques* (identical), which he acknowledges as a utility of isomorphism. The approach to the definition of Jordan's meriedric isomorphism in 1870 suggests the possibility of a correspondence or transmission of *some* properties from one group to another.

Two groups are isomorphic (similar), if it is possible to establish a biunivocal correspondence among their elements in such a way that the product of the *corresponds* is equal to the *corresponding* product; that is, the operation is preserved. The correspondence Jordan refers to is what is called today group isomorphism.

On the other hand, the usefulness of the notion of isomorphism of substitution groups, according to Jordan (1870), concerns "the similarity of properties that isomorphic groups present between each other. […] therefore in many cases replace the direct consideration of a group by that of any of its isomorphs" (p. 60). Today, the importance of isomorphism is difficult for students to recognize; for example, Lajoie (2000) showed that the ideas, images and conceptions that students assign to group isomorphism do not allow them to understand the importance of this concept in mathematics. Students tend to refer to isomorphic groups with expressions such as *similar* or *equivalent*, and in practice, to determine whether two groups are isomorphic, students tend to rely on the literal interpretation of these words, so they try to find similarities between the operations or elements of the groups.

Finally, an important idea for the design of tasks has been extracted from the analysis of the work; this idea includes asking the students to construct a group of permutations isomorphic to a given group G.

Acknowledgments: We thank Jesús Romero Valencia for helpful discussions.

NOTES

[1] de développer les méthodes de Galois et de les constituer en corps de doctrine, en montrant avec quelle facilité elles permettent de résoudre tous les principaux problèmes de la théorie des équations. (Jordan, 1870, p. VII)

[2] **23**. On donne le nom de *substitution* à l'opération par laquelle on intervertit un certain nombre de choses que l'on peut supposer représentées par des lettres *a*, *b*, …. […]
27. On dira qu'un système de substitutions forme un *groupe* (ou un *faisceau*) si le produit de deux substitutions quelconques du système appartient lui-même au système.
Les diverses substitutions obtenues en opérant successivement tant qu'on voudra et dans un ordre quelconque certaines substitutions données A, B, C,… forment évidemment un groupe : nous l'appellerons le *groupe dérivé de* A, B, C,…, et nous le désignerons par le symbole (A, B, C,…). (Jordan, 1870, pp. 21-22)

[3] **67**. Un groupe Γ est dit *isomorphe* à un autre groupe$G$, si l'on peut établir entre leurs substitutions une correspondance telle : 1° que chaque substitution de G corresponde à une seule substitution de Γ, et chaque substitution de Γ à une ou plusieurs substitutions de$G$; 2° que le produit de deux substitutions quelconques de G corresponde au produit de leurs correspondantes respectives.

L'isomorphisme sera dit *mériédrique*, si plusieurs substitutions de G correspondent à une même substitution de Γ, *holoédrique* dans le cas contraire. (Jordan, 1870, p. 56)

[4] Jordan (1870) points out that: "un groupe G est *simple*, s'il ne contient aucun autre groupe auquel ses substitutions soient permutables, *composé* dans le cas contraire" (p. 41).

[5] Supposons, pour fixer les idées, que G contienne *m* substitutions $g_1, \dots, g_m$ correspondantes à une même substitution γ du groupe Γ. Soient γ' une autre substitution quelconque de Γ, g' une substitution de G qui

lui corresponde : $g_1^{-1}g'$ correspondra à $\gamma^{-1}\gamma'$, et par suite chacune des *m* substitutions $g', g_2 g_1^{-1} g', ..., g_m g_1^{-1} g'$ correspondra à $\gamma\gamma^{-1}\gamma' = \gamma'$. Chaque substitution de Γ ayant ainsi *m* correspondantes dans G, l'ordre de Γ sera *m* fois moindre que celui de G.
Le groupe Γ contient la substitution I. Soient $h_1, ..., h_m$ les substitutions correspondantes de G : elles forment un groupe auquel toutes les substitutions de G sont permutables. Car soient g l'une de ces dernières, γ sa correspondante : $g^{-1}h_1 g$ a pour correspondante $\gamma^{-1} I \gamma = I$: elle appartient donc à la suite $h_1, ..., h_m$.
Si $m > \mathrm{I}$, le groupe $(h_1, ..., h_m)$ ne peut se réduire à la seule substitution I : il sera d'ailleurs moindre que G, si l'on suppose que Γ ne se réduise pas à la seule substitution I ; donc G sera composé. D'où cette conclusion : *Les groupes composés ont seuls des isomorphes mériédriques* (non exclusivement formés de la substitution I). (Jordan, 1870, p. 56)

[6] According to Jordan (1870),"un groupe est *transitif* lorsque, en opérant successivement toutes ses substitutions, on parvient à faire passer une des lettres à la place de l'une quelconque des autres; plus généralement, il sera *n fois transitif* si ses substitutions permettent d'amener simultanément *n* lettres données *a*, *b*, *c*, ... aux places primitivement occupées par *n* autres lettres quelconques *a'*, *b'*, *c'*, …" (p. 29).
[7] **68**. PROBLÈME. — *Déterminer les groupes isomorphes à un groupe donné* G.
Le problème se réduit à déterminer ceux de ces groupes qui sont transitifs. […]
69. Cherchons donc les groupes isomorphes à G et transitifs. Nous allons voir que leur détermination se ramène à celle des divers groupes contenus dans G.
Soient en effet $x, x_1, \ldots$ les lettres que G permute entre elles ; $\mathrm{H} = (h_1, ..., h_n)$ un groupe quelconque contenu dans G. Les substitutions de G peuvent toutes être mises sous la forme $h_\alpha g_\beta, g_1, ..., g_\beta, ..., g_m$ étant des substitutions convenablement choisies, dont la première se réduit à l'unité et dont le nombre *m* est égal au rapport des ordres de G et de H.
Soient maintenant F_1 une fonction rationnelle quelconque de $x, x_1, \ldots$, invariable par les substitutions H ; F_s ce qu'elle devient par la substitution *s*. Les *m* fonctions $\mathrm{F}_1, ..., \mathrm{F}_{g_m}$ seront transformées les unes dans les autres par toute substitution de G. En effet, la substitution $h_{\alpha'} g_{\beta'}$ transforme F_{g_β}, par exemple, en $\mathrm{F}_{g_\beta h_{\alpha'} g_{\beta'}}$. D'ailleurs $g_\beta h_{\alpha'} g_{\beta'}$, appartenant à G, peut être mise sous la forme $h_{\alpha''} g_{\beta''}$ et $h_{\alpha''}$ n'altère pas la fonction F_1 : donc F_{g_β} sera transformée en $\mathrm{F}_{h_{\alpha''} g_{\beta''}} = \mathrm{F}_{g_{\beta''}}$.
Chaque substitution de G, effectuée dans les fonctions $\mathrm{F}_1, \ldots$, équivaut ainsi à une certaine substitution effectuée *entre* ces fonctions. Ces dernières substitutions forment évidemment un groupe Γ, isomorphe à G. Ce nouveau groupe sera transitif, les substitutions $g_1, ..., g_m$ permettant de transformer F_1 en l'une quelconque des fonctions $\mathrm{F}_1, ..., \mathrm{F}_{g_m}$.
70. Nous allons montrer réciproquement que tout groupe transitif, isomorphe à G, est identique à l'un de ceux que nous venons de former. […]
Notre proposition se trouve ainsi établie. (Jordan, 1870, pp. 56-59)

[8] THÉORÈME. — *Un groupe quelconque de degré q est isomorphe sans mériédrie à un groupe de degré* $2^{2k} - 1$, *à substitutions linéaires abéliennes, k étant le plus grand entier contenu dans* $\frac{q-1}{2}$.(Jordan, 1870, pp. 364-365)

REFERENCES

Anacona, M. (2003). La historia de las matemáticas en la educación matemática. *Revista EMA*, *8*(1), 30-46.

Barbin, E. (1997). Histoire des Mathématiques: Pourquoi? Comment? *Bulletin AMQ*, *37*(1), 20-25.

Brechenmacher, F. (2012). Un portrait kaléidoscopique du jeune Camille Jordan. *arXivpreprint arXiv:1205.5339*.

Clark, K., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang, X. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceeding of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (pp. 135-179). Montpellier: IREM de Montpellier.

Fauvel, J., & van Maanen, J. (Eds.) (2000). *History in mathematics education: The ICMI study*. Dordrecht: Kluwer Academic Publishers.

Furinghetti, F. (2019, in print). History and Epistemology in mathematics education. *International Journal of Mathematical Education in Science and Technology*. DOI:10.1080/0020739X.2019.1565454

González, M. T. (2002). *Sistemas simbólicos de representación en la enseñanza del Análisis Matemático: perspectiva histórica acerca de los puntos críticos*. Tesis doctoral. Universidad de Salamanca. España. Disponible en http://hdl.handle.net/10366/22651.

Jankvist, U. T. (2009). A categorization of the “whys” and “hows” of using history in mathematics education. *Educational Studies in Mathematics*, *71*(3), 235-261.

Jordan, C. (1870). *Traité des substitutions et des équations algébriques*. Paris: Gauthier-Villars.

Kleiner, I. (2007). *A history of abstract algebra*. Boston: Birkhäuser.

Kuckartz, U. (2014). *Qualitative Text Analysis*: *A Guide to Methods, Practice and Using Software*. London: Sage publications.

Lajoie, C. (2000). *Difficultés liées aux premiers apprentissages en Théorie des groupes*. Thèse de doctorat. Université Laval, Ste-Foy. Retrieved from http://www.collectionscanada.gc.ca/obj/s4/f2/dsk1/tape3/PQDD_0019/NQ56442.pdf.

Lajoie, C. (2001). Students’ difficulties with the concepts of group, subgroup and group isomorphism. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.): *Proceedings of the 12th ICMI Study Conference*: *The Future of theTeaching and Learning of Algebra*, *vol. 2,* University of Melbourne, Melbourne, Australia, 9-14 December 2001 (pp. 384-391). Melbourne, Australia: University of Melbourne. Retrieved from http://hdl.handle.net/11343/35000.

Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *Journal of Mathematical Behavior*, *28*(2-3), 119-137.

Lebesgue, M. H. (1926). Notice sur la vie et les travaux de Camille Jordan (1838-1922). Lu dans la séance du 4 Juin 1923. En Académie des sciences (France) (Ed.), *Mémoires de l'Académie des sciences de l'Institut de France* (pp. XXXIX-LXVI). Paris: Gauthier-Villars.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics*, *29*(2), 153-174.

Matthews, M. R. (Ed.). (2014). *International handbook of research in history, philosophy and science teaching*. Dordrecht: Springer.

Menghini, M. (2000). On potentialities, limits and risks. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education: the ICMI study* (pp. 86-90). Dordrecht: Kluwer Academic Publishers.

Neumann, P. (1999). What groups were: A study of the development of the axiomatics of group theory. *Bulletin of the Australian Mathematical Society*, *60*(2), 285-301.

Schlimm, D. (2008). On Abstraction and the Importance of Asking the Right Research Questions: Could Jordan have Proved the Jordan-Hölder Theorem? *Erkenntnis*, *68*(3), 409-420.

Schubring, G. (2005). *Conflicts between generalization, rigor, and intuition: Number concepts underlying the development of analysis in 17–19th century France and Germany*. New York: Springer-Verlag.

Timmermans, B. (2012). Prehistory of de Concept of Mathematical Structure: Isomorphism Between Group Theory, Crystallography, and Philosophy. *The Mathematical Intelligencer*, *34*(3), 41-54.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies inMathematics*, *56*(2–3), 209–234.

Wussing, H. (1984). *The Genesis of the Abstract Group Concept: A Contribution to the History of the Origin of Abstract Group Theory*. Massachusetts: The MIT Press.

Mathematics and its relation to science, technology, and the arts: Historical issues and socio-cultural aspects in relation to interdisciplinary teaching and learning

THE ART AND ARCHITECTURE OF MATHEMATICS EDUCATION

A Study in Metaphors

Snezana LAWRENCE
Middlesex University, School of Design Engineering and Mathematics, The Burroughs, Hendon, London NW4 4BT, UK
snezana@mathsisgoodforyou.com

ABSTRACT

This chapter presents the summary of a talk given at the Eighth European Summer University, held in Oslo in 2018. It attempts to show how art, literature, and history, can paint images of mathematics that are not only useful but relevant to learners as they can support their personal development as well as their appreciation of mathematics as a discipline. To achieve this goal, several metaphors about and of mathematics are explored.

1 Every story is a journey

This chapter is based on the talk given in Oslo in 2018, and as such it cannot and will not attempt to cover all that was said and not said. During questions after the talk also, much was discussed which would be unfair to summarise here, and so that will not be attempted.What is presented here is therefore a summary of what was said and at the same time a new piece of work. The aim of the chapter is to paint a picture of how history of mathematics can support learning in a personal way, and in the time in which it is becoming increasingly unclear what visions of the future we share – in Europe as well as the US.

The story that follows is that of a personal journey that I have shared with my students and colleagues, and the ways I used metaphors in mathematics education.

So let me begin. It was an honour to be invited to give the lecture on which this chapter is based, and it came at an important time for me personally. I had been, until the summer of 2018, in mathematics education for more than twenty years in various roles from working as a teacher, to being a teacher instructor and educator. Towards the end of that period, and leading up to this lecture, I have moved from teaching secondary to primary teachers. I also knew at the time that this period itself was coming to an end as I was looking to move to another position teaching students in STEM professions, and further away from educating teachers for reasons that will become imminently clear further bellow. This was therefore, a period in which my sense of purpose and the role(s) I had in mathematics education was moving from educating teachers to a wider educationalremit of using mathematics both as an applied discipline and, more importantly for me, as a tool by which my students can learn to think about and solve greater problems of their lives.

This was, therefore, an opportunityfor me to summarise a couple of decades' long experience and through some examples of my research show lessons that the history of mathematics can teach. To be able to do this in a relatively short time, and manifest the ways in whichmathematics history can help, support, and enhance learners' experience of mathematics education, requires a little bit of an art. Theart in question was therefore decided to rely on an underlying structure of a kind, and the use of some tools. I decided to paint some images of mathematics through metaphors that can be used for learning, and that I used in my work with teachers. Let us then, begin our journey of exploring such images.

1.1 Wise women and men (or paying the homage to mathematics and mathematicians)

A couple of years ago I did some experiments with my colleagues mathematicians: I asked a room fool of them to raise a hand if they were a mathematician. This may sound slightly illogical, but there was a preceding discussion to this experiment that made me realisethat some mathematicians don't consider themselves to be mathematicians. The answer to my question was intriguing – only three people out of about seventy admitted to being so. From this came a little research project (Lawrence, 2016) that showed the disparity of the views of mathematicians from within and without the profession. While pursuing the project, it became apparent to me that, as groups of professionals go, this one has been an incredibly *nice* one to do a research on. *Nice* in this case I use to mean unpresumptuous and unassuming, rather than, for example, superior, or rather I found almost no one in this group having images of themselves of being superior in any way.

It turned out that the status of mathematician was so highly rated by the group, that this was only bestowed by individuals from the group on those who changed the field in a major way or had a calling of a professor. To aim high, and to maintain such a discipline (however within the discipline) was I thought quite *nice*. It did not give much favours to the profession or the view of it from the outside, but it showed true humility.This paper is therefore, an homage to all those who do mathematics professionally, and are oriented towards education, pedagogy, and history, and are not even boasting that they do make a huge difference in the world.How? Let's hope what follows may explain that.

1.2 Desert of mathematical education landscape – where has the sea gone?

In 2004, a national enquiry about took place in England and Wales (Smith, 2004) that would have major consequences for mathematics education in this country. A question was posed to young people from around the country on what words describe their experience of mathematics education best. The two most commonly used words were 'boring' and 'irrelevant' (Lawrence & Ransom, 2011). Following this enquiry, the problem being identified, huge efforts have been made by successive UK governments to ensure this is rectified. Finally, from 2014, a large number of further studies, enquiries, and projects, led to a unified solution: to introduce Mathematics Mastery in state schools, therefore learning from the Chinese mathematics education experiences and successes (Mathematics Mastery).

To use a little metaphor, overnight the mathematics educational landscape became an unknown territory for hundreds of thousands of teachers entering the profession. The new system had no, or very little, history or familiarity with the existing. History of Chinese mathematics and history of its mathematics education is unfortunately, not a well-known field among mathematics teachers and educators in England. But there was a help at hand – a nation-wide network of consultancies and projects started to be funded by the government to introduce the system, a process still on-going across the country. Notwithstanding pluses and minuses of this novel system, it is worthwhile mentioning that schools must buy *into* and *subscribe to* it at a considerable expense (Benson, 2016; 2018).

In this context, it became even more important than before to remind how history can, apart from all other aspects of its benefits to mathematics education (Barbin et al., 2015), give teachers and learners opportunities to find their authentic pathways for learning and development.

The following chapter therefore is meant as a metaphorical travelogue from my last couple of years as a teacher educator in this new landscape of mathematics education.

1.3 Metaphors and their role in the world of education

The role of a metaphor has long been recognised in mathematics education (Latterell & Wilson, 2017; Bishop, 2012; Erdogan et al., 2014; Gibson, 1994; Presmeg, 1997).To present the types and uses of such metaphors, I will first begin with a metaphor about the pillars of wisdom – something I picked when researching the history of geometry in England in the early modern period (Lawrence, 2002). In this tradition, there is a reference to Euclid, for example, and the two pillars of wisdom are mentioned as means through which geometrical knowledge was transferred during and beyond the great flood as described in the Bible (Lawrence, 2002).

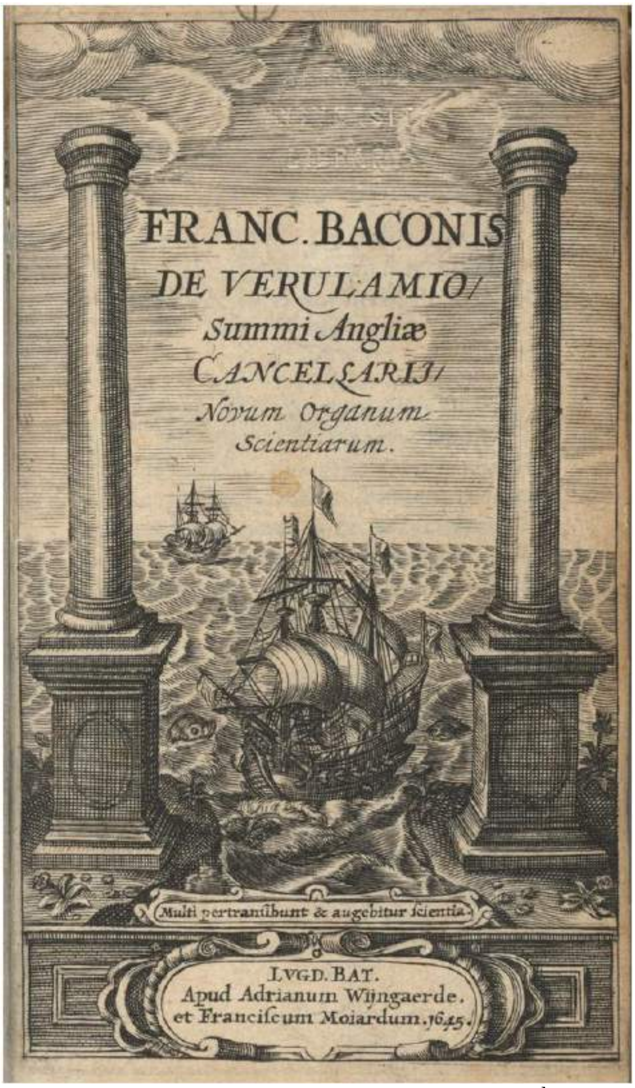


Figure 1.1: Two wise pillars of *Novum Organum* (1645, 2nd edition), London, Francis Bacon

But we can also think of twopillars of wisdom as central symbols of Francis Bacon's (1561-1626) *Novum Organum* (1620) which were about finding the essence of a thing'. This, Bacon suggested, is done by the use of inductive reasoning, which is illustrated by the two mythical pillars of Hercules that stand at the either side of the Straight of Gibraltar, and should be smashed through to open up possibilities to explore new world of exploration.

What relevance is there of a metaphor of the 'pillars of wisdom' to mathematics education? I will explore thismetaphor and use it so we can see how a construction of an edifice that mathematics education can be seen as,could withstand the destructive forces (whatever these may be) and could even be erected in our newly developed metaphorical desert.

2 Solving or resolving: a dichotomy of the current system in mathematics education

In the introduction I mentioned why the history of mathematics is becoming increasingly, in my opinion, more important to mathematics education in the UK. Here I want to identify and name what I believe is going on currently: the moving of the focus from something that teachers have free access to (known sources and resources of mathematics and mathematical cultures) to something that can be done only via an agent (sometimes the state assuming this role) and through controlled exchanges and publishing projects, is a *de facto* process of de-contextualising mathematics education. But how is this detrimental to the outcomes of mathematics education? The question can indeed be discussed for much longer than I intend and am able to here, but we will concentrate on one simple focal point: by de-contextualising mathematics a whole field of appreciation of the discipline, its history, methods, and traditions are lost.

Let us look now at two divergent but perhaps equally important aims of education in any discipline: proficiency and skill on the one hand, and the appreciation of the discipline in question on the other.

At this point it is important to make clear that, in my opinion, appreciation of a discipline could not be the *sole* purpose of education in general, and mathematics education in particular; here I agree with Paul Sally (Sally, 2008). And equally, the other 'pillar of wisdom' of the discipline, the teaching of appreciation, cannot be replaced by a process which leads to acquisition of skills and utility only. Arguments have been made around this question for centuries: let us then through them examine whether we can say that two aims, skill and appreciation, are equally important in mathematics education. In other words, can we use the metaphor of two 'pillars of wisdom' for mathematics education in this way?

2.1 Arguments for appreciation of mathematics in education

Dewey's argument for appreciation is this: aesthetic experiences are ones that promote growth of an individual, and this goal must be central to the conception of the goals of education (Dewey, 1934). Wilson's example is similarly related to the creation of community of individuals, the cornerstone of any peaceful and successful society: he points that the universally shared aesthetic preferences contribute to building of such societies (Wilson, 1998). Žižek sheds some further light on thisparticular aspect (building of society) from a political and ideological stance of a philosopher: and says that there is no movement without a poet (Žižek, 2001). This is, Žižek says, because poetry ‘continues to give us artistic pleasure long after its historical context disappeared; this universal appeal is based in its very ideological function of enabling us to abstract from our concrete ideological-political constellation by way of taking refuge in the “universal” content’(Žižek, 2001: 2). So these are views that relate in general to the importance of aesthetics in societies.

We can further analyse and see how this can actually work in educational practice. Le Lionnais, for example, gives us a classification scheme of facts and methods and points us to another vantage point from which we can see mathematics education.Facts can be seen as artefacts, methods as proofs or techniques, and our choice of study is based on subjectivity of aesthetic experience (Le Lionnais, 1948).

If we think of how important it is that our pupils become individuals who can think for themselves too, Pinker and Dissanayake’s argument can be useful. It goes to tell us of the consequential dimension of human behaviour, the way in which ‘‘making special’’ or ‘‘registering enabling acts’’ forms the basis for aesthetic sensibility as a necessary tool in education (Pinker, 1997; Dissanayake, 1992).

2.2 Aesthetics and elitism

For a book relating to mathematics education though, Poincaré’s view must certainly be somewhere there at the top of the important views. It is quite well known that he promoted the view that the aesthetic dimension of mathematics is the defining characteristic of mathematics, and not the logical. This was met with the charges of elitism as most learners will not have the opportunity to do (real) mathematics, whilst others will not be capable of experiencing the aesthetics of it as they may not know enough of it (Papert, 1978). But does this charge mean that we may not try? And not try to teach the teachers how to find such aesthetic dimension in mathematics either?

2.3 Value of aesthetics in mathematics education

To go back in history, I found a couple of views which were interesting. Alexander Baumgarten (Baumgarten, 1735) emphasised aesthetics as the experience of art, and this subsequently as a means of knowing. In the same vein, Ehrenfels (Ehrenfels, 1897) explored the meaning of value in education. Value, he said, is attributed to something, contingent through desiring it: “We desire things not because we comprehend some ineffable quality ‘value’ in them but we ascribe value to them because we desire them” (Ehrenfels, 1897, p. 44). Certainly this type of thinking has made consumer society flourish. But let us stick to mathematics education: there are many papers and books that have been written on this, too numerous to mention, bar the ones given to begin investigating such claims (Sinclair, 2008; Betts, 2003).

The aesthetics is generally then accepted as an important aspect of social life, and of educational experience. How can then mathematics education flourish and inspire if it is stripped of such experience and searching for it in various examples rich in different contexts, some of which would surely be historical?

3 Metaphors about mathematics

In this section I will first look at some examples of metaphors that can be used in mathematics education.

There are a few metaphors which are often used for mathematics. An example is ‘mathematics is a universal language’. Like all metaphors, this is partly true, depending on the possible interpretation. The quote is a take on the original by Galileo:

> This book (i.e. the universe) is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth…(Galileo, 1623, p. 178).

Then there are other famous metaphors about mathematics: ‘mathematics is a universal language’ is about what mathematics is or can be. ‘Structures are the weapons of the mathematician’ is a metaphor also often used that tells us about how that is possible, a famous quote by Bourbaki (Bourbaki, unknown date). Luckily we have a particular example of this ‘weapon’ in Euclid: “Reductio ad absurdum, which Euclid loved so much, is one of a mathematician’s finest weapons. It is a far finer gambit than any chess play: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game” (Hardy, 1941, p. 12). In the same book, Hardy of course offers a few more metaphors – but perhaps in our context of mathematics and aesthetics, it is most pertinent to mention his view that “mathematician’s patterns, like the painter’s or the poet’s must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics” (Hardy, 1941, p. 84).

In a very short space, we have therefore seen three metaphors: about what mathematics can be described as being, about what tools it has, and about how mathematicians act and use mathematical tools.

3.1 Grounding metaphors

The three previous metaphors about mathematics require some experience and knowledge of what mathematics is like, but are in effect not too complex. They provide the ground upon which mathematics can be further explored through the study of metaphors related to its pursuit and history. Let me however give an example of a complex metaphor – one which can only be understood by having some knowledge of mathematics (in this case geometry).

3.2 A hope from Flatland

The value of such a complex metaphor is that it can provide a basis upon which to explore the mathematical content in education. An example that I have often used in work with teachers was that of *Flatland* (Abbott, 1880; Lawrence, 2015). This mathematical novella uses mathematics to provide opportunity to consider other aspects of life of an individual and society: “I exist in the hope that these memoirs… may find their way to the minds of humanity in Some Dimensions, and may stir up a race of rebels who shall refuse to be confined to limited Dimensionality” (Abbott, 1884, p. 3).

Now this is a complex metaphor – to understand and appreciate Abbot’s text, both an understanding and knowledge of context and history are needed, as well as the understanding of some mathematical ideas (Lawrence, 2015). In fact Abbott used mathematics here to give a message about society and equality.

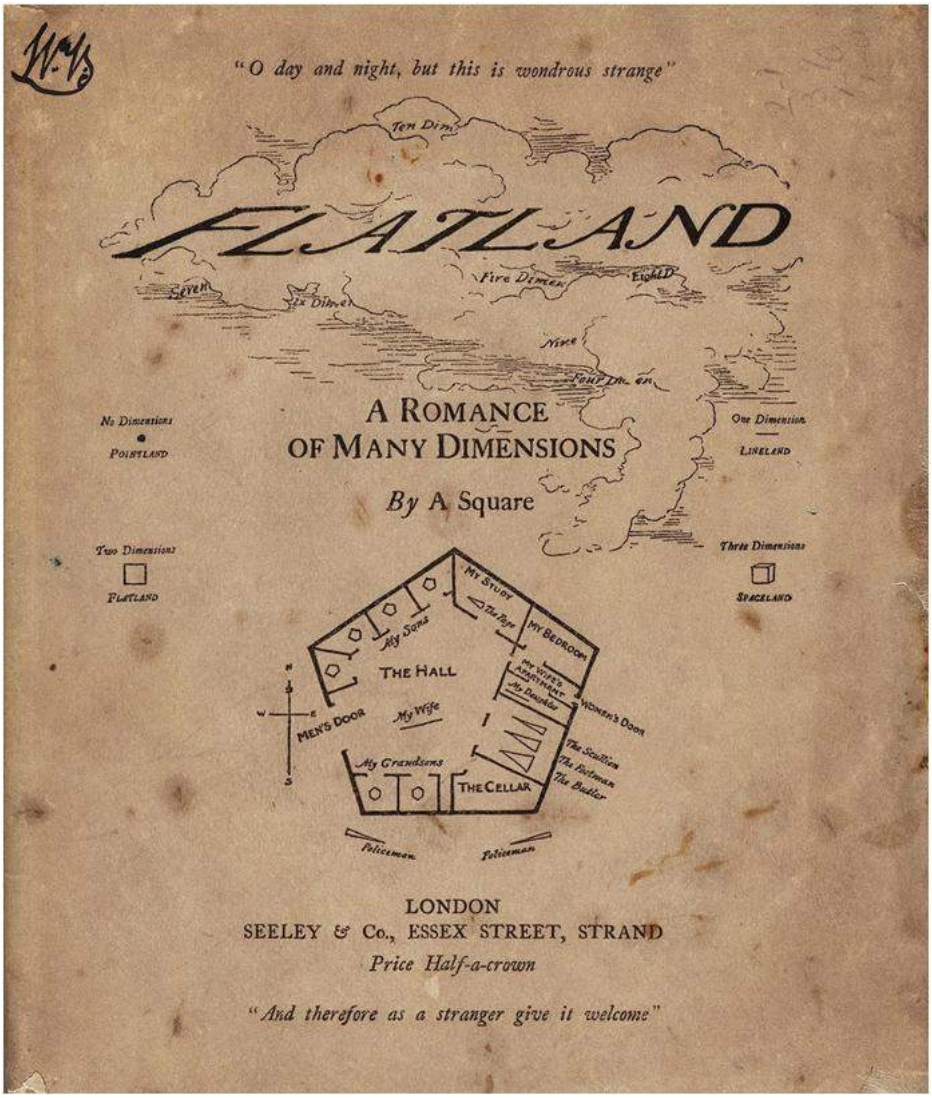


Figure 3.1: Front page of Flatland, 1884, a novel by Abbott. Abbott is a good example of complex metaphor using mathematics to promote a message about equality between people.

Such metaphors are complex to recognise, deconstruct, and use in mathematics education. They are a good pathway to introduce the learning the mathematics and its history which are needed in order to understand the meaning of the text. This in turn offers opportunities not only for the learning of our discipline, but for examining and considering the values of societies of which we are part.

These are then the types of metaphors that I have used and developed in my work with teachers in the past decade. Whilst I have here explained what *kinds* of metaphors can be used, I will now explore two of my projects to show *how* that can be done.

4 Mathematics is a space

One of my projects was about searching for an aesthetically based experience based on a historical piece of art, which would lead students towards learning some new mathematics. The aim of this particular approach was to take my students away from an increasing pressure of (often and increasingly, state-) prescribed approaches and the utilitarian view of mathematics, and to remind them of our ‘second pillar of wisdom’, where they would explore how to teach appreciation of beauty contained in mathematics, and how to attempt to facilitate such experience in aprocess of learning mathematics.

I have describedmy project on Euclid already in other publications (Lawrence, 2016a; Lawrence 2018), so will not go into details about this project, but will need to describe what has been the new lawyer to this existing base.

4.1 Nature of mathematical space

The image on which my *Euclid* project was based was Raphael’s *The School of Athens*, a fresco in the Vatican, painted between 1509-1511. The image portrays most famous scholars from the Greek tradition: Aristotle and Plato walk together and are surrounded by some of the finest minds of their civilisation, finding themselves in an imaginary space in which they discuss their work.

Figure 4.1: Rafael’s *The Academy of Athens*, Vatican, 1509 - 1511

In this painting, I first concentrated (Lawrence, 2018) on the figure of Euclid, and his demonstration of a theorem on a little black-board. From there, the journey began to find which theorem exactly was being demonstrated. This led to consider mathematics from the Euclid’s *Elements* and the late antiquity contributions by pseudo-Euclid, believed to be mathematicians from late antiquity, Hypiscles and Isidore of Miletus (Lawrence, 2018). The journey further continued by the examination of Proclus’ claim that the *Elements* may have been writtenin order to present all mathematics one would need to understand regular polyhedra:

> Euclid belonged to the persuasion of Plato and was at home in this philosophy; and this is why he thought the goal of the *Elements* as a whole to be the construction of the so-called Platonic figures (Proclus, p. 57; Rosán, 1949: pp. 11-35; Sanders, 1990).

Although that was refuted by Heath, (Heath, 1956: I, p.2) it is an interesting and possibly truthful view, and one that is important in the contact of the discovery and re-discovery of semi-regular polyedra in Antiquity and then during the Italian and European Renaissance (Field, 1997).

In this, probably best described as the second leg of the research journey, we worked on Platonic and Archimedean solids and conversed about artists and mathematicians such as Piero della Francesca, Dürer, Luca Pacioli, Barbaro, and later Kepler. An exciting result was to realise that the rhombicuboctahedron, the translucent object subtended from the ceiling behind Pacioli, painted (c. 1500) by de'Barbari (c. 1460-1516) was actually for the first time visualised, ordrawn and recorded, in this image. This is interesting as there have been some charges of plagiarism, by Pacioli, of Piero della Francesca (Stakhov, 2009). The proof that this may not quite be so was to be found (not in the pudding this time, but) in this painting of Pacioli. The project ended up by considering mathematical knowledge needed for such a painting to be made, and the importance of the supremacy of invention in mathematics. This in turn is an important aspect of both the history of mathematics and the understanding of the concept of originality in mathematics, and how these can be explored in education.

Figure 4.2: *Luca Pacioli*, portrait of the famous mathematician by Jacopo de'Barbari, 1500.

One can take a view that the nature of mathematical space is such that it has to be described and constructed in some way. A step towards that would be to be able to construct regular and semi-regular three-dimensional solids which can populate such a space (Sanders, 1990). And that is where the lesson of that particular project, designed for secondary (therefore specialist) mathematics teachers rested. So what could possibly be done further with the primary teachers?

4.2 Euclid and his friends

Whilst I could go into the details of mathematics contained in the original image (Euclid's diagram on Rafael's painting) and subsequent research with specialist mathematics teachers, the primary mathematics teachers in training could not access that mathematical content due to their lack of pre-entrance qualifications in mathematics. The majority of my work with them was therefore to a) enable them in their functional skills in mathematics and b) venture with them into the world of mathematics history to offer views of mathematics that would draw them into further study. In this way, I hope, they would be able to build their own subject-knowledge in years to come. The focus therefore changed slightly. Whilst with the secondary teachers my focus was on the history of mathematics, by the way of investigating mathematical examples, with the primary teachers my focus was shifted onto the method by which they could learn some mathematics through historical research.

The work therefore focused on re-creating the journey I had undertaken in previous couple of years with secondary teachers as if creating a travelogue for the primary school teachers. Avoiding mathematical detail, there was much left from the research that could still be explored. The emphasis turned from understanding mathematics to understanding the relationships between those who create mathematics, whether they are contemporaries or somehow grouped by their interest or their 'pursuit of truth' through mathematics.

An interesting outcome was the exploration of other sources as a consequence. One student mentioned at some point that Pacioli, Rafael, Dürer, could be put in a friendship'group'. There ensued my re-reading of the *Euclid and His Modern Rivals* with the students of this 'year group', which then made a connection between literature (easily accessed by primary school teachers) and mathematics. I of course refer to the author of both *Euclid and His Modern Rivals,* as well as novels *Alice in Wonderland*, and *Alice Through the Looking-Glass*, Charles Dodgson (1832-1898). From here on we could explore many aspects of correspondence between mathematics, logic, and literature. We followedsome of Dodgson's dialogues that are topical in the current global rise of fake news, and imagined where and when we could produce arguments such as these:

> "Alice laughed: 'There's no use trying," she said; "one can't believe impossible things.'
> 'I dare say you haven't had much practice,' said the Queen. 'When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast'" (Dodgson, 1871: 24).

And so Euclid was able to again take us from 300BC to 2018AD, and from finding solutions to mathematical problems to those of contemporary every-day life. What new lessons could we find in Euclid's *Elements,* and how can the history of mathematics support general goals of education? I will come back to this point in the conclusion.

4. 3 How is mathematics 'space'?

As metaphors go, 'mathematics is a space' is not perhaps the most 'catchy' one. But notwithstanding its lack of appeal at first sight, let us now examine what it refers to. When I used that metaphor with my students towards the end of the work with them in both cases when the project run (with secondary and then primary teachers) I was referring to mathematics that transcended cultures, eras, and personal connections and that had made possible the creation of a universal space. This space, which may well be imagined as *The Academy of Athens*, is where ideas can be exchanged, tested, and discussed, just as in the Rafael's painting.

Against this backdrop we may look at some images of modern Academies in the UK. If you conduct search on the Internet for such images (one local to me, Thomas Clarkson Academy for example), you will see fantastic new architecture being made to house the new state-private enterprises that schools are becoming, which are changing not only the landscape of school architecture, but the landscape of educational values too. Within that landscape however, there is not much change in terms of providing 'space' for children, teachers, their supporters – sponsors and business partners of such academies, and parents, to *discuss* mathematics, challenge each others' views, or simply try to agree on their aims of education.

Mathematics education, I propose, and ideas that are contained within mathematics, require space, and offer space, to do just that: discuss, challenge, find common ground. But not, it seems, just now (West and Wolfe, 2018).

My final thought on that for this chapter is that the metaphor is a valuable tool for all involved in mathematics education. *Mathematics is a Space* as a metaphor points to an ideal space where people from different times and backgrounds can test validity of their ideas, so that they are empowered as well as educated, to make their own decisions based on facts and consensus.

5 Mathematics is an ocean

The metaphor of mathematics as an ocean is linked to a project I did some years ago with the students working on the Shakespeare's *Tempest*, as part of the cross-curricular project in the schools in South-West of England. The aim of the project was to enable students, teachers in training, to build framework of cross-curricular explorations, look beyond the curriculum, and design activities that would inspire their pupils of middle-school age(years nine to twelve). The original hurdle was to find how to connect *Tempest* with mathematics. *Tempest is* a story about how a disposed duke, with his daughter, finds himself in a tempest (tempest itself here being a metaphor for both a real-life event and a political turmoil) but manages to return to his rightful place through dialogue, forgiveness, and some magic (Shakespeare, 1611).

Figure 5.1: Prospero disarming Ferdinant, an illustration of the scene from the *Tempest*, Shakespeare's play with same name. Henry W. Banbury and Francesco Bartolozzi, 1793, London.

5.1 Dee's one book for which he is most famous

The problem of how to connect such a story with mathematics for the middle school children and their teachers was luckily short lived. It transpired that John Dee (1527-1608) was most probably the inspiration for the central character of the play, Prospero (Grant 1976, Lawrence, 2011). Dee was a mathematician, astronomer, philosopher, and advisor and code maker for Queen Elizabeth I. Most famous for his introduction to the first English-language translation of Euclid's *Elements* by Henry Billingsley (d. 1606), Dee's most famous work is his *Mathematical Preface*. This preface is a treasure-trove to discuss, with mathematics teachers and students alike, a classification of mathematical sciences, including some 'jabberwocky' disciplines such as 'traumaturgike' (Lawrence, 2011). Discussions about views of whether such disciplines belong to contemporary mathematics can shine a light on the understanding of what constitutes mathematics, and its importance for a society as well as individual.

Dee spent six years journeying Eastern Europe, visiting Poland, and Bohemia, and staying in Prague for some years. He was not only an avid traveller but also had a special interest in scientific exploration of the New World. He gave instruction and advice to pilots and navigators of the English travellers conducting exploratory voyages to North America, and even conjured angels to help them on their voyage. The angels also suggested to Dee to name the new colony 'Atlantis', which, as we know, was not to be.

During his travel to Poland in 1583 however, his library was ransacked, with most books and objects stolen, never to be found in one place again.

5.2 Bibliotheca Mortlacensis

It is known that Dee had amassed a large library of books, manuscripts and curious objects. We know this as he compiled the inventory of this collection of about 3000 items before he set on his journey, and listed books on mathematics, mechanics, astronomy, optics, military and naval sciences, philosophy and magic. Among these books was a copy of Apollonius of Perga's (late 3rd – early 2nd centuries BC) *Conics* (Apollonij, 1537).

This book had an interesting page written in Dee's hand facing the title-page: it was Dee's attempt at classifying the mathematical sciences, a sketch that would serve as a basis for his later diagram in *Mathematical Preface*, his *Ground plat*.

5.3 To survive a tempest

This book survived the destruction of Dee's library (at the time of writing it is on sale with a New York antiquary for half a million US dollars). Strangely enough, Dee had a premonition that his library will be destroyed and wrote about this in his diary:

> I dremed that I was dead, and after my bowels were taken out I walked and talked with diverse, and among other the Lord Thresorer who was come to my house to burn my bokes… (Dee, 1582, as reported by Sherman, 1995, p. 51).

When talking about the burning of his books, Dee's dream, and ransacking of his library, a reality that was to befall him (and by all accounts contribute to his demise), is a point to note in its own right. The 'burning of the books' as a metaphor in dangerous times may turn into action. Dee was aware of this possibility - and we would do well to also remember and teach this - the power of knowledge contained in such books and the desire to destroy it by those who fear knowledge is an archetypal energy and event that is repeated throughout history.

There is of course a happy ending for the mentioned copy of Apollonius' *Conics*. The book was bought by John Winthrop the Younger, in the year when he was to leave for America to join his father, the first governor of the Massachusetts Bay Colony, in 1631. Through Winthrop Jr., Dee's work spread to Puritan New England (Woodward, 2011; Calis, et al. 2018), and hence some of Dee's library survived the tempest of his life and was passed on to the new land of which he cared so much and which he wished to call Atlantis.

5.4 How is mathematics an ocean?

History teaches us not only of the successes, but also of lost and burnt libraries and books, of suffering and tempests. But mathematics survives such events, and knowing of such historical instances can give us not only hope but structure. Like in Shakespeare's story about tempest, it is good to know that keeping a clear mind can get us out of a big trouble. In mathematics education, teaching how to do that is a valuable lesson.

6 Conclusion

To complete my chapter, I chose to mention finally our pillars of wisdom again. In this case though, I want to mention my namesake's famous *Seven Pillars of Wisdom* Lawrence, 1935). For anyone who has read this book, the first question that comes after reading close to some seven hundred pages, must surely be: where are these seven pillars to be found? There really are no pillars to speak of that are explicitly mentioned in Lawrence's book.

Could we really describe the pillars of wisdom that mathematics is (using a metaphor), and for that matter what is then mathematics education? I would suggest that we could certainly try, and by using metaphors, history, art, and literature, we can attempt to paint a picture of mathematics that is enchanting, inspiring, and gives hope to our young people. It also offers some concrete instructions, through mathematical history as well as through learning how to solve mathematical problems, on how to use mathematical thinking and make it a safe place, even a harbour in the troubled, even tempestuous times. Such education gives examples and practices the mind on how to conduct a conversation in a mathematical and logical space that the peoples of our era share with many who have explored similar ideas before us. We can paint the pictures of mathematics as a space to think and live in, and through which one can communicate and make friendships.

One of the threads that connect these metaphors that I did not have the time to go into detail here but had discussed in my talk in Oslo, is also one that is about the overarching purpose of education, and of mathematics. Love of a discipline, of learning, and of truth:

> *I loved you, so I drew these tides of men into my hands*
> *And wrote my will across the sky in stars*
> *To earn you Freedom, the seven-pillaried worthy house,*
> *That your eyes might be shining for me…* (Lawrence, 1935: dedication).

REFERENCES

Apollonii Pergei (1537). *Conica (translated in Latin by* Giovanni Battista Memo). Venice: Bernardino Bindoni.

Barbin, É., Jankvist, U. T., & Kjeldsen, T. H. (Eds.) (2015). *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU*. Copenhagen: Danish School of Education, Aarhus University.

Baumgarten, A. G. (1735). *Meditationes philosophicae de nonnullis ad poema pertinentibus*.Venezia (original text and translation by K. Aschenbrenner & W. B. Holther, University of California Press, Berkley, 1954).

Benson, D. (2016). Maths mastery: The Key to pedagogical liberation? In *Mathematics Teaching*, vol 254. Association of Teachers of Mathematics. (https://derby.openrepository.com/handle/10545/622916)

Benson, D. (2018). *Maths mastery: a blessing or a curse?* https://researchschool.org.uk/derby/blog/guest-blog-maths-mastery-a-blessing-or-a-curse, (accessed 3rd April 2019).

Bishop, J. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom.*Journal for Research in Mathematics Education*, *43*(1), 34–74.

Calis, R., Clark, F., Flow, C., Grafton, A., McMahon, M., & Rampling, J. M. (2018). Passing The Book: Cultures of Reading in the Winthrop Family, 1580–1730. *Past & Present*, *241*(1), 69–141.

Dewey, J. (1934). *Art as Experience*. New York: Penguin.

Dissanayake, E. (1992). *Homo Aestheticus: Where Art Comes from and Why*. Seattle: University of Washington Press.

Dodgson, C. L. (1871).*Alice Through the Looking-Glass*. London: Macmillan.

Ehrenfels, C. (1897). *System der Wertlehre*. Leipzig: Reisland.

Erdogan, A., Yazlik, O. D., and Erdik, C. (2014). Mathematics teacher candidates' metaphors about the concept of ‘mathematics’. *International Journal of Education in Mathematics, Science and Technology*, *2*(4), 289–299.

Field, J. V. (1997). Rediscovering the Archimedean Polyhedra: Pieroella Francesca, Luca Pacioli, Leonardo da Vinci, Albrecht Durer, Daniele Barbaro, and Johannes Kepler. *Arch. Hist. Exact Sci. 50*, 241–289.

Galileo, G. (1623). *Il Saggiatore.* Rome: Giacomo Mascardi. English translation: *The Assayer* (translated by S. Drake in *Discoveries and opinions of Galileo*(pp. 230-280). New York: Anchor Books, Doubleday (first English translation by Thomas Salusbury in *Mathematical Collections and translations*, London 1661).

Gibson, H. (1994). Math is like a used car. In D. Buerk (Ed.) *Empowering students by promoting active learning in mathematics: Teachers speak to teachers*(pp. 7–12). Reston, VA: National Council of Teachers of Mathematics.

Hardy G. H. (1941). *A Mathematician's Apology*. London: Cambridge University Press.

Heath, T. L. (1956). *Euclid: The Thirteen Books of Elements*, 3 volumes. New York: Dover.

Latterell, C. M., & Wilson, J. L. (2017). Metaphors and Mathematical Identity: Math is Like a Tornado in Kansas. *Journal of Humanistic Mathematics*, *7*(1), 46-61.

Lawrence, S. (2002). Geometry of Architecture and Free masonry in 19th century England. PhD thesis, Open University, UK.

Lawrence, S. (2015). Life, architecture, mathematics, and the Fourth Dimension. In *Nexus Network Journal*, *17*, 587–604.

Lawrence, S. (2016). What are we like… In B. Larvor (Ed.), *Mathematical Cultures: the London meetings 2012-2014, Trends in the history of science* (pp. 111-125). London: Birkhauser/Springer.

Lawrence, S. (2016a). The Old Teacher Euclid and his Science in the Art of Finding One's Mathematical Voice. *MENON* ©online *Journal of Educational Research*, 2nd Thematic Issue - The Use of History of Mathematics in Mathematics Education (pp. 146-158).

Lawrence, S. (2018). Learning New Mathematics from Old – Euclid's Art after Bath. In K. M. Clark, T. H. Kjeldsen, S. Schorcht, & C. Tzanakis (Eds.), *Mathematics, Education and History: Towards a harmonious partnership* (pp. 367-382). Cham: Springer.

Lawrence, T. E. (1935). *Seven Pillars of Wisdom*. Garden City, NY: Doubleday, Doran & Co.

Le Lionnais, F. (1948). *Le grands Courants de la Pensée Mathématique*. Marseille: Éditions des “Cahiers du Sud”. English translation: *Great currents of mathematical thought* (Vol. I, translated by R. A. Hall & H. G. Hermann; Vol. II, translated by C. Pinter, H. Kline). Mineola, N.Y.: Dover, 1971.

Mathematics Mastery (accessed 4th April 2019) https://www.mathematicsmastery.org/

Papert, S. A. (1978). The mathematical unconscious. In J. Wechsler (Ed.), *On Aesthetics in Science*, Cambridge, MA: MIT Press.

Betts, P., & McNaughton, K. (2003). Adding an Aesthetic Image to Mathematics Education. *International Journal for Mathematics Teaching and Learning*. (CIMT - Centre for Improvement of Mathematics Teaching, Plymouth, University of Plymouth), April 2003 http://www.cimt.org.uk/journal/index.htm (accessed 7th August 2019).

Pinker, S. (1997). *How the mind works*. New York: W. W. Norton & Co.

Presmeg, N. C. (1997). Reasoning with metaphors and metonymies in mathematical learning. In L. D. English (Ed.) *Mathematics reasoning: Analogies, metaphors, and images* (pp. 267–280). Mahwah, NJ: Erlbaum.

Proclus. (1970). *A Commentary on the First Book of Euclid's Elements*. Translated by G. R. Morrow. Princeton: Princeton University Press.

Rosán, L. J. (1949). *The Philosophy of Proclus*. New York: Cosmos.

Sally, Jr. P. J. (2008). *Tools of the Trade: Introduction to Advanced Mathematics*. Providence, RI: AMS.

Sanders, P. M. (1990). *The Regular Polyhedra in Renaissance Science and Philosophy*. PhD Thesis, University of London, Warburg Institute.

Sherman, W. H. (1995). *John Dee: the politics of reading and writing in the English Renaissance*. Amherst, MA: University of Massachusetts Press.

Sinclair, N. (2009). Aesthetics as a liberating force in mathematics education. *ZDM Mathematics Education*. *41*(no 1-2), 45-60.

Stakhov, A. P. (2009). *Mathematics and Harmony From Euclid to Contemporary Mathema*tics. London: World Scientific.

West, A., & Wolfe, D. (2018). *Academies, the School System in England, and a Vision for the Future*. LSE Academic Publishing. Available from http://www.lse.ac.uk/social-policy/Assets/Documents/PDF/Research-reports/Academies-Vision-Report.pdf (accessed 12th April 2019).

Wilson, E. O. (1998). *Consilience, The Unity of Knowledge*. New York: Vintage Books.

Woodward, W. W. (2011). *Prospero's America, John Winthrop, Jr., Alchemy, and the Creation of New England Culture*, 1606-1676. Chapel Hill: The University of North Carolina Press.

Žižek, S. (2001).*Repeting Lenin*. Zagreb: Arkzin.

17TH CENTURY FORTIFICATION AND GEOMETRY

A military and mathematical revolution

Frédéric MÉTIN
University of Burgundy, ESPÉ & IREM, Dijon, France
frederic.metin01@u-bourgogne.fr

ABSTRACT

After the French Italian wars, military architects tried to reply to the power of cannons by creating new shapes taking into account the straight lines of cannonballs trajectories: the bastions. Different models were invented by a variety of authors, but none of them gave real reasons for their constructions. The first mathematical proofs were given by Jean Errard de Bar-le-Duc, a military engineer of King Henry IV of France. His works were well received in the Low Countries, where they inspired a new Dutch "manner of fortification" (i.e. shapes with algorithms of construction). Among the authors linked to the Leiden school of engineers, Samuel Marolois was the first to explain the use of trigonometry to compute the distances and prove the shapes to be fitted for defense. This paper compares different manners of constructing bastions and different uses of geometry to be found in the original texts.

1 A short history of "modern fortification"

Fortification is a matter of geometry, as anyone would agree considering the star shaped forts, citadels or cities enclosures that are still visible around the world including Norway, with the famous Akershus fortress in Oslo. All of them are visible from the sky through Google Earth or other satellite imagery services[1]. The reasons for that geometric choice must be sought in the history of war technologies, as it is a response to the progress of artillery in the 15th century. Let's give the floor on this subject to a Dutch writer of the golden age, Matthias Dögen, whose *Architecture militaire moderne ou fortification* (Dögen, 1648) gives one of the best short histories of fortification.

According to Dögen, the oldest usual way to protect people in the cities was to build large stonewalls around them, but there was a huge drawback: the assailants could come under the cover of the walls, when the defenders had to get uncovered to try to repel them. In the Antiquity, architects invented the principle of flanking: they built outer towers to allow defenders to shoot their enemies from behind when these enemies dared to approach the walls. This principle was still in use in the medieval castles with their round towers at the edges of the main walls.

But as Geoffrey Parker showed (Parker, 1988), when the Western world discovered the use of gunpowder, it was a real revolution in attacking as well as in defending and building fortresses. The high medieval towers were ideal targets for the newly melted cannons and their easy remote destruction caused casualties inside the cities without any damage for the assailants. Height had become a tactical disadvantage; bastions were to be born. The bastions are a kind of low pentagonal towers, intended to host the defenders' canons and cannonballs, hence their names *bulwarks* (from the German *Bollwerke*). The

[1] For regular polygons, we recommend travelling by Google Earth in this order: Forte San Pedro, Cebu, (Philippines); Vardøhus, Finnmark (Norway); Fort Belgica, Banda Neira (Indonesia); the citadel of Saint Tropez (France); the Alba Carolina citadel in Alba Iulia (Romania); the cities of Neuf-Brisach (France) and Palmanova (Italy). Finally, the old city of Nicosia (Cyprus) with its eleven bastions might hold the record.

geometric shape of the bastions was generated in such a way as to force the trajectories of gunfire and to reduce damage on the walls, but as we can see on heritage monuments or original maps, there is a variety of sizes and angles according to their creators and time of their conception. The set of consistent shapes, along with their algorithms of constructions and even their justifications, were called *manners of fortification*. Early modern writers distinguish three principal manners at the turn of the 17th century: the Italian, the French and the Dutch[2]. All of them find their origin in the Italian one, but we can question the reasons of their differences and the use of mathematics for the generation and justification of shapes, especially the use of Euclidian geometry. In order to render unto Caesar the things which are Caesar's, we start with the Italian architects and follow their inventions towards North, to France and the Low Countries, where military architecture would find their (temporary) perfection through mathematics. Then we give examples of constructions described in texts designed for officers to study or practice the "new" fortification.

2 The creators of modern fortification

Modern fortification was invented in Italy around the end of the 15th century, during the Italian Wars. Architects like Michelangelo or Leonardo took part to a large movement of research about new shapes of city enclosures. The first inventor of the bastion remains unknown, if he ever existed, but this shape (see for example Figure 3.2) was widely accepted in Italy as a response to the power of cannons. Everywhere in Europe, new bastions would be built in the Italian manner of fortification, undertaken by Italian engineers themselves. Many local authorities had connections with Italy. All the Kings and Dukes, every City Council urged their recently engaged Italian counselors to secure their places with new enclosures *à l'italienne*[3] (Rogers, 1995; Parker, 1988, ch. 1).

Though universally acknowledged as taking its origins in Italy, modern fortification can't be reduced to the presence of bastions at the corners of city walls. As an architectural artefact, the bastion, and more generally the fortified enclosure would know many enhancements, practical as well as theoretical. Before the end of the 17th century, when the manner of Monsieur de Vauban becomes hegemonic, at least two major stages of successive improvement are notable as far as geometry is concerned: in France, initiated by Jean Errard with his *Fortification reduicte en art et demonstrée* (Errard, 1600) then in the Low Countries after the publication of Samuel Marolois's *Fortification* (Marolois, 1615). Let's describe in short the national contexts.

2.1 Italian authors, before 1600

Architects-artists like Leonardo and Michelangelo didn't publish their researches about the shapes of city walls. Before the mid-16th century, only a few books were published (De la Croix, 1963), but we find in Book 6 of Tartaglia's *Quesiti et invention diverse* (Tartaglia, 1546) a first attempt of reflection about the rules every architect should follow in fortifying a city. While admitting not to be a practitioner,

[2] Some authors add a fourth one, the Spanish manner, which can somehow be seen as a variant of the Italian manner.
[3] For a critical point of view on the expression *Italian trace*, see Bragard, 2014.

Tartaglia approaches the problem of fortification with his intellectual tools, and he determines six principles about the shape, size and areas of the cities enclosures.

Tartaglia's followers won't extend his reflection, but mostly publish their methods to draw the star shaped fortresses. The reasons for particular designs are kept secret, giving way to discussion on their concrete fulfillment on the field. Some of the treatises which are published will be translated into foreign languages, mainly in French (for example: Cataneo, 1574; Theti, 1589). In general, these books deal with architecture more than geometry. However at the very end of the century, an interesting controversy occurs between two architects in charge of building the Palmanova fortress, Giulio Savorgnano and Buonaiuto Lorini (La Penna, 1997). Having rather different conceptions about the final shape of Palmanova, both had had to advocate their own views before the Venitian Senators. Lorini published his rules as an appendix of his *Delle fortificationi* (Lorini, 1597), while Savorgnan left his unpublished. But Savorgnan's manuscript was found amongst Galileo's papers, which indicates a certain consideration.

In order to show the significant demand for knowledge on fortification, let's mention the private courses given by the same Galileo in Padova around 1590 (Valleriani, 2015). The manuscript of the course, *Trattato di fortificazione*, begins with many usual techniques of practical geometry, such as the drawing of perpendiculars, parallels, regular polygons, and so on. No doubt that there was a need for a theorization of fortification practices. Finally Jean Errard arrived.

2.2 Jean Errard and the French geometric School on fortification

At the end of the Italian Wars, the conflicts moved from Italy to the North, concentrating on the frontiers of the Spanish Habsburg Empire. Following the same path, many unemployed Italian engineers were recruited here and there in Europe to build new ramparts for fragile cities.

In France for example, when King Francis I ordered the construction of a fortified harbor at the mouth of the river Seine, this task was attributed to Girolamo Bellarmato, an Italian architect who worked in several other cities in France. Along with Castriotto and Marini, Bellarmato was one of the major military architects in France at the time. Till the end of the 16th century, it seems that no French engineers had been able to supervise the creation of important fortresses, and even more so to write something substantial about fortification. But there was a need for an elite corps of French engineers. Little by little, the Italians were replaced by local military architects, and King Henry IV ordered his favorite engineer, Jean Errard de Bar-le-Duc, to write the first French book on Fortification. The title of this book is indicative of its project: *La fortification reduicte en art et demonstrée* (Errard, 1600) meaning that the whole process will be established on a detailed analysis of the needs, then fully described, and finally justified by mathematical demonstrations. In the preface, Errard himself justifies the title (Errard, 1600, p. 1):

> *I dared undertake what every engineer so far hasn't dared or wanted, at least nothing was written about that science. Because the discourses on mechanical things do not deserve this Title, not being here a matter of strokes which for*

someone could succeed by accident, but a matter of Geometrical demonstrations that will give infallible certitude to anyone[4].

As one can read the specificity of Errard's approach lies in his attempt to turn the practices of fortification into a true science. For that purpose, the book starts with a description of all the cannons in use in the French armies, and the conversion of their power into the men's working days which would be necessary to reconstruct the collapsed walls. Four principles, called the *Maxims of fortification*, are then presented, the principal two dealing with the flanked angle, which must be right, and the line of defense, whose length must not exceed the reach of the defender's weapons (see figure 3.6 for a glossary). The algorithms of constructions are given in detail and the different steps are well illustrated. Some places in the North of France were fortified according to Errard's principles, especially the citadel in Amiens, still visible now, except for the two Eastern bastions, which were destroyed to leave space for an enlarged road.

Errard's manner was abandoned shortly after his death in 1610, but his *Fortification demonstrée* kept a reference book for many authors, especially teachers, till the end of the 17th century (Métin, 2016). His legacy was received and prospered by Dutch engineers related to the famous school of engineering at the University of Leiden.

2.3 The Leiden School

During the Eighty Years' War, many cities were besieged, even alternately by one and the other side, namely the Spanish Empire and the Dutch Republic. The young Dutch Republic had to face an experienced enemy, but Maurice of Nassau lacked time, money and trained engineers to fortify towns and protect citizen from the Spanish *furia*. Following the advice of his closest counsellor Simon Stevin, he founded in 1600 the *Duytsche Mathematique*, an engineering training course in theoretical and practical mathematics taught in Dutch (Dijksterhuis, 2017). Stevin had written a treatise on fortification (Stevin, 1594), but apparently no fortresses was built on the field according to the shapes he created.

Of course, fortification practices already existed before the foundation of the *Duytsche Matematique*. Italian engineers such as Paciotto and Marchi worked for the sovereigns of the Spanish Low Countries. But we also find non-Italian mathematics practitioners at work in Antwerp, such as Michel Coignet, a mathematics teacher and an instrument maker (Meskens, 2013) who became the equivalent of Stevin in the court of the Archdukes Albert and Isabelle of Austria, governors of the Habsburg Netherland. Several subsisting manuscripts show that Coignet gave lessons in French, Italian and Spanish. His taught *manners* are revealing of the transition between Errard and the Dutch fortifiers.

Michel Coignet's explanations on the practices of his time and military side are exposed in a French manuscript course on trigonometry (Coignet, 1612). The author gives a first method of fortifying polygons, which deals with lengths and no angles at all, but he is not satisfied with it. Asserting that the flanked angle needs to get a unique value of 90°, he gives a second method starting based on angles, following Errard's view. Meanwhile in the opposite camp, the disciples of Stevin free themselves from the constraint of the right

[4] J'ai osé entreprendre ce que tous les Ingénieurs, jusqu'à présent, n'ont voulu ou osé, au moins n'en paraît-il rien par aucun écrit traitant de cette science. Car les discours des choses mécaniques ne méritent point ce Titre, n'étant ici question des traits, qui à quelqu'un pourraient réussir à l'aventure, mais de démonstrations Géométriques qui donnent à tous assurance infaillible {with modernized spelling}.

angle, thanks to trigonometry. Before 1650, publications will follow one another, more or less inspired by the first of them, Marolois' *Fortification* (Marolois, 1615).

Marolois' book was published at the time he was unsuccessful in applying to the job of professor at Leiden school of engineers. It is the third and last part of a complete mathematical textbook for the use of engineers including geometry, the use of surveying instruments, gauging, trigonometry, perspective and fortification (Marolois, 1616). The *Fortification* is composed of two parts, the first of which consists in a case study of a variety of shapes, from the square to the dodecagon, each of them being divided into several sub-cases. In his attempts to find the perfect shape is, Marolois explores the different values he can assign to the proportion between face and curtain, or flank to gorge, or even flanked angle to flanking angle (see glossary on figure 3.6). After having studied fifty or so cases, he opts for simple proportions (3 to 2 or 4 to 3) and gives the eleven maxims which will define the first Dutch *manner* of fortification. His successors will more or less follow the same rules, leaving aside explorations to focus on now universally accepted shapes.

We'll show in concrete terms in §3.3 the importance of trigonometry in Marolois' works and more generally in Dutch fortification. In Western countries, trigonometry had found his corner stone in Regiomontanus's *De Triangulis* (Regiomontanus, 1533), including the Law of Sines and the resolution of triangles. But Regiomontanus's main goal was to provide the astronomers with useful theorems (Maior, 1998, 41-46). Grattan-Guinness also points out the prominent role of trigonometry during the period 1540-1660 that he names "age of trigonometry' (Grattan-Guinness, 1998, 174-233), but he mostly mentions astronomy, surveying and navigation. According to him, one of the most important books for this period is Pitiscus's *Trigonometria*, (Pitiscus, 1600). A close examination of the different editions shows the growing range of applications of trigonometry: a first version in 1595 contained only two theoretical parts, but from 1600 on, it would be completed with the tables of sines, tangents and secants, plus an appendix on applications of trigonometry to a variety of problems in surveying, geography, gnomonic and astronomy. In the 3rd edition (Pitiscus, 1612), the appendix is extended to a new domain: architectonic, that is military architecture. The huge influence of this work amongst engineers and fortifiers can be measured by its numerous quotations and even reproductions of its contents in many books for decades in Europe.

Now we examine the three different *manners* we introduced above.

3 The *Manners* of fortification

Our paper reports a workshop, the "working part" of which consisted of studying different *manners of fortification* with ruler and compass. The main goal was to question the role of geometry in these several *manners*, beyond the simple use of instruments to draw the required shapes. We essentially describe the participant's procedures.

We have chosen to focus on the three major steps in the researches on fortification: the Italian use of diagonals in polygons, Jean Errard's hexagon "reduced into art and demonstrated", and Samuel Marolois's "trigonometricky" hexagon. Due to their minor role in history, we had to leave aside Michel Coignet's propositions (Coignet, 1612), despite their interest as transition markers between Errard and his Dutch followers.

3.1 Two examples of Italian architects

Despite Tartaglia's investigations on the fundamentals of fortification and his expression of what should be its principles, we find but few real justifications in the Italian books on military architecture of the 16th century. Even if the fortresses were fully described, using many diagrams, they were more detailed as stone and earth real life fortresses than as ink and paper diagrams.

In a majority of Italian books and manuscripts, the construction algorithms keep undisclosed and the reader has to make sense of the figure himself. This non-didactical aspect might be due to the identity of this targeted reader, who may have been an engineer, or at least a well-skilled person. To give a typical illustration of this specific *manner*, let's examine first a manuscript we found in the Jagiellonian Library in Cracow (Anonymous, 16th century). Its anonymous author writes in Italian, but he favors the figures: the only comments describe lines and angles in terms of stone and earth fortresses, but don't explain the generation of the shapes. For instance, below a diagram of a fortified hexagon we can only read: "All lines are drawn from and through the points and the intersections. Thus, measuring is not needed."[5] (Anonymous, 16th century, fol. 12v).

Let's try to apply this principle to reconstructing one of the numerous fortified shapes in the manuscript, for example the fortified square on fol. 24v (Fig. 3.1). Our intuitive view leads us to start with the largest square ABCD and its diagonals (AC) and (BD). The side of the square is twice the opening of the compass, that is to say the diameter of each arc drawn inside ABCD. These arcs, centered at A, B, C and D, give the midpoints of the sides, respectively E, F, G and H, so the medians (EG) and (HF) can be drawn. If we then draw the arcs centered at midpoints E, F, G and H, we obtain useful points, namely I, J and K, L, which allow us to draw further symmetrical straight lines, as do points M, N and O, P. Our initial square is now divided into sixteen smaller squares.

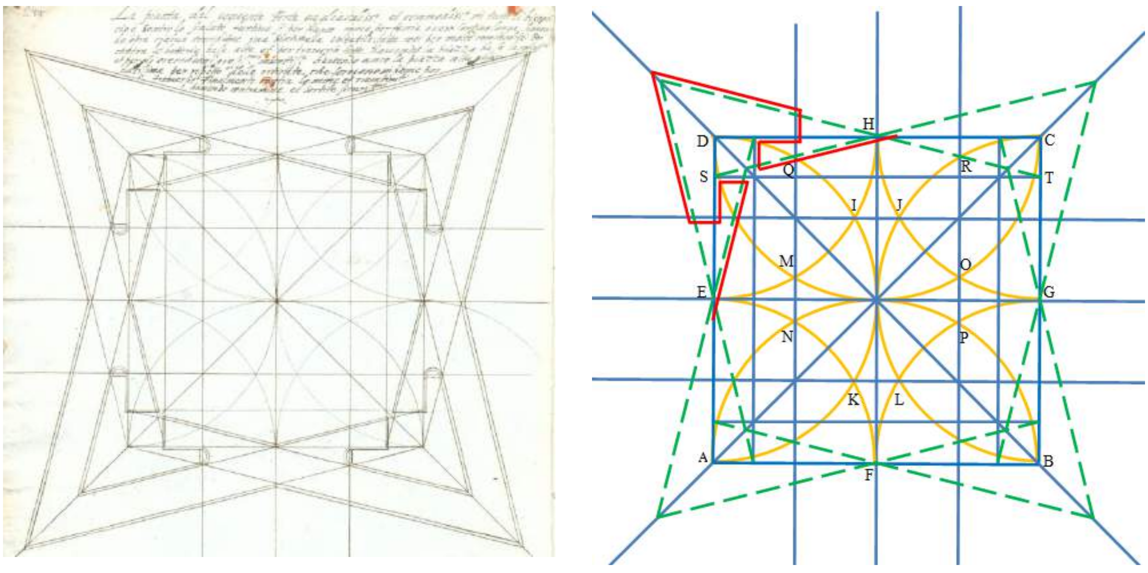


Figure 3.1: Original drawing and attempted reconstruction (Anon., 16th century, fol. 24v)

5 *Tutte le line sono tirate per i punti et intersegationei regolari. Et peró non c'é di bisogna la misure* (All translations into English were made by the author of this paper).

But the arcs centered at E and G intersect vertical lines (MN) and (OP) at Q and R respectively. Lines (HQ) and (HR) meet two opposite sides of the square at S and T respectively. It looks like (ST) bisects the formerly drawn squares. Indeed S and T could have been found as midpoints of segments, but no indication is clearly given on the original figure. To complete the construction, we just have to produce rays [TH) and [SH) to their intersection with the diagonals of the initial square, and repeat this operation three times to get the four vertices of the bastions. Finally, we pick the appropriate segment lines to obtain the desired shape, as shown on the top left of the reconstruction in Figure 3.1.

We take our second example from a very rare book (Scala, 1598) in which the constructions are based both on unmeasured lines and on lines measured by a scale. It has been written by Giovanni Scala, a Roman mathematician and engineer, otherwise known as the geometer who completed the edition of Pomodoro's *Geometria prattica* (Pomodoro & Scala, 1599). It is likely that Scala gave lessons on fortification to several German and Polish officers in Roma before 1600. His book on fortification was first published as a portfolio of plates, and we can find copies of it with hand-written comments in Poland and in Paris. A frontispiece and a preface to Henry IV King of France were added in 1598 in order to make it a real book. It contains more than 50 various shapes of bastions, the constructions of which are more or less explained. As far as we know, none of them were actually built on the field.

In Figure 3.2, we show the bastion on plate 6.

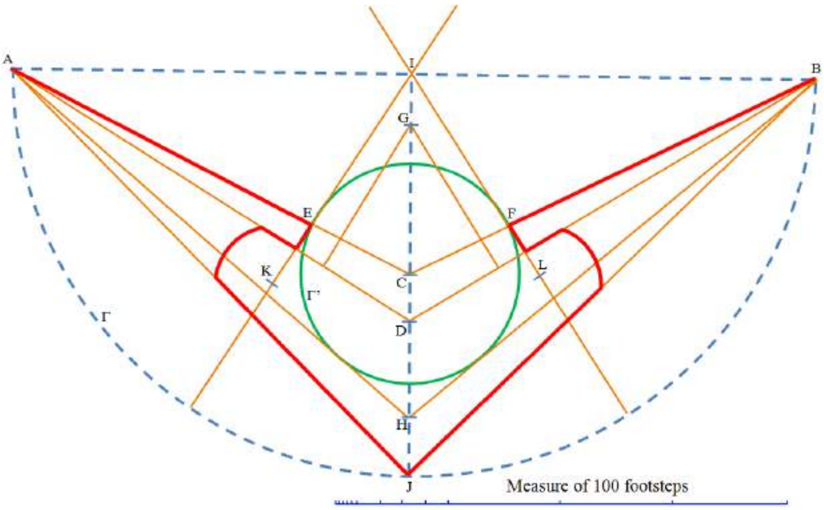


Figure 3.2: Giovanni Scala's bastion (reconstructed from Scala, 1596, plate 6)

Here is our reconstruction of the figure: take segment [AB] of 180 steps[6]; on [AB] draw the half-circle Γ and the perpendicular bisector (IJ), which intersect at J. Join the midpoint C of [IJ] to A and B; from C as a center and with radius 25 steps, draw the circle Γ', which cuts (CA) and (CB) at E and F respectively. Put point D on [JC] at 10 steps from C, and G on [IC] at 30 steps from C; join both points to A and B; segments [CA] and [CB] cut circle Γ' at respectively; produce lines [IE] and [IF] to the circumference of Γ and draw segments [DA] and [DB]; from A and B, draw the tangents to circle Γ', which meet at H;

[6] The *passo*, or geometric footstep, was the thousandth part of the Roman mile, i.e. around 1.5 meters. For our reconstruction, we took the numerical values from the figure, using the scale engraved on it. Our starting point was the value of 180 footsteps, which is the explicit basis of several other Scala's constructions.

from G, draw two segments parallel to (IE) and (IF) respectively, ending on [AD] and [BD] respectively; K is the midpoint of the segments cut on line [IE) by lines [AD] and [AI]; draw an arc with center K and radius 12.5 between these two lines, and do the same symmetrically with L. The bastion is obtained by selecting the appropriate arcs and line segments among the various ones that have been drawn (in bold red on Figure 3.2).

Of course, we can't be sure of our reconstruction, especially when they deal with tangents or measures, as the engravings in the portfolio are sometimes imprecise, or show hesitations. Nevertheless, we can assume that the engraver, and consequently the inventor of the shape (they may be the same person), were aware of construction programs. These programs may have had to keep illegible for non-specialists, which would explain the lack of instructions in the texts. Anyway, the shapes do not look like they had been created responding to scientific principles, but rather in an aesthetic aim. The way of generating them indicates a global idea of the bastion profile, but no mathematics is used, except the usual geometrical concepts and drawing procedure. This will be changed in 1600 by French engineer Jean Errard de Bar-le-Duc.

3.2 Jean Errard, and the Euclidean proof

Errard's fundamental example is the regular hexagon, explained on one of the six equilateral triangles it is composed of (cf. Fig. 3.3). In fact, the author considers the triangle, the square and the pentagon as unfit for receiving right-angled bastions, what can be justified nowadays by the impossibility of applying Errard's general algorithm of construction to these particular shapes.

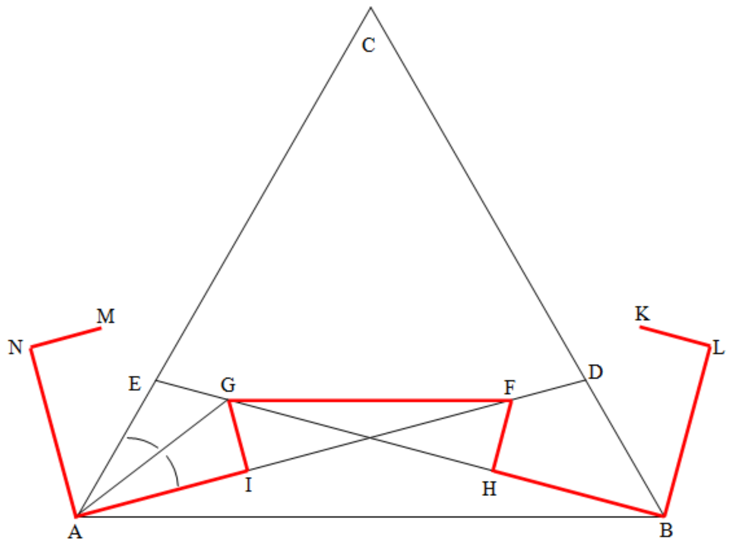


Figure 3.3: Errard's hexagonal construction (after Errard, 1620, p. 40)

Here is the text given in the posthumous edition made by Alexis Errard, Jean's nephew and also an engineer, supposedly according to his uncle's will. We give this version preferably to the very first one, as it clearly separates the construction and the proof (Errard, 1620, p. 40):

> *Let be proposed to fortify a hexagon, as far as the hexagon can be divided into six equilateral triangles. On AB describe the equilateral triangle ABC, and angle CAD of 45 degrees. Draw line AE equal to line BD, then drawn line BE. Divide angle EAD into two equal parts by AG, & let DF be taken equal to EG. Draw the curtain*

wall GF, as well as FH perpendicular to BE. Let AI be taken equal to BH, and GI be drawn perpendicularly as FH. So are described the two half-bastions AIG & FHB[7].

Since Errard based his construction algorithms on scientific principles, he had to prove that the results met his requirements, the most important being about the line of defense (i.e. AF on Fig. 3.3 or BF on Fig. 3.4), whose length must not exceed 100 toises[8]. Here follows a modernized version of Errard's demonstration (after Errard, 1600, p. 24; Errard, 1620, p. 41-42; see Fig. 3.4).

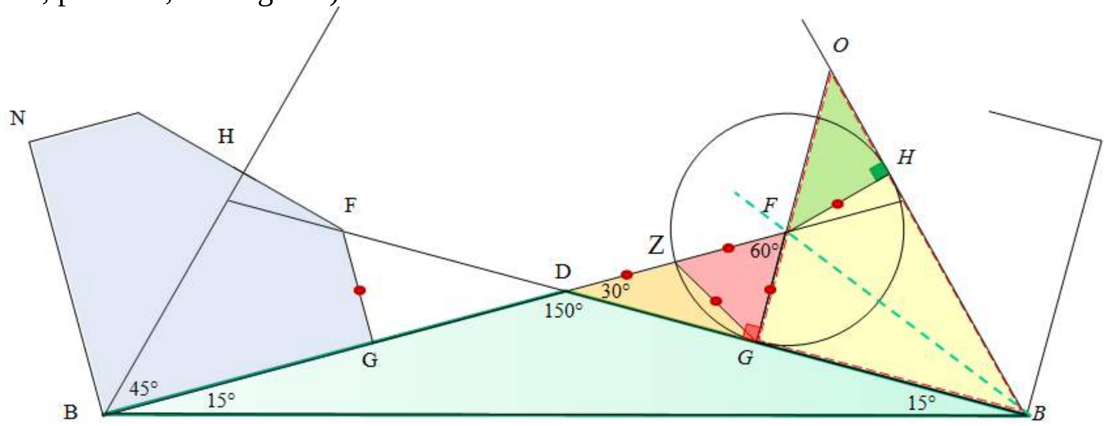


Figure 3.4 : Errard's demonstration (after Errard, 1600, p. 25 & Errard, 1620, p. 42)

Remembering that F is on the bisector of angle *O*B*G*, let's draw a circle centered at F and tangent to the sides [*BO*) and [*BG*) of angle *OBG* at *H* and *G* respectively. The circle cuts [*F*D] à Z. We first consider triangle DB*B*: since ∠HB*B* = 60° and ∠HBG = 45°, then ∠DB*B* = 15°, same for ∠D*B*B, by symmetry. Thus ∠BD*B* = 150° and consequently ∠*F*D*G* = 30°. Now let's examine triangle *F*D*G*: right-angled at G, it has an angle of 30° at D, so ∠D*FG* = 60°. But Z*F* = Z*G*, so *F*Z*G* is equilateral and *F*Z = Z*G* = *FG*; moreover, ∠*F*Z*G* = ∠*FG*Z, so ∠Z*G*D = 30°, ΔZ*G*D is isosceles and Z*G* = ZD (= FH).

Having linked these lengths together, Errard takes FG = *FG* = 16 toises as a common unit for all of them. Using the Pythagorean theorem in right triangle *FG*D, he obtains $\mathrm{D}G = \sqrt{32^2 - 16^2} \approx 27.713$ (he takes $27\frac{3}{4}$). in the isosceles right triangle *OHF*, $OF = 16\sqrt{2} \approx 22.63$ (Errard gives $22\frac{3}{5}$), and finally, in the isosceles right triangle *BGO*, $BG = GO = GF + FO = 38\frac{3}{5}$. The line of defense F*B* (or *F*B) can now be evaluated: $\mathrm{F}B = \mathrm{FD} + \mathrm{D}G + GB = F\mathrm{D} + \mathrm{D}G + GB = 32 + 27\frac{3}{4} + 38\frac{3}{5} \approx 98\frac{1}{2}$, which is less than 100.

Thanks to his choice of appropriate angles, Errard needed only basic Euclidean propositions to systematically find what was missing. No trigonometric lines there but essentially the Pythagorean Theorem. We would call this demonstration Euclidean, no doubt it would have pleased Jean Errard. But this pleasant aspect of the right angle has major drawbacks on the field: the defenders on the flanks are turned towards the walls instead of the counterscarp, and the faces of the bastions are too large to resist a long time

[7] *Soit proposé à fortifier un Hexagone, d'autant que l'Hexagone se divise en six triangles équilatéraux. Soit sur AB décrit le triangle équilatéral ABC, puis soit fait l'angle CAD de quarante-cinq degrés. Soit faite la ligne AE égale à la ligne BD, en après soit tirée BE. Soit divisé l'Angle EAD en deux également par la ligne AG, & soit prise DF égale à EG, & tirée la Courtine GF : comme aussi FH perpendiculaire sur la ligne BE. Soit prise AI égale à BH, & soit tirée la ligne GI perpendiculairement comme FH. Ainsi seront décrits les deux demi Bastions AIG, & FHB* {modernized spelling}.

[8] An old unit, roughly corresponding to the human height (approximately 195 meters).

to the pounding of artillery. Errard's heirs in Holland would use the newly invented methods of trigonometry to get rid of the right flanked angle and set the generation of bastions free of it.

3.3 Samuel Marolois's "trigonometricky" hexagon

Unlike Errard, Marolois doesn't establish his constructions on necessity and principles drawn from the field practices. The first few pages of his *Fortification* state the values of angles for any regular polygon from the square to the dodecagon. The flanked angle is not right; on the contrary its value depends on the angle at the center of the polygon. The different values given in a table (Marolois 1638, 5) correspond to a simple algorithm, which is guessable at first glance as in a modern spreadsheet, but not formally expressed. In our time, it would be: flanked angle = half-angle at the center + 15°. Knowing that the angle between the flank and the curtain is always right, all the other angles are determined. It is only after a variety of case studies according to diverse proportions of lengths that Marolois gives "the manner how to describe succinctly the designs or Plots of some regular Fortifications" (*Idem*, 27) and finally the *Maxims* of regular fortification (*Ibid*., 43).

Here is a slightly modernized version of the English translation of the first example, the design of a hexagonal fortress (*Ibid*., 29-30, to be followed on figure 3.5, using the glossary on figure 3.6 for specific terms[9]):

> *Let there be given a Hexagonal Fortress to be fortified, whereof the face AC makes 24 rods[10], and the flanked angle 80 degrees, according to which the interior flanking angle will make 20 degrees, and the exterior 140 degrees. Let the curtain be 30 [i.e. 32] rods, which gives the reason of the face to the curtain as 3 to 4. To do this, we shall draw the infinite covered line AB, by the help of a graduated instrument, the other angle CAD of 20 degrees (of 20, because the interior flanking angle, which is always equal to it, makes here 20 degrees) by means of the indefinite line, upon which you make the length of the face 24 rods, as from A to C; from which point C, the perpendicular CD being drawn upon the line AB, shall be placed from D the length of the curtain, which is here 32 rods as from D to E. Finally the distance AD from E to B, and the perpendicular EF the distance of CD as from E to F. Drawing the line FB, you have the other face, so that all the part of the given reason are described; […] we make the angle HKA only of 35 degrees, according to which the gorge in the flank will be almost as 4 to 3, or somewhat more by reason of the line HK, cutting the diagonal line AG at H, from which point H the line HN being drawn parallel to AB, you shall have the interior Polygon, upon which the lines CL and FN being drawn in length, the lines DC to L and EF to M. In doing so, all the essential parts of the said fortress will be described.*

This excerpt needs some comments. The first four lines remind us of the previously calculated angles, of the chosen length of 24 rods for the face, and of the proportion of the face to the curtain, that is 3 to 4. This being established, the construction program is described step by step: on line AB as a basis, with an angle of 20 degrees and a length of 24 rods, we draw the face AC of the left bastion; we put on line AB the following points:

[9] According to Marolois's notations, the figure shows lowercase letters, while the text in written in capitals. Moreover, the letters in figure 3.6 are not consistent with those in figure 3.5.
[10] The unit of measurement is the Rhineland rod (a 12-foot rod, approximately 3.8 meters), used during the reign of Maurice de Nassau. The contemporary English rod is equivalent to 5.5 yards (around 5 meters).

D, by orthogonal projection of C on AB, then E and B such as DE = 32 (length of the curtain wall), and EB = AD. Symmetry is at work, even if not mentioned; point K is determined on line AB by ∠AKC = 35°, then point H as the intersection point of (KC) and the side AG of equilateral triangle AGB (G is not visible on figure 3.5); (HN) is drawn parallel to (AB), then C is orthogonally projected on (HN) to get L, and the complete shape ACLMFB is obtained by symmetry.

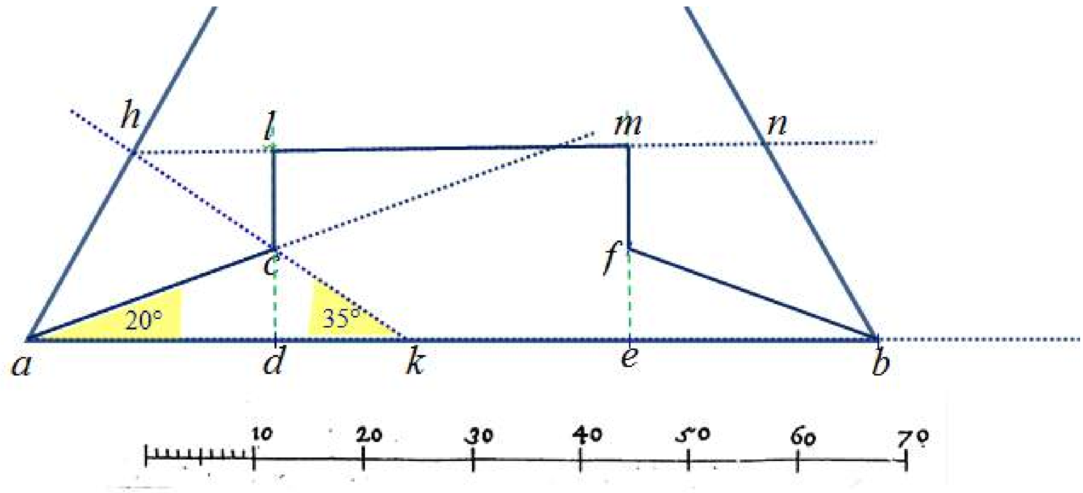


Figure 3.5 : Marolois's construction (after Marolois, 1615, plate 13, fig. 72)

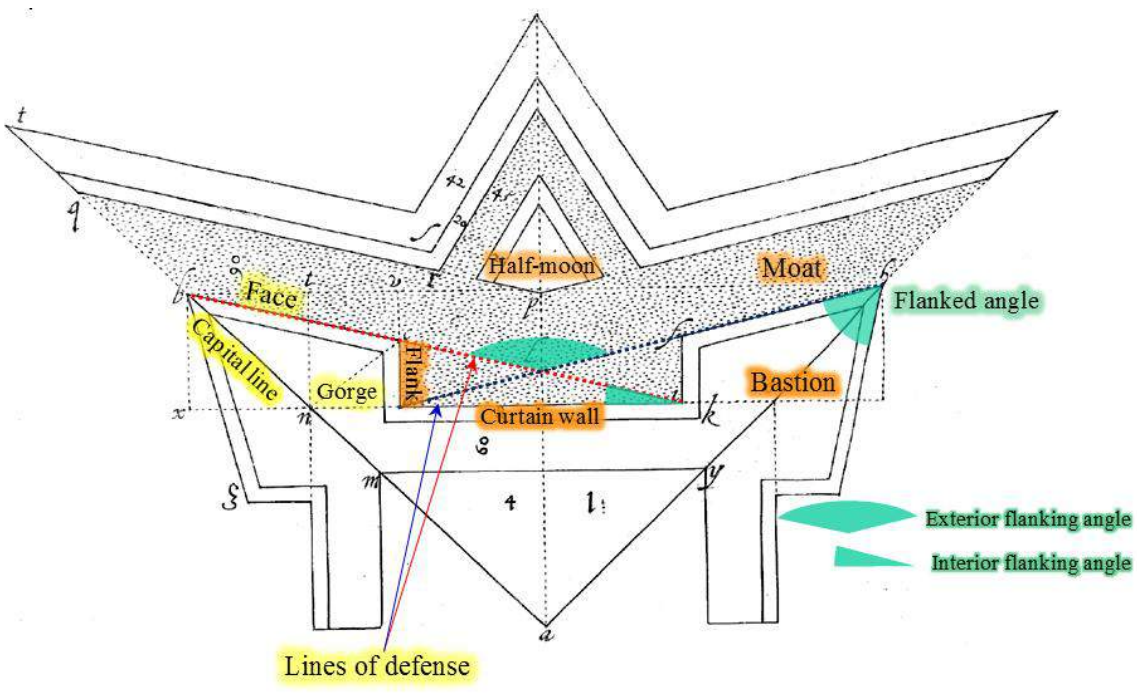


Figure 3.6: Glossary (after Marolois, 1615, plate 1, fig. 1)

The profile being constructed, Marolois doesn't give any calculation or demonstration, because all of them have been detailed before. His translator Henry Hexham is even terser, writing for example (*Ibid.*, 20):

> *Here is nothing but that which is ill calculated by the Author, or rather by his disciples, as from the beginning (without all doubt) seeking to help themselves with*

the figures put here under, which was needless, supposing that they are skilled in Trigonometrie.

In fact, trigonometry is excessively used by Marolois to calculate every length and distances in every case study in the first part of his book. The hexagonal fortress we showed the construction above had already been the subject of calculations in the 44th example (Marolois, 1615, fol. Tv). For example, DL is take, as the sine of $\angle$DAC, providing that the face AC (of 24 rods) corresponds to the *sinus totus*[11] of 100 000 parts. Then AD is computed using the sine rule in triangle ACD. Unfortunately, the flanked angle (i.e. twice $\angle$HAC), chosen of 80° according to the rectified table of angles of part 1, is in fact taken of 75° according to the unrectified table. The demonstration is thus entirely in conformity with another case, what turns the readers into confusion. This may be the reason why it doesn't belong to the English version.

4 Conclusion: towards an European military architecture

From the Italians to the Dutch we have shown changes in the way of using geometry. In a certain way we could infer that Italian architects were driven by the images of what should be the final shapes, together with a bright idea of symmetry and even beauty. Following this supposition, Errard may be seen as a direct heir of their way of thinking. It is quite clear that the final shape of the bastion led his thinking throughout its establishment process. Indeed Errard's bastions having three right angles are truncated squares. Demonstrations and calculations are based on the existence of these right angles, which allow the use of classical Euclidean theorems without need for trigonometric tables. But this makes the difference between Errard and his Italian predecessors: the rigorous approach of Errard promotes scientific discussion and anticipation for future adaptation to the improvements of attacking practices. The Dutch, thanks to their virtuoso practice of trigonometry, do not seem to be stopped by a closed vision of the final shapes, but create them with more liberty. The use of the sine rule and trigonometric methods allow them to master the distances, lengths and angles, whatever the proportions they choose. It was a necessity for them to keep their fortresses suitable for new material conditions of sieges, especially the use of explosives and the trench approaches.

Even off the field, the question of adapting the shapes of fortresses was taken seriously. For instance mathematics teachers of that time had links to the milieu of military architects. The numerous courses on fortification of the 17th century, printed as well as manuscripts, that we found in France echo the discussion of engineers about the qualities of one or another angle. Teachers expose and compare the constructions, generally drawing their examples from Errard, Marolois and the next generation of French military engineers, especially Antoine de Ville and Blaise-François Pagan. The manners of fortification were not unified yet, but it would be the case at the end of the century, when everywhere in Europe engineers would fortify places according to the *manière* of Monsieur de Vauban (or Menno van Coehoorn, his counterpart in the Low Countries). Unfortunately for us, Vauban didn't promote a mathematical approach of fortification. Quite the contrary, he claimed that geometry was useless for that purpose.

[11] That is: the radius of the trigonometric circle.

For the reader to realize how simple and non-mathematical it was, here is a summary of Vauban's manner of bastionning a line AB of 180 toises[12] (Muller, 1746): AB being perpendicularly bisected at C, set point D on the perpendicular 30 toises off from C; then put E on [AD] 50 toises from A and H on [BD] 50 toises from B; G and F are the symmetrical points of H and E with respect to D. The profile of the two half-bastions is given by AEFGHB (readers are invited to drawn this profile themselves; as an easy exercise). As one can see, there are no angles to consider here, but only distances, which are to be taken with the compass on a scale of 180 toises.

In fact, the former period is much more interesting for nowadays mathematic educators. The military revolution process is more attractive than his final results about fortification, especially when you consider the role played by geometry. At the beginning of the 17th century, things were not entirely determined and many controversies happened, on the field as well as in the offices, in the mansions or even at the Royal Courts. Nobles, whether officers or not, had to know the terms, concepts and practices of fortification perfectly. Many noble families needed mathematics skills to be taught to their teenagers, because geometry was the language of fortification. For our present math classes, studying fortification in the original texts can bring the students a good example of a concrete use of geometry and justify the learning of this vanishing discipline.

REFERENCES

Anonymous (16th century), *Collection of fortification drawings in Italian*. Ms 465a (DD XV 4) Jagiellonian Library, Cracow, Poland.

Bragard, P. (2014). La «Trace italienne». Réflexion sur une expression infondée. In N. Faucherre, P. Martens, & H. Paucot (Eds.), *La genèse du système bastionné en Europe (1500-1550)* (pp. 49-52). Navarrenx, Cercle Historique de l'Arribère.

Cataneo, G. (1574). *Le capitaine de Jerosme Cataneo, contenant la manière de fortifier places, assaillir, & defendre*… Lyon: Jean de Tournes.

Coignet, M. (1612), *Recueil de traités mathématiques de Michel Coignet: Des Tables de Sinus, Tangentes, Sécantes*... Manuscript II 769, Royal Library of Belgium.

De la Croix, H. (1963). The Literature on Fortification in Renaissance Italy. *Technology and Culture, 4*(1), 30-50.

Dijksterhuis, F. J. (2017). Duytsche Mathematique and the Building of a New Society: Pursuits of Mathematics in the Seventeenth-Century Dutch Republic. In L. B. Cormack, J. A. Schuster, & S. A. Walton (Eds.), *Mathematical Practitioners and the Transformation of Natural Knowledge in Early Modern Europe* (pp. 167-181). Studies in History and Philosophy of Science, vol 45. Cham: Springer.

Dögen, M. (1648). *L'architecture militaire moderne ou Fortification. Confirmée par diverses histoires tant anciennes que nouvelles, & enrichie des figures des principales Forteresses qui sont en l'Europe*. Amsterdam: Louys Elzeviers.

Errard, J. (1600). *La Fortification reduicte en art et demonstrée*. Paris: publisher not identified.

Errard, J. (1620). *La fortification demonstree et reduicte en art par feu J. Errard de Bar-le-Duc. 3rd edition*. Paris: Pierre Delon.

Grattan-Guinness, I. (1998). *The Norton history of the mathematical sciences: the rainbow of mathematics*. New York: W. W. Norton & Company.

La Penna, P. (1997). *La fortezza e la città. Buonaiuto Lorini, Giulio Savorgnan e Marcantonio Martinengo a*

[12] Approximately 350 meters.

Palma (1592-1600). Firenze: Leo S. Olschki editore.

Lorini, B. (1597). *Delle fortificationi di Buonaiuto Lorini, nobile fiorentino Libri Cinque...* Venise: Giovanni Antonio Rampazetto.

Maior, E. (1998). *Trigonometric delights*. Princeton, NJ: Princeton University Press.

Marolois, S. (1615). *Fortification ou architecture militaire tant offensive que deffensive, suputee & dessignee par Samuel Marolois*. La Haye: Hendrik Hondius.

Marolois, S. (1616). *Opera mathematica ou oeuvres mathématicques. Traictans de Geometrie, Perspective, Architecture et Fortifications par Samuel Marolois. Ausquels sont ajoins les fondements de la Perspective, & Architecture de J. Wredm. Vriese. Augmentée & Corrigée en divers endroicts par le mesme Auteur*. La haye: Hendrik Hondius.

Marolois, S. (1638). *The Art of Fortification, or Architecture militaire, as well Offensive as Defensive, compiled & set forth by Samuell Marolois. Reviewed, Augmented and Corrected by Albert Girard Mathematician. Translated out of French into English By Henry Hexam*. Amsterdam: John Johnson.

Meskens, A. (2013). *Practical Mathematics in a Commercial Metropolis. Mathematical Life in Late 16th Century Antwerp*. Dordrecht: Springer.

Métin, F. (2016). *La fortification géométrique de Jean Errard et l'école française de fortification, 1550-1650* [Jean Errard's geometric fortification and the French School of fortification, 1550-1650]. Doctoral dissertation, defended on December 6, 2016 under supervision of Dr. Evelyne Barbin at Nantes University (to be published).

Muller, J. (1746). *A treatise containing the elementary part on fortification, regular and irregular*. London: L. Nourse.

Parker, G. (1988). *The military revolution: military innovation and the rise of the West*, 1500-1800. Cambridge: Cambridge University Press.

Pitiscus, B. (1600). *Trigonometria sive De dimensione Triangulos Libri Quinque. Item Problematum variorum nempe Geodaeticorum, Altimetricorum, Geographicorum, Gnomonicorum et Astronomicorum Libri Decem*. Augsburg: Michaël Manger.

Pitiscus, B. (1612). *Bartholomaei Pitisci Grunenbergensis Problematum variorum nempe Geodaeticorum, Altimetricorum, Architectonicorum, Geographicorum, Gnomonicorum et Astronomicorum Libri Undecem*. Frankfurt: Nikolaus Hoffmann.

Pomodoro, G., & Scala, G. (1599). *Geometrie prattica tratta Dagl' Elementi d'Euclide et altri autori...* Roma: Stefano de Paulini.

Rogers, C. J. (1995). *The military revolution debate: readings on the military transformation of early modern Europe*. Boulder: Westview Press.

Regiomontanus (pseud. Müller, J.) (1533). *Johannis de Regio Monte De Triangulis omnimodis libri quinque…* Nuremberg: Johannes Petreius.

Scala, G. (1596). *Delle fortificationi di Giovañi Sala Mathematico. Libro primo. Al Christianissimo Re di Francia e di Navarra Henrico IIII*. Roma: publisher not identified.

Stevin, S. (1594). *De Sterctenbouwing, beschreven door Simon Stevin van Brugghe*. Leyden : Francoys van Ravelenghien.

Theti, C. (1589). *Discours sur le faict des fortifications du seigneur Charles Tetti…* Lyon: Barthelemy Vincent.

Valleriani, M. (2010). *Galileo engineer*. Boston Studies in the Philosophy and History of Science, vol. 269. Dordrecht: Springer Netherlands.

THE GEOMETRY OF THE DAMBSUTERS

A cross-curricular approach using history in the mathematics classroom with students and teachers

Peter RANSOM
The Mathematical Association, UK
pransom@btinternet.com

ABSTRACT

Since 1992 I have developed many lessons that use history in the mathematics classroom. These lessons are cross-curricular using events from history to enthuse students in their learning of mathematics. They show that the mathematics used today had, and continues to have, applications to everyday life. As well as being classroom lessons these episodes form mathematics master classes done on Saturday mornings around the UK. The lessons use old and new technology to provide students with an insight into how they are made and used. Recent developments with the STEM (science, technology, engineering and mathematics) initiative in the UK mean these lessons are even more relevant. One episode is described here; others have been presented at previous conferences.

1 The research impetus

I incorporate events from history into mathematics lessons because I find it very interesting to see the practical applications of mathematics set into the period when it was used. My research questions are qualitative and are concerned with whether students and teachers find the mathematics interesting and clear. I chose to write about this episode since it has been developed since 2003 and is very rich in both mathematics and history.

Tzanakis & Thomaidis (2011) classify the arguments and methodological schemes for integrating history in mathematics education and my episodes fit into the two-way table mainly as History-as-a-tool and Heritage though there are overlaps into the History-as-a-tool and History cell. The over-riding concept in my work is History-as-a-tool. The distinction between history and heritage was made by Grattan-Guinness (2004) where he writes in the abstract of his article:

> Their [mathematicians'] normal attention to history is concerned with heritage: that is, how did we get here? Old results are modernized in order to show their current place; but the historical context is ignored and thereby often distorted. By contrast, the historian is concerned with what happened in the past, whatever be the modern situation. Each approach is perfectly legitimate, but they are often confused.

The reliability of this research in the sense of reproducibility by someone else is impossible to quantify, since teachers use such episodes in different ways with different students and probably not in costume! Every session I do with students is different according to local conditions and the knowledge students bring to the sessions, so results will vary.

2 The evolution of the episode

Back in the 1990s I was introduced to the Royal Institution mathematics masterclasses for gifted and talented 13/14-year-old students. These took place (and still do) on Saturday mornings and last for 2½ hours. They are organised by local groups of interested teachers and generally take place at prestigious places covering schools within travelling distance. They put on a programme of 4 to 8 sessions (though most places do 6) inviting staff at local schools to nominate a couple of students who would benefit from these master classes. Members of staff are also invited to attend and without these volunteers the sessions would be difficult to run since they attract between 30 and 60 students each session (I prefer to work with groups of about 30, because it is difficult making good contact with larger groups to ensure all remain focussed for the whole session.). These volunteers help when students are involved in the tasks since the sessions involve a lot of interactivity. The topics offered vary considerably and cover a wide range of mathematical topics. As the years progressed I felt that there was plenty of material that was suitable for general classroom work and so by developing the master classes I do in my own classroom I was able to trial the work with a wide variety of ages and abilities. In the 1990s I was inspired by John Fauvel to include various historical activities in my classroom and this was at a time when there was far more autonomy in the UK on what to teach and how to teach it: a description of the situation at the time can be found in Fauvel & van Maanen (2000) on page 16.

In my opinion the work should have relevance to the students, so I always try to include material that covers mathematics primarily and history secondly. However, science, engineering and geography also play significant parts and all these are brought together to provide a holistic coherent scenario in which to develop the mathematics.

Sometimes the opportunities arise by chance: a suitable anniversary, an interest or an opportunity can be the spur to spending many hours pursuing threads which gets sewn into a rich tapestry of mathematics. The STEM (science, technology, engineering and mathematics) initiative in the UK is aimed at developing an interest in those subjects since they have shown a fall in numbers at degree level over the years. By tackling these areas with scenarios that show how they were dealt with in the past I believe it gives students enthusiasm for mathematics and provides them with a rich background of general knowledge. The episode I mention here has been trialled with teachers and students in the classroom and then improved after feedback from them and my peers in the last year. The research questions used in this paper are based on three sessions with 107 13/14-year-old students in November 2017; 35 trainee teachers in August 2017 and August 2018 together with 46 similar trainees in January 2018; and with 16 participants at my ESU8 workshop in July 2018.

3 Description of the episode

Full details of this can be found in Ransom (2004). The workshop presented at ESU8 focuses on fewer activities due to time constrictions. The mathematics involved stems from an event that took place on the night of 16 May 1943 when a significant raid took place during World War II. Full details of this can be found in Sweetman (2003).

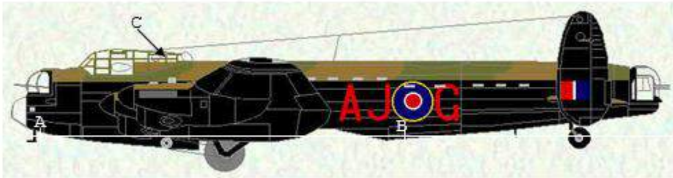
The workshops are always presented by Flight Sergeant ‘Kidnap’ Ransom in 1940’s Royal Air Force uniform and participants work in twos. They hear about the Dambusters raid of 1943 and how mathematics was used to navigate the Lancaster aeroplanes over Germany. Participants plot the route on A3 maps, working with scales and bearings then look at how the lights were arranged on the plane so that it travels at 60 feet above the surface of the water, using geometrical constructions to find the angles at which the lights were inclined. They also problem solve by finding the position of the plane when the bomb needs to be released and the time taken for the plane to pass over the edge of the dam. These two worksheets are illustrated in Figures 3.1 & 3.2:

Pilot's Sheet

Your job is to fly the Lancaster at 60 feet above the ground.

To do this you have to adjust the angle of the Aldis lamps (at A and B) so that their light meets 60 feet below the front edge of the wing so the navigator could look out of window C and see the where they meet.

The scale of the Lancaster is 1 cm to 5 feet.



Draw in the ground first.

The Lancaster is ……………. feet long.

The tail fins are ……………. feet high.

The Aldis lamps are ………………… feet apart.

Lamp A is angled at ……….° to the line.

Lamp B his angled at ……….° to the line.

Use trigonometry to check if you know how.

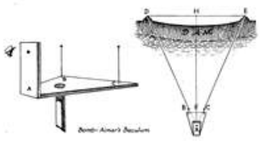
Figure 3.1: Working with scales and construction of a perpendicular to a line from a point

Bomb aimer's Sheet

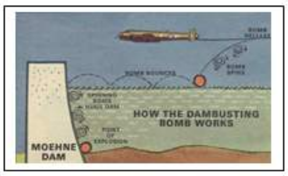
Your job is to draw a plan of the position of the Lancaster when Upkeep is to be dropped. It should be dropped 450 yards from the dam and on the perpendicular bisector of the towers. The Lancaster should be travelling at 220 mph when Upkeep is dropped.

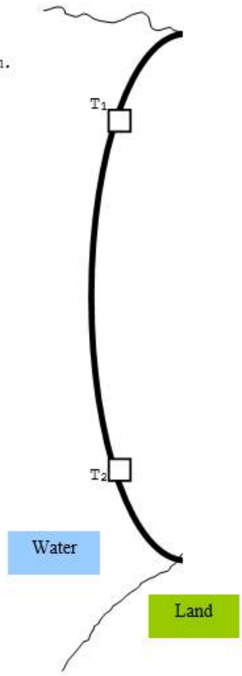
1 yard = 3 feet
1 mile = 1760 yards

The scale of this plan is 1 cm to 100 feet.



This is the device used to ensure Upkeep was released at the right distance from the dam.





The distance between the towers (T_1T_2) is ……………. feet.

The dam is ……………. feet wide.

The angle at the bomb aimer's eye is ……….°

The Lancaster takes ……….seconds to reach the dam after releasing Upkeep.

Figure 3.2: Problem solving, using construction of a perpendicular bisector

These two activities involve ruler and compass constructions and are excellent problem-solving activities. They develop their mental geometry with a short exercise that should improve their 3D coordination. With clips from the Dambusters film they then make a bomb-aimer's sight and test it. This activity involves creativity in the mathematics classroom. As far as possible, the workshops recreate, in the form of mathematical theatre, what happened with Squadron X (later known as Squadron 617, the Dambusters squadron) in 1943. There is also a civilians' sheet (Figure 3.3) which deals with the coinage of the time and how that relates to the decimal system used today. That sheet is illustrated here.

Civilian's Sheet: The coinage of 1943: see if you can fill in the table!

Coin	Number of old pence	Number of coins/notes in £1	Number of new pence
farthing	¼		
halfpenny			
penny	1	240	
threepenny	3		
sixpenny			
shilling		20	
florin	24		10
half crown	30		
10 bob note		2	
pound note			£1

This airman received 6/6 (six shillings and six pence) out of 9/3 for each day worked.

What do you notice if you multiply the number of old pence by the number of coins in £1?

....

Figure 3.3: Working with historical objects (coins) and inverse proportion

Figure 3.4: Teachers test their bomb-aimer's sights with Flight Sergeant 'Kidnap' Ransom

The other benefits to this class' learning included the following: working cooperatively and developing their creative skills.

Did the historical aspect make a difference? I have no evidence to say 'yes' or 'no' at this point since no question was directed at this aspect. (This links in with the History side of the two-way table mentioned in Tzanakis & Thomaidis (2011)). However, there are some comments about the history in the feedback analysis section of this paper.

When trialling new work with my students I never know what will happen, but with over 30 years' experience of the 11-18-year-old classroom I think I can judge reasonably well what will be accepted. In the plenary at the end of a session students are asked about what they learnt and it is only the comments that students offer at the end of the session that give me an insight into whether these sessions are successful or not. Comments such as 'The part I enjoyed least was going home' and 'It was interesting because we were given a realistic scenario for the maths.' encourage me to continue with such sessions adapting them for different groups. Comments such as 'The civilian's sheet wasn't very clear'; 'I didn't enjoy the mental geometry at the start' means I make amendments to the work if something is not enjoyed by most students.

4 Sensitivity issues

The UK is a multi-cultural society with students from many different nationalities and this subject is dealt with as sensitively as possible. War sets nation against nation and is not to be treated lightly or jingoistically, which is why I explain to the students the fact that although the raid on the dams was seen as a success at the time, most of the casualties were civilians, rather than militia. I also mention that this had an effect on the Geneva Convention, resulting in an amendment that said dams, being a non-military target, were not allowed to be attacked under the articles of war.

It is unfortunate that mathematics and science both make progress during wartime. We see how the mathematics of fortification improved during the 17^{th} and 18^{th} centuries in France when Vauban played a major part in fortifying strategic towns and cities. The study of projectiles during this time also developed. Then we see the development of the computer under Alan Turing during WW2, with the original intention of cracking the Enigma Code. This is a great benefit, less so the development of nuclear warfare.

I also mention the fact that Barnes Wallis, the inventor of the 'bouncing bomb' that allowed the mine to bounce along the surface of the water and run down the inner face of the dam, felt that the success of the raid was far out-weighed by the loss of eight Lancasters and their crew – a total of 56 young men of the RAF.

5 Feedback analysis from 13/14-year-old students

No formal survey is conducted after every workshop, but for ESU8 I thought it would be useful to collect some simple data in a format that was easy for students to complete. Bearing in mind that students who have survived 150 minutes (with a 10-minute break) of working with a partner they have not met before do not want to spend more than a couple of minutes filling in a feedback form, this is kept simple.

Dambusters Evaluation Form

Thank you for attending a Dambusters masterclass. I hope that you found it interesting.

Your feedback helps me to develop the session so I would be grateful if you could complete my evaluation form. Please include any additional comments on the back of the form.

Sussex masterclass – Saturday 11 November 2017

Please rate the session	**Poor**	**Average**	**Good**	**Very good**	**Excellent**
How interesting was the session?					
How clear was the work?					

	Too low	**Just right**	**Too high**
The level of the session was …			
What part did you enjoy most?			
What part did you enjoy least?			
Please circle your gender	Female	Male	

Figure 5.1: Feedback form used with students

It is important to me that the mathematics I present is accessible by both genders, hence the last part of the form. This was used with 107 students over two sessions in November 2017 (one with 45, the other two with 32 and 30) and the data gathered (not everyone returned a feedback form) are shown in the following tables:

Female results

Please rate the session	Poor	Average	Good	Very good	Excellent
How interesting was the session?			6	24	12
How clear was the work?		4	13	21	4
	Too low		Just right	Too high	
The level of the session was …	1		41	0	

Male results

Please rate the session	Poor	Average	Good	Very good	Excellent
How interesting was the session?		2	10	29	10
How clear was the work?		4	13	25	9
	Too low		Just right	Too high	
The level of the session was …	3		48	0	

So, overall, 95% of the replies indicated that the level of the work was ‘just right’, which indicates to me that the work is most suitable for the ability of the students. This is a 10% difference in the genders between how interesting they found the session: 86% of the females found it very good or excellent, compared to 76% of the males. I thought that if there would be any difference it would be the opposite way around. There is a 7% difference in the genders between how clear they found the work: 60% of the females found it very good or excellent, compared to 67% of the males.

As a further check on whether there is any significant difference in the interest between the genders, I started with the null hypothesis that there would be no difference in the interest of the material between the genders and used Fisher’s exact test to see whether this hypothesis should be accepted or not. This produced the following two-way table:

	Male	Female	*Row total*
Very good / Excellent	39	36	*75*
Poor / Good	12	6	*18*
Column total	*51*	*42*	*93*

This gives a probability of 0.270, which is well above the probability of 0.05, below which the null hypothesis would be rejected. Therefore, I accept the null hypothesis that there is no difference in the interest of the material between the genders.

To keep the wording at a minimum on the worksheets I keep the instructions as concise as I can because these tasks are the type of problem-solving activities that is being promoted in the UK. Students are expected to tackle problems using the mathematics they have learnt and for sessions like this where students are working in pairs with others from different schools it encourages them to communicate mathematically and suggest approaches to the problems that they have not seen before.

I found it interesting that 44% overall (48% of the females and 41% of the males) enjoyed the making of the bomb aiming sight the most. This was by far the most enjoyable part that was mentioned in the feedback form. However, there were six comments (6.5%) about the history (or history related) being the most enjoyable part: here are some of their replies to the question ‘Which part did you enjoy most?’ with the gender of the responder.

- *Learning the back story* (female)
- *It was interesting because we were given a realistic scenario for the maths* (female)
- *History behind lesson* (female)
- *Learning about the history of it* (male)
- *Historical math*(male)
- *It being maths + history*(male)

6 Feedback analysis from trainee teachers

This analysis consists of feedback from three groups. These consisted of 35 Quantum Scholars (trainee teachers who have come from different countries to teach in the UK, some of whom have taught for a small number of years) in two cohorts, one of 13 in August 2017 and the other of 22 in August 2018. The third group was 46 Mathematics Scholars in January 2018. These are trainee teachers who have been through a rigorous selection process to gain the prestigious title of Scholar. They are given no extra money, but receive two years' free membership of The Mathematical Association, the London Mathematical Society and the Royal Statistical Society as access to various resources and a community entitled to attend various events for their benefit without charge. The Quantum Scholars sessions lasted 2½ hours, like the 13/14-year-olds, the Mathematics Scholars had a shorter 1¼ hour session.

The data gathered from these groups (again, not everyone returned a feedback form) are summarised in the following tables. This time, because there was little difference between the results for each gender (and two sheets were returned without any gender identification), I have not separated out the results based on gender, but on the different types of trainee since they had sessions of different lengths.

Quantum Scholars results

Please rate the session	Poor	Average	Good	Very good	Excellent
How interesting was the session?			1	12	22
How clear was the work?		1	6	20	8
	Too low		Just right	Too high	
The level of the session was …	0		27	0	
Are you likely to use any of this material in the future?	No	2	Yes	33	

Mathematics Scholars results

Please rate the session	Poor	Average	Good	Very good	Excellent
How interesting was the session?			8	23	15
How clear was the work?		3	17	17	9
	Too low		Just right	Too high	
The level of the session was …	0		45	1	
Are you likely to use any of this material in the future?	No	3	Yes	43	

For these trainee teachers the level of the session was ‘just right’ for all bar one, which indicates to me that the work is most suitable for the ability of the trainee teachers! This is a 14% difference in the two groups between how interesting they found the session: 97% of the Quantum Scholars found it very good or excellent, compared to 83% of the Mathematics Scholars. I thought this difference was probably due to the different time spent with each group. This is important feedback for me as it indicates that one should not try to rush to include as much as possible in the session, but to allow people more time to think and discuss the work with others. There is a 23% difference in the groups between how clear they found the work: 80% of the Quantum Scholars found it very good or excellent, compared to 57% of the Mathematics Scholars.

As a further check on whether there is any significant difference in the interest between the genders, I started with the null hypothesis that there would be no difference in the clarity of the material between the two groups and used Fisher’s exact test to see whether this hypothesis should be accepted or not. This produced the following two-way table:

	Quantum scholars	Mathematics scholars	*Row total*
Very good / Excellent	7	28	*35*
Poor / Good	20	26	*46*
Column total	*27*	*54*	*81*

This gives a probability of 0.016, which is well below the probability of 0.05, below which the null hypothesis would be rejected. Therefore, I reject the null hypothesis that there is no difference in the clarity of the material between the two groups and accept that the Mathematics Scholars found the work less clear than the Quantum Scholars. As mentioned in the previous paragraph, this is probably due to the shorter time the Mathematics Scholars had to digest the material.

To keep the wording at a minimum on the worksheets I keep the instructions as concise as I can because these tasks are the type of problem-solving activities that is being promoted in the UK. Again, I think this is due to the time spent and lack of the discussion time.

The bomb aiming sight was not made with the Mathematics Scholars due the time restriction. An extra question “Are you likely to use any of this material in the future?” that was not used with the 13/14-year-old students was added to the feedback form as I wanted to know whether any of the trainee teachers would find the materials useful when they are teaching full-time. 94% of each group said they would use some of the material in the future, which I find most encouraging as I always hope to pass on useful material.

There were 31 comments (38%) about the history (or history related) being the most enjoyable or interesting part, with 34 comments (42%) about it being real-life applications of mathematics. Here are some of their feedback comments that I enjoyed reading most!

- *How the women were involved and how maths made a difference in the war.*
- *Learning about something I had no idea about.*

- *The nice mix of maths topics.*
- *Really nice contextualisation of a few topics.*
- *History of it really brought it [the mathematics] to life.*
- *Presentation of material – very good delivery. The chap was a character.*
- *The way Peter created learning without noticing – hands on with real life connections.*
- *Hands on, real world content – highly engaging and collaborative.*

7 Feedback analysis from ESU-8 participants

Feedback was received from 16 participants from various countries at my workshop on July 21, 2018. This was timed for 1½ hours, but since it was timed against other 2-hour workshops most (if not all) participants stayed for 2 hours. The feedback here is far more varied than for the previous two sets of feedback as those present have generally spent far more time in different branches of mathematical education. Therefore, I have listed more comments as I find these very helpful in developing my teaching. Due to some incomplete feedback forms the numbers do not always add up to 16.

Participants' results

Please rate the session	Poor	Average	Good	Very good	Excellent
How interesting was the session?				4	12
How clear was the work?			3	6	7
	Too low		Just right	Too high	
The level of the session was …	0		14	0	
Are you likely to use any of this material in the future?	No	3	Yes	11	

So, overall, 100% of the replies indicated that the level of the work was 'just right', which tallies well with the feedback from previous sectors. A higher proportion of this group (21%) were unlikely to use any of this material in the future, which is what I expected, given that few of this group are teaching 13/14-year-olds. Here are their replies on what they found most useful:

- *The connection between mathematics and military (the real problem).*
- *The stories in the history. They can be used to teach mathematics.*
- *Change of perspective, transformation geometry, estimation.*
- *Range of mathematical tasks the context applied to.*
- *The turning card. That was nice and surprising. I think this is especially adequate for teacher training.*

- *The use of simple, even primitive material, to solve significant and interesting problems.*
- *To see how the approach was a hands-on approach. It was a wonderful performance! Thank you!*
- *Mathematics teaching is teaching a story. The more authentic we can make the story, the more students appreciate its utility.*
- *I am no more teaching mathematics, but it will inspire me with my grandchildren!*

Here are some of their replies on what they found most interesting:

- *The use of mathematics to solve the actual problem.*
- *Measure and draw the bomb aimer's sheet.*
- *Different kind of math concept applied in one lesson.*
- *The whole story and the connection with real sources as well as the personal bonding.*
- *The authentic setting was both interesting in itself and quite relevant to the kinds of problem needed to solve.*
- *I liked the concrete calculations and it inspires me to do similar things in other contexts.*
- *All the different applications of math (trig, data analysis, spatial reasoning, constructions) all in one thematic context.*

8 Development

I am always looking for new material to include in these sessions and since doing the research in this paper I have come across a navigator's mapping board, shown below.

Figure 8.1: Navigator's mapping table with movable arm allowing a course to be plotted and detail of the North pointing arrow (circled) and movable protractor

The board itself is made of wood and is hidden underneath the map. The metal arm allows the movable protractor to be put on any position on the map and is a very interesting linkage, consisting of two sets of parallel rulers, joined by a right-angled isosceles triangle. Thus, since the sides of the triangle always remain parallel to the sides of the board, the small metal arrow always points North and the course can be read from the protractor. So, this board will be shown at future student workshops and students will be allowed to use it, if time permits.

9 Conclusion

My experience with using history in the mathematics classroom is in relating the mathematics students do with episodes from history when it was used. I believe students need to see where mathematics has been used to appreciate its importance. I imagine that teachers will not run a similar session in its entirety as few mathematics lessons are long enough to do that. However, some of the activities are stand-alone and can be used when appropriate. This activity was developed for use in the classroom and master classes, not as a research project, so the evidence of their success (or failure) is mainly ephemeral through comments received by participants after each session.

Finally, I wish to express my thanks to all the students and colleagues who have provided the feedback on the sessions, and my reviewers whose comments (especially with regard to using Fisher's exact test) on this paper have been invaluable and very much appreciated.

REFERENCES

Fauvel, J. & van Maanen, J. (Eds.) (2000). *History in Mathematics Education: The ICMI Study.* Dordrecht: Kluwer

Grattan-Guinness, I. (2004). The mathematics of the past: distinguishing its history from our heritage. *Historia Mathematica 31*, 163-185).

Ransom, P. (2004). The Maths Busters: The Geometry of the Dam Busters. *Mathematics in School 33*(2), 22-24.

Sweetman, J. (2003). *The Dambusters*. London: Time Warner Books.

Tzanakis, C., & Thomaidis, Y. (2011). Classifying the arguments & methodological schemes for integrating history in mathematics education. In M. Pytlet, T. Rowland, E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 1650-1660). Rzeszow: University of Rzeszow.

HOW TO USE CULTURALLY RELEVANT TRANS-DISCIPLINARY ACTIVITIES TO IMPROVE STUDENT ATTITUDES AND LEARNING IN SCHOOL MATHEMATICS

Jennifer Marie RODIN
Oglala Lakota College, USA
jennirodin@gmail.com, jen@olc.edu, jrodin@uwyo.edu

ABSTRACT

Mathematics is a human activity (Gardiner, 2009) which combines pattern analysis, logical reasoning, creative problem solving and communication. Every human is unique, making sense of mathematics in their own way, thus good teachers incorporate students' existing funds of knowledge (Moll et al., 1992) as well as culturally relevant activities (Ladson-Billings, 1995) and community involvement (Civil & Bernier, 2006) to improve learning opportunities, experiences and satisfaction for students of all cultural backgrounds. Combining teacher/student and student/student interaction patterns (Mohatt & Erickson, 1981), improving mathematical practice and discourse, as well as finding connections with other disciplines and with social issues (Boaler, 2000), lead to animated classes and a strong student knowledge and cohort base. Thirty years of teaching mathematics to underrepresented students of diverse backgrounds has informed a pedagogy rich with these methods, producing students who feel mathematically confident and empowered. These students have increased life opportunities going forward. Teachers implementing these methods appear to have increased satisfaction. After a brief introduction of background theories, this constructivist, interactive, hands-on, activity-based workshop will model classroom behaviors and methods that lead to positive student and teacher outcomes rich multifaceted mathematical learning.

During this exciting workshop, participants will learn briefly about Indigenous beadwork history and designs in North America. Resources for further study will be provided. Traditional Sami textile designs will also be discussed. I am not an expert in Sami designs, so I will elicit comments from participants. Subsequently, each participant will design his or her own simple beadwork piece with colored pencils on graph paper. Partners will work together to come up with functions, relations, or patterns which could generate their specific designs. Participants will share their work and give each other constructive feedback on their process, comparing symmetries of different design patterns, and exploring the possibilities of color and dimension. If time permits, they will build their own simple, Native American, bead looms to actually create their beadwork pieces with string, cardboard, tape and colorful glass beads provided. Lastly, we will explore the Virtual Bead Loom, designed by Dr. Ron Eglash, and its applications in school mathematics. https://csdt.rpi.edu/culture/legacy/na/loom/index.html

This activity can be simple or advanced. I have used it with students from the middle level, approximately aged ten years, to the advanced level in which we explore university-

level graphing, functions, relations, geometry, symmetries, basic group theory, and abstract algebra concepts. Students love this activity, because it empowers their individual creativity, shows them that mathematics abounds in visual designs, encourages them to communicate with each other, and highlights often-overlooked, indigenous, mathematical, reasoning. There are numerous adaptations and extensions for this activity.

I have been teaching mathematics at Oglala Lakota College for 13 years. This is a tribal college, serving indigenous students in North America. 99% of our students are Lakota people, and our college requires that all students learn Lakota history, language, culture, and arts. Instructors at Oglala Lakota College are encouraged to learn as much as possible about Lakota history and culture in order to better serve our students. I have been taking classes in Lakota arts, language, and psychology since 2011. I am not Lakota myself, but I have found that the more aware an instructor is of the cultural origins of her students, the better she is able to teach them and work with their learning styles. (Ladson-Billings, 1994; Leonard & Guha, 2002; Kilman, 2006 ; Oglala Lakota College School of Education Vision Statement, 2017; Eglash, 2007). Research into underperforming and marginalized groups of students in mathematics is showing increasingly that when multiple ways of sense-making in mathematics are valued, students thrive. In my personal experience, the student evaluations of my teaching are consistently excellent. Students in my classes learn the standard mathematics topics required by our curriculum, and also gain a confidence in mathematics that they did not have prior to working with me. I create a supportive, discourse-full, culturally responsive, learning environment that really leads to success.

I grew up in New Jersey, learned to speak French at a young age, was fascinated with and embraced diverse cultures. As an undergraduate at Smith College, I learned about symmetry pattern classifications of three-dimensional crystals and wall pattern designs. I continue to enjoy such explorations. Upon earning my Bachelor's degree in mathematics, I taught middle level mathematics in the Central African Republic for two years at the C.E.G. d'Ippy. It was there that I became interested in the intersection of mathematics, arts, and culture. I learned about the field of Ethnomathematics in the late 1990s, and joined the International Study Group for Ethnomathematics, where I met highly-esteemed scholars such as Ron Eglash, Jerry Lipka (who has done extensive work with the Inuit people of Alaska), and Claudia Zaslavsky. In the autumn semester of 2000, I began incorporating cultural relevance and ethnomathematics in the Mathematics for Educators courses that I taught at Chadron State College. The students gave me very positive feedback about this, and requested that we share our activities with the community preschools and at the Earth Day fair, both of which served children of diverse backgrounds. I gave presentations on our activities at the National Council for Teachers of Mathematics Annual Meetings in 2004 and 2011. I have continued to develop activities, with the help of my students and colleagues. I have been too busy teaching to write extensively on this subject, but I began writing an activity book last summer, and I am currently working on a doctoral degree at the University of Wyoming, which will focus on equity and culture in mathematics education. In my doctoral program, I am gaining proficiency at researching and writing articles, with the goal of improving the mathematics learning and teaching experience for all people.

REFERENCES (10pt fonts)

Boaler, J. (2000). Mathematics from another world: traditional communities and the alienation of learners. *Journal of Mathematical Behavior*, *18*(4), 379-397.

Civil, M., Bernier, E. (2006). Exploring images of parental participation in mathematics education: challenges and possibilities. *Mathematical Thinking and Learning, 8*(3), 309-330.

Eglash, R. The fractals at the heart of African designs (2007). *TED Talk,* https://www.ted.com/speakers/ron_eglash. Retrieved 20/12/2017.

Gardner, M. (2009). An interview with Martin Gardner. *The College Mathematics Journal*, *40*(3), 158-161.

Kilman, C. (2006). Learning Lakota. *Teaching Tolerance, Issue 30,* https://www.tolerance.org/magazine/fall-2006/learning-lakota (retrieved, 15/7/2018)

Ladson-Billings, G. (1994). What we can learn from multicultural education research. *Educational Leadership*, *51*(8), 22.

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Association*, *32*(3), 465-491.

Leonard, J., Guha, S. (2002). Creating cultural relevance in teaching and learning mathematics. *Teaching Children Mathematics, 9*(2), 114-118.

Mohatt, G. & Erickson, F. (1981). *Cultural Differences in Teaching Styles in an Odawa School: A Sociolignuistic Approach*. Rowley, MA: Newberry House.

Moll, L., Amanti, C., Neff, D, & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory Into Practice*, *31*(2), 132-141.

Oglala Lakota College Vision Statement (2017). http://www.olc.edu/departments/education.htm (retrieved 20/12/2017).

DISCOVERING NEGLECTED SYNTHETIC GEOMETRY ON SOCIAL NETWORKS

Learning Maths as in the Historical Italian Academies

Maria Giuseppina ADESSO[1,3], Roberto CAPONE[1,4], Oriana FIORE[2,5], Francesco Saverio TORTORIELLO[1,6]

[1]Maths Department, University of Salerno, [2]L.S. "P.E. Imbriani", Avellino
[3]mapinadesso@gmail.com, [4]rcapone@unisa.it, [5]orianafio@gmail.com,
[6]fstortoriello@unisa.it

ABSTRACT

In this paper, we present the work of an ongoing project, devoted to 13–15 years old students in southern Italy, aiming to improve their mathematical literacy.

Starting point of our research was the way to spread the culture in the sixteenth century Academies.

There were about 800 Academies in Italy in 1525-1700 years: they were fundamental for the development of intellectual networks and the dissemination of ideas in Europe. In these Academies, interdisciplinary paths were developed, including literature, arts, natural sciences, medicine and mathematics.

In Mathematics, there was a fervent discussion about "new geometry", i.e. calculus vs "old" synthetic geometry. Because of calculus success, some very interesting results about synthetic geometry were neglected.

Aim of present work is planning and experimenting an interdisciplinary learning unit, focused on some hidden historical theorems of synthetic geometry developed for 1500-1800 years about "Cevian, Orthic and Pedal Triangles", by simulating "old Italian Academies" in a modern key, also by using ITC.

These theorems are quite interesting from a Maths and Physics point of view, too. We studied, in particular, Ceva's theorem and Fagnano's problem.

Both Ceva and Fagnano theorems may be generalized, under some conditions, to orthic quadrilaterals of a convex quadrilateral and Fagnano's problem may be extended to "polygonal billiard" physics, too.

In order to re-discover these hidden theorems, blended learning and flipped lessons methodologies have been also used.

The planned learning unit has been experimented with about thirty students, attending the second year of a Secondary School in South of Italy. Italian Academies have been simulated in two ways: virtual and blended academies were founded, respecting main characteristics of ancient Academies. English has been chosen as universal language, instead of Latin used in the past. Nevertheless, some historical documents (e.g. Ceva theorem) have been studied in Latin and in archaic Italian, to critically analyze original fonts.

Information Technologies can provide significant advantages in this learning – teaching approach: they have been used to share and communicate main results, to create "virtual Academies" and to realize multimedia materials (a final video, summarizing all the activities), too, which are "modern fonts" to be archived on cloud software's.

The experimented educational path shows that, although students generally consider Geometry as a difficult branch, they may excite if they study it in an innovative way and in an historical environment.

1 Introduction

A scientific revolution arose during the seventeenth centuries, thanks to the introduction of a new Maths branch with respect to traditional "Synthetic geometry": the "calculus" (1666-1687), also known as "the new Geometry", due to the mathematicians Isaac Newton (Newton, 1687) and Gottfried Leibniz, who independently invented it, with a different symbolism and approach. Because the success of calculus, some very interesting results about synthetic geometry were neglected.

This scientific revolution depended not only upon the introduction of new ideas, but also upon the development of new institutions for the acquisition and dissemination of knowledge. The most important of these new institutions were the princely court and the informal Academy (Ruscelli et al., 1984).

Over 800 academies flourished in Italy in the period 1525-1700, forming a significant and influential aspect of social and intellectual culture. Interdisciplinary in their interests, bridging literature, arts, medicine, and sciences, the Academies operated outside, but were often interconnected with official institutions like universities, courts, political and religious bodies, and offered a more flexible, apparently ‘free’ and ‘equal’ form of association. Members or affiliates could sometimes include socially marginal figures like women and artisans. Academies also attracted foreign intellectuals and their networks extended across Europe.

Scientific Academies were in all the Italian towns, first of all in Salerno and in Padua.

In scientific Academies, some “geometric problems” were also discussed, taking into account both calculus and synthetic geometry solutions. Few details may be found about these “discussions”, also due to the general “academic rules”: the members could only have access to all the research material developed inside the Academy.

In the present work, we show an interdisciplinary educational path, in which “hidden” (i.e. absent on the schoolbooks) synthetic geometry theorems have been re-discovered by simulating scientific Academies in a modern key, in a second class of a Secondary School, by using ITC (in particular social networks and GeoGebra), analyzing historical texts and using different maths approach. The class was subdivided into three Academies, where members were connected between them by using a social network. In these modern Academies, students also researched some information about ancient Italian Scientific Academies, evidencing the social connections between them, by using historical research methods in an intuitive way. Rarely, teacher scaffolding has been necessary.

Our research starts from these questions:

- May a “social” comparison between students improve their motivation in studying Mathematics?
- May an activity based on an historical, students self-managed, research intrigue and arouse interest in Mathematics, and especially in Geometry?
- May the use of an informal language encourage the students to discuss about Maths?

2 Italian Scientific Academies: main characteristics

In order to better simulate Italian Academies, main characteristics have to be outlined. Italian Academies of sixteenth and seventeenth centuries were a knowledge dissemination model, which was emulated in all the Europe.

Some common features may be found in all the Academies: an effective name, a logo, a motto, generally a noble or a “celebrity” who favored its growth, interdisciplinary discussions and freedom of thought.

Testa (2012), by analyzing these main characteristics, suggested that “Academies were the first intellectual networks of early modern Europe”.

Their membership included women as well as men, and representatives of all social classes. In addition to their intellectual pursuits, the Academies had a more playful aspect, including the delivery of orations based on paradoxes, the performance of games, or the invention of amusing names for the Academy.

Members of Academies frequently published, for many different reasons, under their Academy nicknames, which often reflected the Academy name.

For 1525-1700 years, Academies became more specialized: some literary, artistic and scientific Academies were born in different Italian towns.

A study on academies in Bologna, Florence and Naples (Irace and Panzarelli Fratoni, 2011) analyses most relevant topics in the Academies: 7,7% is about science and maths.

Nevertheless, Scientific Academies were often in contrast with the Universities. This contributed to the University crisis. New topics and new methodologies were developed inside the Academies, which were absent in the University learning, in particular, an increasing freedom of thought characterized the Academic studies.

Mathematics discussions were mainly focused on Geometry, taking into account the “new branches” (calculus and projective Geometry) and “the old one” (synthetic geometry).

3 Educational Path “Academy 2.0”

An interdisciplinary educational path has been planned rounding about re-discovering of “hidden” geometry theorems and problems, which are missing in maths schoolbooks, but are largely diffused on the web.

Arts, Latin, English, History and Literature have been involved in addition to Maths, to effectively organize and realize this learning unit.

The present path has been experimented in a second class of scientific high school (about thirty students).

We simulated two different types of Academies:

a) Virtual Academy, in which members communicate between them only by using social networks;

b) Blended Academy, with both virtual and live meetings.

For each Academy we chose: a logo, a motto, some communication and sharing rules and a specific dress, for the blended Academies, too, as in the old Academies.

Students used their nicknames to login to social networks, in these modern Academies.

4 Methodologies and Information Communication Technologies

Integration between Flipped Classroom (Bergmann and Sams, 2012) and Blended Learning methodologies, with a *blended-on line* and in-class format approach (Novak et al. 1999) in a *Student-Centered Active Learning Environment* has been used.

We didn't choose a devoted platform, but we preferred social networks (*WhatsApp*) for a free learning materials exchange and for an effective communication between members. Main advantage of the chosen social network is that it is possible to create closed groups, doing their research “secret”. Moreover, all the students daily use it.

An interactive and dynamical geometry software, GeoGebra, has been used, to verify main theorems. Students used their smart-phones and/or tablets, as in Bring Your Own Device (BYOD) practices, whereas a LIM and a PC have been used in the classroom, too. In order to verify geometry theorems, Information and Communication Technology (ICT) laboratory has been often used.

5 Experimental Activities

The educational path has been subdivided in three phases:
1) Learning activities in virtual Academies
2) Live meeting in Blended Academies
3) Final discussion and products realization.

5.1 Phases 1: Virtual Academies

The experimentation started by creating three virtual Academies (each one with about ten students as members), named as their specific research theme:

- Cevian Triangles;
- Pedal Triangles;
- Orthic Triangles.

Whats App groups were the "virtual places", where students met and had their initial learning activities. The "ImageGroup" was the Logo, chosen in agreement with all the members (which were all administrators, so nobody was the leader). Teacher was also added to each Academy group, but only as a moderator.

The "virtual members" posted on the groups their web researches, video, images, idea, suggestions and comments.

Communication was asynchronous, students feel free to have their research in each place and time, so these *e-learning* activities were effective, and students were enthusiast. Virtual debates were very useful to "have a trace" about free discussions.

An introductive message was posted in each Academy/group by the teacher, suggesting main topic, as an example:

"*27/10/17, 18:14 Hi guys, now you are Pedal Academy members. Search on the web and post here all you can find about pedal triangles, both historical and geometric aspects*".

Informal language has been used by the teacher, too. No indications about websites to be used have been furnished by the teacher.

Dialogues posted by students prove the effectiveness of e-learning activity, because peer to peer education gave the opportunity to work in the zone of proximal development (Vigotsky, 1978).

Just as an example, let's analyze a brief virtual dialogue, between students and teacher, in the Pedal Academy Group:

27/10/17, 18:28 Mirko*: I found and interesting link*
http://web.mclink.it/MC2113/geometria/java/Tpedali.html

28/10/17, 19:02 Paolo*: I found definition of pedal triangle: in Geometry a pedal triangle of a point with respect a triangle is identified by the point projections on the triangle sides.*

[...]

05/11/17, 19:29 - Teacher: *Question: May I choose all the points I want?*

05/11/17, 19:42 Andrea: *Yes you can, orthocenter, incenter and circumcenter, too*

05/11/17, 19:43 Gabriele: *No, you can't choose all the points, they have to be inside the triangle.*

05/11/17, 19:47 Mirko: *I agree with Gabriele, just for an example, I think that the circumcenter of an obtuse triangle is out of the triangle, so I think it isn't OK.*

Andrea: *used an emoticon image to say "I LIKE"*

05/11/17, 20:05 Teacher: *Have you tried to represent it with some special points?*

Andrea posted 3 Geogebra files (as you can see in the Topics section)

05/11/17, 20:12 Teacher: *You have just studied circumference, look if there is a connection between pedal triangle and circumference.*

05/11/17, 20:14 Gabriele: *Prof, I'm trying... I draw the pedal triangle of the orthocenter (D), by Geogebra, I draw a circumference passing by D and a vertex C, CD is a diameter...*

05/11/17, 20:20 Andrea: *Gabriele, I put the Pedal of the Incenter, CD is always the diameter...*

05/11/17, 21:05 Giampietro: *Diameter is perpendicular to the cord, it is obvious*

Teacher: *Wonderful*

05/11/17, 21:50 Paolo: *I'm searching for historical origin of pedal word, but I didn't find anything...who does invent this word?*

By analyzing all the web debates, we may answer to some research questions; in particular, it is evident that:

- ✓ students discuss about Mathematics between them, without any fear to make a mistake;
- ✓ students use an informal language, also including "emoticon" to approve or disapprove, as in a friendly chat;
- ✓ in this free scheme framework, students have the opportunity to do mathematical and historical research and rediscover some results, which are not present in their schoolbooks, but are strictly connected to standard geometric topics;
- ✓ students are very interested in the topics, so they continue to study Maths also late in the evening.
- ✓ in the research/discovery phase, GeoGebra has a relevant role, as a flexible instrument to immediately verify some student's hypothesis and some theorems they found on the web.

Moreover, for each group, teacher suggested an extra topic: to find some information about a specific Scientific Academy. For this historical purpose, students feel enthusiastic and posted their link late in the evening, too.

Nevertheless, we noted that, in this research phase, a great students' astonishment was when they discovered that on the web all the historical texts concerning the studied theorems and numerous news concerning the Academies to be discovered could be consulted.

5.2 Phase 2: From Virtual to Blended Academies

After the "discovery phase", students had some problems to virtually organize a coherent sub-unit about their own results: it was difficult, in a virtual discussion, to individuate main elements and synthesize them.

For this reason, all the students agree to change the "virtual Academies" in "blended" ones: members met at school, discussed about their topics and each Academy summarized its results in a poster, (see for example Fig.5.1 for Cevian Poster). No specific difficulties have been evidenced in the student's work.

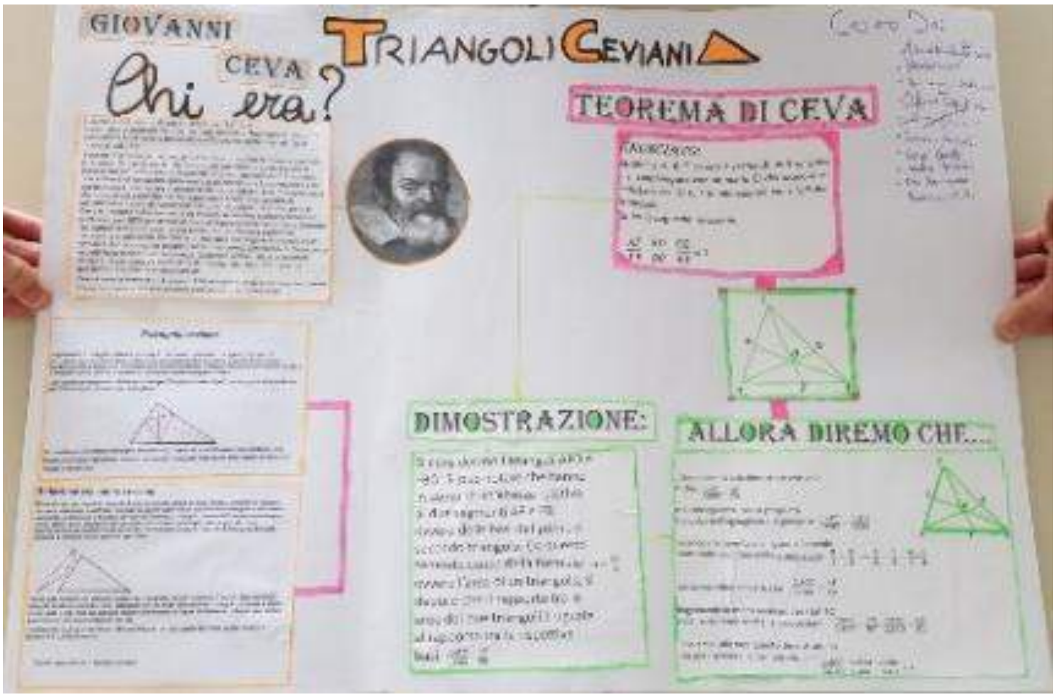


Figure 5.1: Posters realized by “Cevian, Pedal and Orthic Triangles Academies”

As you can see in Fig. 5.1, both historical and maths results have been organized in the posters, together with the GeoGebra verification's theorems. Some theorems proofs have been also posted, as the students learned, directly on the web.

5.3 Discussions and Final Product

In a final meeting, students belonging to each Academy, also wearing a specific uniform (with red, blue and white shirt, respectively for Cevian, Pedal and Orthic Academies), described their “research results” to all the “community”.

Literature, Physics and Mathematics teachers also were present at this meeting. They discussed about the realized posters and showed some geometric results by using GeoGebra. Some of them, in particular Cevian Group, discussed about historical documents they analyzed, in Latin language, from the book “*De lineis rectis se invicem secantibus, statica constructio (1678)*”, directly consulted on the web.

All the students were surprised about some common features in the analyzed topics, which will be outlined in the Topic paragraph.

At the end of this experience, students were invited to realize a multimedia product (video), summarizing their results and their own point of view about this adventure. The realized video (about 10 minutes long, that you can find on YouTube in the Italian version) was effective, too. They also included in it some comments, which are similar to “slogan”, i.e.:

– “From the past to the future: we worked as Academy members”
– “Our Academies 2.0: comparison, dialogues and research”
– “Learning all together is special!”

6 Topics

Analyzed topics were geometry theorems about triangles, discovered after 1500, which are not classified, not well dated, not always organized, not present in maths school books, but they are very useful, interesting and easy to be studied at High School and directly connected with curricula.

We focused on three sub-units (corresponding to the three virtual Academies). Just some results, which have been individuated by the students, are shown here.

6.1 Cevian Triangles

A Cevian is any line segment in a triangle with one endpoint on a vertex of the triangle and any other endpoint on the opposite side.

Given a point G, interior of a triangle ABC, the Cevian triangle A'B'C' is defined as the triangle composed of the endpoints of the cevians, being G the Cevian Point (see Fig.6.1).

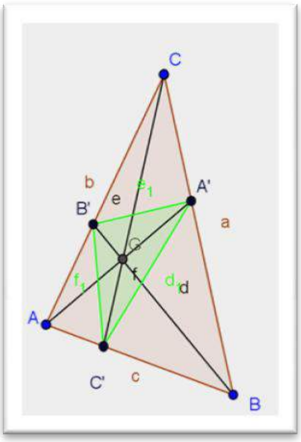


Figure 6.1: A'B'C': Cevian triangle of the main ABC triangle, G = Cevian Point.

The condition for three general Cevians from the three vertices of a triangle to concur is known as Ceva's theorem**:**

In a triangle ABC, three lines AQ, BR and CP intersect at a single point G (i.e. they are concurrent) if and only if:

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RA} = 1$$

As laboratorial activities, students verified it by using GeoGebra (See Fig. 6.2), in the ICT laboratory.

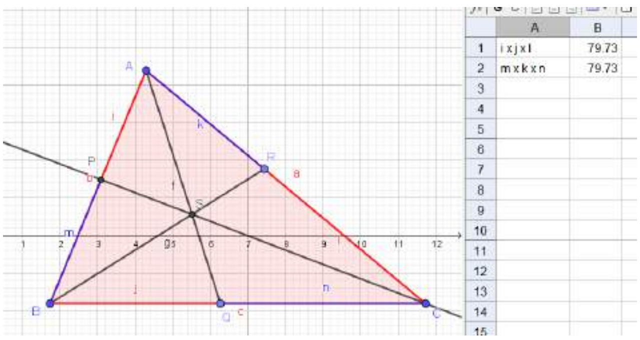


Figure 6.2: Test of “Ceva's theorem” by using GeoGebra

The use of Ceva's theorem is an effective way to introduce remarkable points (barycenter, incenter and orthocenter points), in a nontraditional way: medians, angle-bisectors and heights are cevians, all concurrent in a specific point, as we show here:

Theorem 1 (Existence of Barycenter) - Medians are cevian, concurrent in a point named Barycenter.

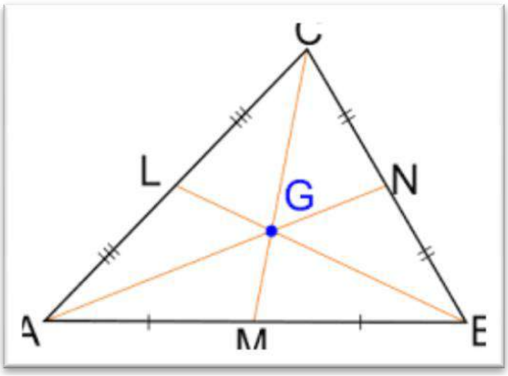


Figure 6.3: Medians concur in the Barycenter

Proof:

Medians connect vertices with the midpoints of the opposite sides.

Therefore,

$$AF/FB = BD/DC = CE/EA = 1.$$

Each of the ratios is 1 and so is their product.

Theorem 2 **(**Existence of Incenter) - Angle bisectors are cevian, concurrent in a point named Incenter.

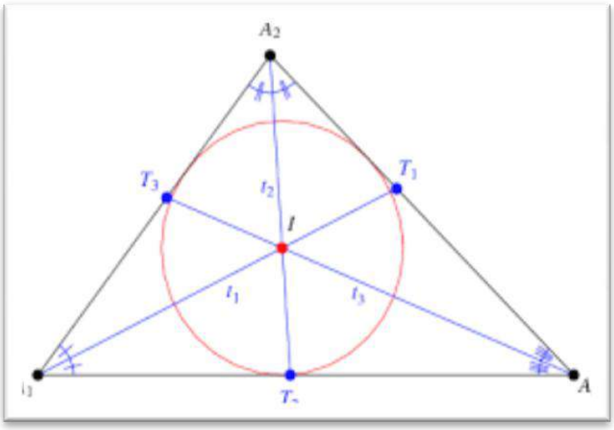


Figure 6.4: Angle bisectors concur in the Incenter

Proof:

For angle bisectors theorem,

$$AF/FB = AC/BC, BD/DC = AB/AC,\ CE/EA = BC/AB.$$

Multiplying the three yields 1.

Theorem 3 **(**Existence of Orthocenter) - Heights are cevian, concurrent in a point named Orthocenter.

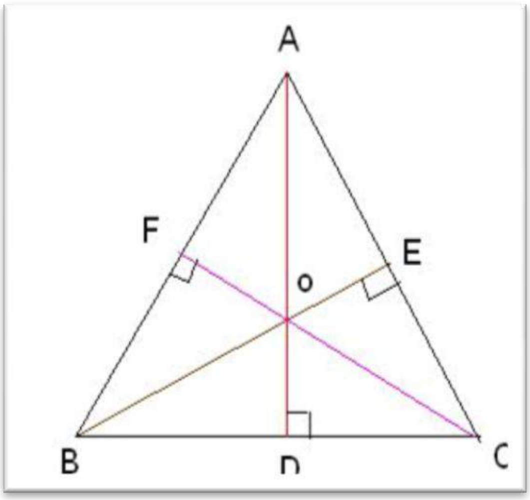


Figure 6.5: Heights concur in the Orthocenter

Proof:
Indeed, right-angled triangles ACD and BCE are similar.
Therefore $\frac{CE}{DC} = \frac{BE}{AD}$

Analogously, $\frac{AF}{EA} = \frac{CF}{BE}$

and $\frac{BD}{FB} = \frac{AD}{CF}$

Indeed: $AF/FB \cdot BD/DC \cdot CE/EA = CE/DC \cdot AF/EA \cdot BD/FB = BE/AD \cdot CF/BE \cdot AD/CF = 1$

6.2 Pedal Triangles

A pedal triangle is obtained by projecting a point onto the sides of a triangle.

More specifically, let's consider a triangle *ABC*, and a point *P* which is not one of the vertices *A*, *B*, *C*. Let's drop perpendiculars from *P* to the three sides of the triangle (these may need to be produced, i.e., extended) and label *L*, *M*, *N* the intersections of the lines from *P* with the sides *BC*, *AC*, *AB*. The pedal triangle is the *LMN* one (see Fig. 6.6)

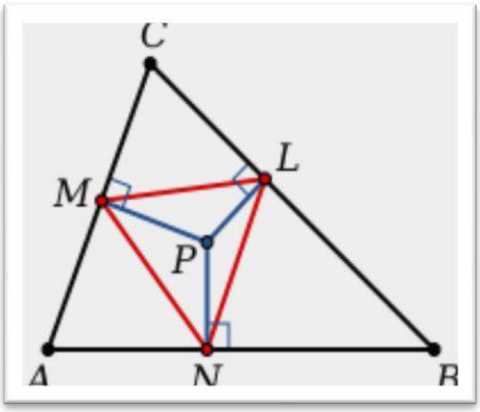


Figure 6.6: LMN: Pedal Triangle of main triangle ABC

Several properties can be proved about pedal triangles, all starting from a "main property":

Main Pedal Property: "Given an ABC triangle and an $A_1B_1C_1$ pedal triangle respect to P point, then A, C_1, P and B_1 belong to a same circumference, with AP as a diameter" (Fig.6.7).

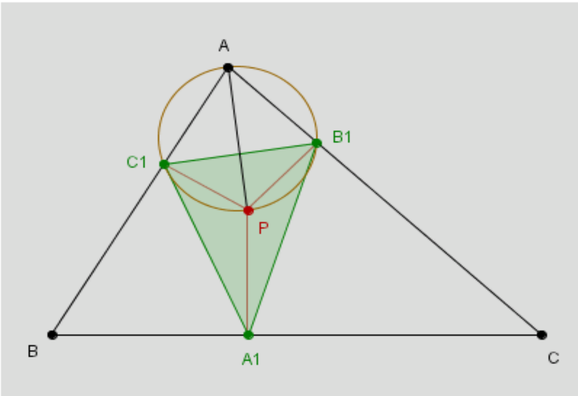


Figure 6.7: Pedal Triangles Main property: A, C1, P and B1 belong to a same circumference, being AP the diameter

6.3 Orthic Triangles

Given a triangle A, B, C, the triangle $H_AH_BH_C$, whose vertices are endpoints of the altitudes from each of the vertices of ABC is the orthic triangle. Orthic triangle is both the pedal and the cevian triangle of a specific point, the orthocenter.

An important property of "orthic triangle" is the following one:

Theorem 1. If DEF is the orthic triangle of ABC, then ABC heights are DEF angle bisectors, i.e. ABC Orthocenter is the Orthic triangle (DEF) Incenter.

Students also verified this theorem in the ITC laboratory, by using GeoGebra, as shown in Fig. 6.8.

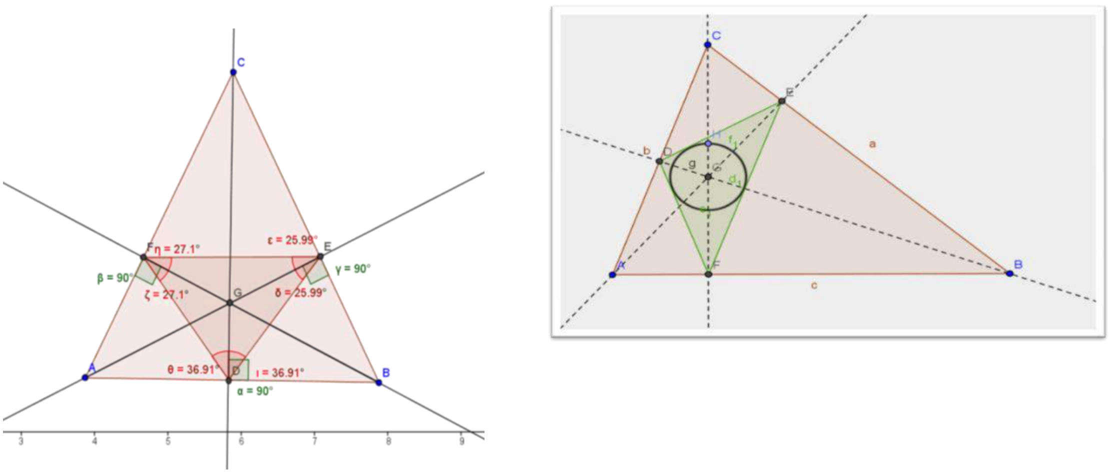


Figure 6.8: Test of the property "ABC orthocenter is the Orthic Triangle incenter"

During 1700s, because of calculus inception, main discussions were about minimum and maximum problems.

A known minimum problem about these topics was introduced by Giovanni Fagnano (Giulio's son) in 1775:

"For a given acute triangle determine the inscribed triangle of minimal perimeter". The solution is "the orthic triangle".

Fagnano' solution used the "calculus", whereas L. Fej´er and While H. A. Schwarz gave a proof by using synthetic geometry (axial symmetries), in an independently way.

Let's notice that Theorem1 does not immediately follow from Fagnano's Problem, since Theorem 1 is valid both if main triangle is an acute-angled and an obtuse-angled triangle, whereas the Fagnano's problem is true only in the acute-angled triangle case.

It is possible to extend these triangle results to "orthic quadrilaterals" (Mammana et al., 2010) and to billiard physics, too (E. Gutkin, 1997).

7 Extended Topics: Scientific Academies

In each Academy, students were invited to find some information about a Scientific Italian Academy. In particular, their researches were focused on Academia Secreta, Academia degli Infiammati and Accademia dei Lincei. Some news about these Academies are here summarized, as organized by the students in a final booklet.

The first Scientific Academies in the world were: Academia Secreta, probably in Salerno, and Accademia degli Infiammati in Padua, both founded in 1540.

Academia Secreta was founded by Girolamo Ruscelli; scientific interests were mainly about chemistry, alchemy and medicine. Few news can be found about this Academy (Ruscelli et al., 1984), the only reference book being "*Secreti Nuovi di Maravigliosa Virtù*", written by Ruscelli, under the nickname "*Reverendo Alexis Piemontese*", containing all the "recipes" about experiments they practiced in the Academy. The *Proemio* to "*Secreti Nuovi*" contains a description of Academy members and place where they met can be found, with no specific references.

Accademia degli Infiammati was founded by Leone Orsini, Frejus' bishop, Ugolino Martelli and Daniele Barbaro (6 June 1540).

Its motto was "*Arso il mortale, al ciel n'andrà l'eterno*", referring about Hercules in the fire on the Oeta mountain, as also shown in the Academy logo (Fig.7.1).

Figure 7.1: Accademia degli Infiammati, Logo

Some members of this Academy were: Giovanni Corner, Galeazzo Gonzaga, Alessandro Piccolomini, Sperone Speroni, Bembo, Lodovico Dolce, Torquato Tasso.

Aim of Accademia degli Infiammati was writing about phylosofical (including scientifical) and literary topics, both in prose and verse, by using the vernacular*: «vera et natural idea» di scrivere «compiutamente», in prosa e versi in volgare, su argomenti filosofici e letterari.*

An important, both scientifical and literary, contribution was due to “Sperone Speroni”, in its “*I Discorsi del modo di studiare, La difesa del volgare*”. Speroni outlined the importance of Mathematics (mainly Geometry) to be happy:

“Solamente le discipline matematiche (come la geometria) abbiana una utilità, perché servono come exercitio. La matematica allora, insieme alla logica, è fondamentale per il raggiungimento della felicità costituita dall’unione di sapienza e eloquenza: sono due quasi prohemii, o previe dispositioni alla felicità de’ mortali”

In 1603, the first European Scientific Academy was founded by Federico Cesi, Accademia dei Lincei.

Some members were: Giambattista Della Porta, Galileo Galilei, Francesco Stelluti, Anastasio De Filiis, Johannes van Heeck. .

Generally, Academies were ephemeral, often without programs and specific organizations. Nevertheless, Accademia dei Lincei had a specific program and admission rules, but it didn't survive when its prince, Cesi, died (1630). Nowadays, a renovated “Accademia dei Lincei” is going on, with specific scientific interests, this year under direction of a famous physician, Giorgio Parisi.

Scientific contributions of this Academy were mainly in “Astronomy”, thanks to Galileo Galilei.

8 Conclusions

An interdisciplinary learning unit has been planned and experimented in a second class of a secondary school, which turns around “synthetic geometry theorems, absent in the Italian maths schoolbooks”. This unit has been realized by simulating sixteenth-seventeenth century Scientific Italian Academies. About 800 Academies were founded in Italy in 1525-1700 years, being a worldwide, very important phenomenon that introduced a new way to acquire and disseminate knowledge.

Simulating an Academy also gave a considerable boost to the “Maths social use”, which is very important to increase motivation to study scientific matters, and in particular the Mathematics.

Flipped lessons and Blended Learning were effective methodologies to better organize and share research results.

Students feel enthusiastic about this new learning way and their feelings were also evidenced in all their dialogues and in a final video they realized to summarize all the activities about this learning unit.

REFERENCES

Bergmann, J., & Sams, A. (2012). *Flip your classroom: Reach every student in every class every day*. Philadelphia: International society for technology in education.

Gutkin, E. (1997). Two Applications of Calculus to Triangular Billiards. *The American Mathematical Monthly 104*(7), 618-622.

Irace E., & Panzanelli Fratoni M. A. (2011). Le Accademie in Italia dal Cinquecento al Settecento. In S.

Luzzatto, & G. Pedullà (Eds-in-chief) *Atlante della letteratura italiana. II. Dalla Controriforma alla Restaurazione* (pp. 314-322). Torino: Einaudi.
https://www.academia.edu/22838423/_2011_Le_accademie_in_Italia_dal_Cinquecento_al_Settecento_In_Atlante_Letteratura_Italiana_Einaudi (accessed 18/8/2019).

Mammana, F., Micale, B., & Pennisi, M. (2010). Orthic quadrilaterals of a convex quadrilateral. *Forum Geometricorum*, *10*, 79–91.

Newton, I. (1687). *De Philosophiae naturalis principia mathematica*. London: Jussu Societatis Regiae ac Typis Joseph Streater.

Novak, G. M., Patterson, E. T., Gavrin, A. D., Christian, W., & Forinash, K. (1999). Just in time teaching. *American Journal of Physics*, *67*(10), 937-938.

Ruscelli, G., Eamon, W., & Paheau, F. (1984). The Accademia Segreta of Girolamo Ruscelli: A Sixteenth-Century Italian Scientific Society. *Isis*, *75*, 327–342. https://doi.org/10.1086/353485.

Testa, S. (2016). Italian Academies and their "facebooks"'. In J. Everson, D. Reidy, L. Sampson (Eds.) *The Italian Academies 1525-1700: Networks of Culture, Innovation and Dissent - Proceedings of the International Conference* (The British Library, September 2012). Oxford: Legenda.

POTENTIAL FOR COLLABORATION BETWEEN HISTORY AND MATHEMATICS TEACHERS

An investigation and framework based on a text by Abu’l-Wafaʾ Buzj'ani

Abdelrahman AFFAN, Michael. N. FRIED
The Center for Science and Technology Education, Ben-Gurion University of the Negev, Beer-Sheva, Israel
affanabd@gmail.com, mfried4489@gmail.com

ABSTRACT

The main goal of our research study is to gain an understanding of mathematics and history teachers' beliefs and considerations towards the integration of history of mathematics, to investigate the potential for interaction and collaboration between them and what characterizes such a collaboration while they participate in a course of history of mathematics and in conducting and transferring a joint learning unit in class. To achieve our aims, a course on the history of Islamic mathematics based on Abū’l-Wafāʾ Buzj'ani book "On the Geometric Constructions Necessary for the Artisan" was designed for a group of history and mathematics teachers in the Arab sector of northern Israel. The research presented in this paper comes out of a project which is ongoing; therefore, only theoretical and methodological aspects of it will be discussed.

Keywords: History teachers, mathematics teachers, beliefs and considerations, Abū’l-Wafāʾ Buzj'ani

1 Introduction

The integration of history of mathematics into mathematics education has been a persistent interest in mathematics education. The importance of history of mathematics in schools' curriculum has been pointed out in documents of many professional councils. The HPM (History and Pedagogy of Mathematics) itself, of course, was founded on the basis of feasibility and potential of integrating history of mathematics into mathematics teaching and learning. This paper, which describes a current ongoing project with teachers, makes the further assumption that since history of mathematics is both mathematics and history, it is relevant both to mathematics teachers and history teachers. Indeed, we see teaching history of mathematics as a join effort on the part of these two communities. Before we get to this point, however, we shall consider some of the general issues against which our teaching project is set.

1.1 Justifications for Integrating History of Mathematics in Mathematics Education

For some researchers, history of mathematics should be taught and introduced into the curriculum hand in hand with mathematics since the history of a subject is an integral part of the subject (Sui & Tzanakis, 2004; Heiede, 1992), and, studying history of mathematics in conjunction with mathematics is a matter of general mathematical culture (Schubring, 2011).

In the last three decades, some scholars attempted to gather and list explicit reasons for integrating history of mathematics in mathematics' lessons. Fauvel (1991) listed some twenty reasons for integrating history of mathematics, and later, Tzanakis et al. (2000)

adduced 17 arguments for integrating history of mathematics into classroom.

Fried (2001) observed that arguments in favor of integrating HOM in mathematics education generally fall into five categories: 1) Humanizing subject matter; 2) Adding variety to teaching; 3) Showing alternative approaches to scientific ideas; 4) Analyzing students' understandings and misunderstandings; and 5) Deepening a sense of the nature of the discipline. Elsewhere, Fried (2014) categorized these justifications into three groups: 1) The Cultural theme; 2) The Curricular theme; 3) The Motivational theme.

For the project, the group of justifications most relevant are those under the cultural theme "recognizing the enterprise of mathematical inquiry as part of students' cultural heritage […] it is both a reflection on general culture as including mathematics and also on mathematics as being cultural" (Fried, 2014, P.680). These are also those must connected to history as such. Pursuing such a justification, on the other hand, is not always easy—not only from a practical point of view but from a theoretical point of view as well.

1.2 Difficulties in Integrating History of Mathematics: A Challenge for Teachers

Fried (2001) claimed that beyond the practical issues such as lack of finding time for fitting historical material into an already crowded curriculum or inaccessibility to correct and appropriate resources, the integration of history of mathematics hides a theoretic difficulty reflecting the tension between anachronism and relevance, and, between useful rational reconstruction and faithful historical analysis. He points out a dilemma whereby teachers must choose between taking a genuine approach to history – in which case, they end up spending time on things that are not part of the mathematics curricula—and trivializing history or taking an anachronistic (Whig) approach in order to use history as a tool for learning relevant mathematics.

> The main difficulty is that while one wants to see historical topics in the classroom or an historical approach in teaching, the commitment to teach the modern mathematics and modern mathematical techniques necessary in the pure and applied sciences forces one either to trivialize history or to distort it. In particular, this commitment forces one to adopt a "Whiggish" approach to the history of mathematics (Fried, 2001, p. 391)

The argument was that since teachers need to teach contemporary mathematics and its applications—for the power of contemporary mathematics is the predominant justification for its emphasis in school education— efforts to integrate the history of mathematics leads to a history that guided and measured by its relevance to the modern era. Thus, the teacher is compelled to adopt an unhistorical anachronistic history (Fried, 2001). The mathematics teacher's inner commitment to modern mathematics actually acts against history of mathematics into mathematical education, because the teacher has to filter out from history of mathematics the relevant and useful material, this the teacher is pushed to adopt a "whiggish" approach to history (Fried, 2001), that is, where the teacher measures what is important and what was the meaning of this thing in the past through the viewpoint of the present (Kragh, 1987). The term "Whig history" of course was a notion introduced into historiography by the British historian Herbert Butterfield (1931/1951), and to describe history in which the present is the measure of the past so that what is considered significant in history is precisely what leads to something deemed significant today, and, Butterfield's point was that that kind of history is not history at all.

1.3 Really a Matter of Two Communities

The difficulty described by Fried (2001) was conceived in terms of a hypothetical mathematics teacher who wants both to teach mathematics within the framework of a typical mathematics curriculum and also introduce history of mathematics into the lessons. In this way, he brought out a dilemma which the entire community of mathematics teachers interested in history must confront. In our opinion, this dilemma arises because the question is actually posed to one community only, namely, mathematics teachers. In fact, it can be thought of as referring to *two* communities:

- Mathematicians or mathematics teachers
- Historians or history teachers

The difficulty of the first community, that of the mathematics teachers, is been already stated. But, what about the history teachers? History teachers might do better than mathematics teachers in avoiding the trivialization of history if they took up the history of mathematics in their classrooms; on the other hand, history teachers cannot be expected to understand the spirit of mathematical thinking and may need to be convinced that history of mathematics is as relevant to history as it is to mathematics. So both communities have something to contribute and their difficulties to confront. They may not each by themselves be able reconcile the conflicts producing those difficulties. However, they may be able to do this working together. Therefore, we think of introducing the history of mathematics in mathematics teaching as a fundamentally multi-disciplinary effort that requires enlarging the community which deals with it. For since history of mathematics belongs both to history and to mathematics, it takes in both the communities of history teachers and mathematics teachers. Bringing together history teachers and mathematics teachers in the context of history of mathematics—even the specific context which we have chosen, namely, the work of Abu'l Wafa— we should point out has been attempted before by Marc Moyon and his colleagues (Moyon, 2013). However, our work takes up the problem in light of the theoretical issues described above.

2 Aims of the Study

Our study brings history teachers and mathematics teachers together in order to determine, first**,** the presuppositions of history and mathematics teachers, respectively, regarding the teaching of history of mathematics. We wish to see whether the presuppositions which are implicit in the disciplinary commitments of each community can be seen in the actual work of these teachers as they consider an appropriate historical topic. At a somewhat more basic level, before we ask mathematics or history teachers to integrate history of mathematics in mathematics education, we may need to stop for a moment and ask ourselves: what they think about this integration, who should teach history of mathematics, what kind of knowledge (along the lines of Shulman, 1986) mathematics teachers need to know about history, and what history teachers need to know about mathematics, and what they know about the nature of the two disciplines? Answering these questions is one aim of the study.

The second aim is to determine whether history teachers and mathematics teachers are genuinely able to work together to produce a classroom unit on the history of mathematics and what are the characteristics of this collaboration. This in effect asks the question whether the two communities truly can settle together the difficulties that cannot be settled each alone.

We should point out here at the outset that the study is ongoing and we still do not have all of the results. Hence, this paper will focus mostly on theoretical and methodological issues.

3 Approach

Given the aims stated above, we need to investigate the following research questions.

3.1 Research Questions

1. What are the considerations, presuppositions, and beliefs of teachers for mathematics and history regarding the integration of history of mathematics into the classroom? and, is there any significant difference in these considerations and beliefs, in either population or both, according to years of experience, gender, education, grade level and school level (elementary, middle school, high school and teacher teachers)?

2. How and to what extent might a course in the history of mathematics impacts the considerations, and beliefs of the two teacher populations regarding mathematics, history, and the history of mathematics? and what are the characteristics of the learning experience (cognitive, interpersonal and mental emotions aspects) of the mathematics and history teachers during their joint participation in a course in the history of mathematics?

3. How feasible is an interdisciplinary collaboration between mathematics and history teachers? To the extent that it is feasible, what are the characteristics of this collaboration, and how it is reflected in the construction and transfer of a joint learning a unit in history of mathematics?

3.2 Participants and Research Tools

The participants in the first stage (Question1) in this research are **approximately 350 in-service mathematics and history teachers** from elementary, middle and high Arab schools in Israel. A questionnaire will be given to the teachers.

In order to answer the second question, we will examine how a course which is based on the history of Islamic mathematics impacts the consideration and beliefs of mathematics and history teachers, and how they act during participating in a common course and what are the characteristics of learning and thinking process. The researcher of this study is also the instructor of the course and the participants are **20 teachers** (10 mathematics teachers and 10 history teachers) from four schools (two elementary schools, one middle school, one high school) will participate in a course-based on history of Islamic mathematics.

The course has been designed to engage both communities at once. The specific historical material is based on texts published by the Islamic mathematician and astronomer, Abū'l-Wafāʾ Buzj'ani, who lived in the 10th century (940–998 CE) *"Kitāb fī mā yaḥtaj ilayh al-ṣāniʿ min al-aʿmāl al-handasiyya"* (Book on what is necessary from Geometric Construction for the Artisan). For this project we will use Istanbul manuscript [in Arabic] which was copied in the first half of the 15th century and Abū'l-Wafāʾ Buzj'ani book edited by Salih (1979). As mentioned above, this was also the text used by Marc

Moyon in his work, and, Moyon has discussed its importance in more than one paper (see Moyon, 2011, 2013). We were unaware of Moyon's work in this connection until recently and considered Abū'l-Wafāʾ as a central text independently: this only underlines the striking and suggestive character of that text.

In the third stage, about 10 mathematics and history teachers will work actively together from a pedagogical standpoint. The participants will be divided into pairs (**five couples**, each couple consist one mathematics teacher and one history teacher who took the course of Islamic Mathematics). Each pair will be asked to teach a short lesson along the lines of the material in the course, but not necessarily concerning Abū'l-Wafāʾ specifically. We chose this approach since we would like ultimately for the mathematics teachers and history teachers to work together in their own schools: this activity then allows them to experience this kind of cooperative effort and allows us to see the difficulties that may be involved. Accordingly, during the lesson, we will make observations and keep a log that will document events, interpretations, significant utterances, as well as personal thoughts and feelings. Our intention is also to incorporate video recording in the observations according to visual ethnography principles.

Abū'l-Wafāʾs treatise is a collection of 171 problems of geometry, divided into 11 chapters; it includes 150 problems of plane geometry and the rest on spheres and polyhedrons. The introduction in the book gives 13 chapters; two chapters are missing "On the division of scalene figures" and "On tangent circles".

Chap. I. Introduction

Chap. II. Basic constructions

Chap. III. Construction of polygons

Chap. IV. Inscription of polygons in the circle

Chap. V. Circumscription of the circle around polygons

Chap. VI. Inscription of the circle in polygons

Chap. VII. Inscription of polygons with each

Chap. VIII. Division of triangles

Chap. IX. Division of quadrilaterals

Chap. X. Division and composition of squares

Chap. XI. On dividing spheres

Abū'l-Wafāʾ Buzjani tells us in the introduction of his book, that he attended meetings between geometers and artisans in Baghdad, such meetings were a widespread phenomenon in the Islamic world. This book provides insights into how mathematicians and artisans collaborated in the Islamic culture.

> I was present at some meetings in which a group of geometers and artisans participated. They were asked about the construction of a square from three squares. A geometer easily constructed a line such that the square of it is equal to the three squares, but none of the artisans was satisfied_with what he had done. The artisan wants to divide those squares into pieces from which one square can be assembled, as we have described for two squares and five squares.

(Abū'l-Wafāʾ in Alpay, 2000, p. 174)

In his book, Abū'l-Wafā᾽ understood the needs and problems of the artisans and was motivated by such meetings and by his efforts to advance Islamic art (Alpay, 2000), he displayed knowledge of pure geometry, familiarity with practical applications and skill in teaching theoretical subjects to practical-minded people.

3.3 Why this Book?

This particular work was chosen for three main reasons:

i) It contains significant mathematical content

Many of the examples in the book were original and some were borrowed from ancient Greek writings of Euclid, Archimedes, Theodosius of Berga, and Pappus. Abu'l-Waf¯a' gave instructions on certain geometric constructions of two or three-dimensional ornamental patterns and also he gave advice on the application of geometry to architectural construction by using cut and paste methods as a didactical tool in teaching geometry to artisans (Alpay, 2000). Most of the constructions given in the book, however, are meant to be carried with compass and straightedge.

The range of problems is very wide, from the simplest planar constructions (the division of a segment into equal parts) to non-trivial problems concerning polyhedrons inscribed in a given sphere

ii) The mathematics had a clear social and cultural context,

Since the mathematics involved had clear, social, and cultural aspects, it touches upon Islamic history in addition to Islamic mathematics. Evidence to the second reason, we can see with the relationship between Islamic tradition/ religious and the Islamic sciences. The Prophetic tradition" May God protect us from useless knowledge" may shed a light on the treatment of the Islamic scholars with sciences. In this spirit, the philosopher and theologian Abu Hamid Al-Ghazali wrote:

> The problems importance of physics (*he was referring to aristotelian natural philosophy*) are of no importance for us in our religious affairs or our livelihood, therefore, we must leave them alone[…] Man has created only to know, but the knowledge man has been created to seek is that which brings him closer to his creator[…]this means, not only that religious knowledge is higher in rank and more worthy of pursuit than all other forms of knowledge, but also that all other forms of knowledge must be subordinated to it
>
> (Al-Ghazali in Sabra, 1987, p. 239).

iii) The treatise concerns the work of ordinary people, the artisans, and is therefore, directed towards a greater set of societal concerns.

In this work by Abū'l-Wafā᾽, there are problems concerning the division of a figure into parts that satisfy certain conditions, and problems on the transformation of squares (for example, the construction of a square whose area is equal to the sum of the areas of three given squares). In proposing his original and elegant constructions, Abū'l-Wafā᾽ simultaneously proved the inaccuracy of some methods used by "artisans."

4 Significance of the study

Since the project described above is ongoing, we still do not have definite conclusions to report. In lieu of that, we cite three points that we think may be significant outcomes of the study when it is completed.

i. The integration of history of mathematics, according to many studies, significantly contributes to students. However, this subject is insufficiently integrated in schools. Assuming the teachers have central role in such integration, the current work might shed some light on the consideration of the teachers that lead to this situation and how those considerations may differ with respect to whether a teacher is a mathematics teacher or a history teacher (who indeed may never have considered history of mathematics as history at all!).

ii. Understanding what teachers should know about their profession when integrating the disciplines of mathematics and history, and what didactic skills are needed for them in order to generate an intellectual experience for their students. The research, in this regard, may open a new aspect of pedagogical content knowledge. Again whether that aspect of pedagogical content knowledge can take on a unified form across the disciplines of mathematics and history.

iii. Uniqueness of the study – to date, a study that examines the considerations mathematics teachers in comparison to those of history teachers regarding the integration of history of mathematics into classroom, in particular and the different practical and theoretical assumptions, and their openness to the interdisciplinary approach, in general, except a few isolated cases, such as the studies of Moyon, has not been done in intensive way.

Acknowledgment

We are grateful to Marc Moyon for informing us about his work in this connection and sending several of his papers to us.

REFERNCES

Abū'l-Wafā̃ʾ Buzj'ani. *Book on what is necessary from Geometric Construction for the Artisans*, Istanbul: Suleymaniye Kutuphanesi, Ayasofya 2753, copied in the first half of the 15th century. [Arabic manuscript], available online at https://www.alukah.net/library/0/88512/

Alpay, O. (2000). Mathematics and arts: Connections between theory and practice in the medieval Islamic world. *Historia Mathematica 27*, 171–201. doi:10.1006/hmat.1999.2274, available online at http://www.idealibrary.com.

Butterfield, H. (1931/1951). *The Whig interpretation of history*, New York: Charles Scribner's Sons.

Fauvel, J. (1991). Using History in Mathematics Education. *For the Learning of Mathematics* (Special Issue on History in Mathematics Education), *11*(2), 3–6.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391-408.

Fried, M. N. (2014). History of Mathematics and Mathematics Education. In Michael Matthews (ed.), *History, Philosophy and Science Teaching Handbook* (Volume I, pp.669-705). New York: Springer

Heiede, T. (1992). Why teach history of mathematics? *The Mathematical Gazette, 76*, 151–157.

Kragh, H. (1987). *An Introduction to the Historiography of Science.* Cambridge: Cambridge University

Press.
Moyon, M., (2013). Diviser en multipliant les approches. Quand les mathématiques remontent aux sources, *Repères-IREM*, *93*, 47-77.
Moyon M., (2011). Practical geometries in Islamic countries: the example of the division of plane figures. In M. Kronfellner, É. Barbin, & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education. Proceedings of the 6th European Summer University* (pp. 527-538). Vienna: Verlag Holzhausen GmbH.
Sabra, A. I. (1987). The Appropriation and Subsequent Naturalization of Greek Science in Medieval Islam: A Preliminary Statement. *History of Science*, *25*, 223–243.
Salih, A. (1979). *Abū'l-Wafā᾽ Buzj'ani, Book on what is necessary from Geometric Construction for the Artisans*. University of Baghdad, Baghdad [in Arabic].
Schubring, G. (2011). Conceptions for relating the evolution of mathematical concepts to mathematics learning—epistemology, history, and semiotics interacting. *Educational Studies in Mathematics*, *77*(1), 79-104.
Siu, M. K., & Tzanakis, C. (2004). History of mathematics in classroom teaching – appetizer? Main course? or dessert?. *Mediterranean Journal for Research in Mathematics Education* (Special double issue on the role of the history of mathematics in mathematics education) *3*(1-2), v–x. (proceedings from TSG 17 at ICME 10).
Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher. 15*(2), 4–14.
Tzanakis, C., Arcavi, A., de Sá, C. C., Isoda, M., Lit, C.-K., Niss, M., de Carvalho, J. P., Rodriguez, M., & Siu, M. K. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study* (pp. 201 – 240). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Appendix

Constructing a square from three equal squares

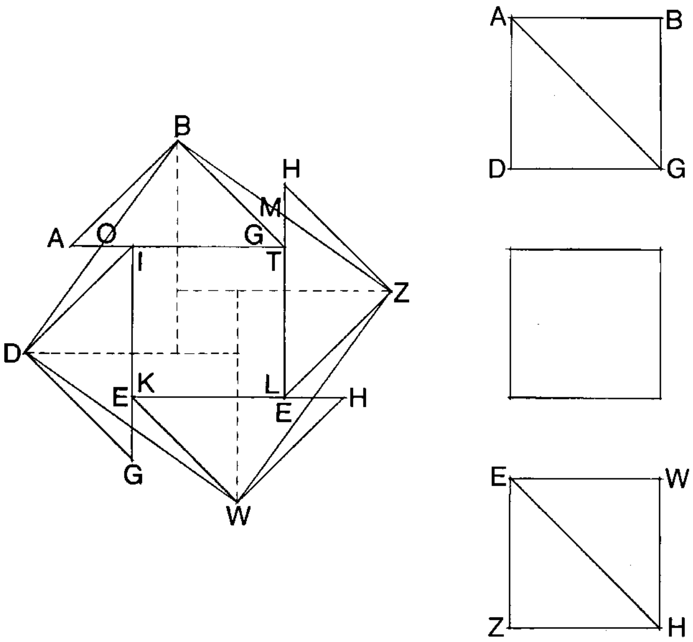
فأما المهندس فإنه إ ذا سئل عن عمل مربع من مربعات قلت أو كثرت فإنه يحدس الخط الذي هو على تلك المربعات ولا يبالي بتقطيع المربعات كيفما كانت.

وذلك أنه إذا سئل عن عمل مربع من ثلاث مربعات فإنه يوصل قطر أحد المربعات ويقيم على احد طرفي القطر خطا يكون عمودا عليه مساويا لضلع المربع المؤلف من ثلاث مربعات

مثال ذلك

فإذا أردنا أن نعمل مربعا واحدا مساويا لثلاث مربعات كل واحد منها مساو لمربع أ ب ج د , أخرجنا قطر ا ج فيكون أ ج ضلع المربع المركب من مربعين ثم أقمنا على نقطة أ من خط أ ج عمود أ ه مساويا لخط ا ج ووصلنا ه ج فيكون خط ه ج ضلع المربع المساوي لثلاث مربعات كل واحد منها مساو لمربع أ ب ج د (فإذا حصل عند المهندس هذا الخط لم يبال بعد ذلك كيف كان تقطيع المربعات , وقال أنه متى عمل على خط ه ج مربعا كان مساويا للمربعات الثلاثة)

If we want to construct a square from three equal squares *ABGD;EWZH; TIKL*, we bisect two of the squares at their diagonals, by means of lines *AG*, *EH*, and we transport [them] to the sides of the [third] square. Then we join the right angles of the triangles by lines *BZ*, *ZW*, *WD*, *DB:* On either side [of the straight line], a small triangle has now been produced from the sides of the [two big] triangles. That [empty position of the triangle] is equal to the triangle which has been cut off from the big triangle. Thus triangle *BGM* is equal to triangle *MZH*, since angle *G* is half a right angle, angle *H* is half a right angle, the two opposite angles of the triangles at *M* are equal, and side *BG* is equal to side *ZH*. Therefore, the remaining sides of the triangles [*BGM*, *MZH*], and the triangles are equal. Thus, if we take triangle *BGM* and put it in the position of triangle *MZH*,18 line *BZ* is the side of the square constructed from three squares. This is a correct method, easier than what was constructed [by the artisans], and the proof of it has been established. This is the figure for it (translated by Alpay. O, in Alpay, 2000, p. 183)



BOREL'S APPROACH TO MATHEMATICS, PROBABILITY AND CITIZENSHIP

Alain BERNARD
University of Paris Est Créteil (ESPE), rue Jean-Macé, 94380 Bonneuil sur Marne, France and Centre A. Koyré (UMR8560)
alain.bernard@u-pec.fr

ABSTRACT

This paper proposes to reflect about contemporary issues on the place accorded to probability and statistics in the French mathematical curriculum, from an examination of the issues raised one century earlier by mathematicians like Emile Borel. For this purpose, we present and comment a series of selected quotes taken from two interesting papers, published by Borel in 1906 and 1908 in the *Revue du Mois,* a journal he had just founded with his wife. The first paper concerns the practical value of probability calculation, while the second bears on issues of its psychological and social acceptability. We show that the two papers are structured around the same kind of opposition between the mathematician's view on probability theory and its value, and the point of view of laymen opposing the latter's pretention to reduce decision making or social facts to calculation. We explain how Borel defines an original compromise between these two views, leading to a definition of the limits and strength of the application of probability to reality. We furthermore show that Borel's argumentation should be understood against the political and epistemological doctrines of his contemporaries. For this we comment on two papers published in the *Revue du Mois* of Alfred Croiset on the teaching of secular ethics, and Célestin Bouglé on the political doctrine of solidarism. Beyond showing the similarities between their argumentation and Borel's, we propose to explain this through the editorial background behind the publication of Croiset's paper. We conclude with some possible perspectives on present-day teaching concerns, that could be drawn from this historical interpretation: the first touches on the ambiguities of the probabilistic vocabulary; the second on the search for problems combining math teaching with the education to critical thinking; the last bears on the possibility of making interdisciplinary bridges between the political history and the rise of probability as a subject of public concern.

1 Mathematics, probability, citizenship: contemporary concerns and old issues

1.1 A contemporary dispute about probability and statistics in the French curricula...

One of the most significant changes in the French mathematical curriculum for secondary schools in the past two decades is the introduction of a new approach to statistics and probability. The latter combines a frequentist conception of probability and an experimental methodology to problems involving random processes, with the use of traditional models based on epistemic probabilities (Dutarte, 2011).

This change has been controversial from its very beginning, because it diminished the place of other important subjects in the mathematical curriculum, like geometry and geometrical thinking. This represented, therefore, a significant change in the underlying philosophy of mathematics: less emphasis on geometrical demonstration and calculus, more on model-making and experience in relation to the call to introduce and study of "real-life" problems. Among these problems, those calling for a critical reflection on the role of statistics and probability in society were given a significant role[1].

[1] See, for example, the following resources produced by the IREM of Paris Nord (Dutarte et al., 2007).

1.2 ... reminding old issues

What is striking in retrospect is that this mixture of epistemological and political debate on the very nature of mathematical thinking on the one side, with a reflection on the importance of statistics and probability for the modern notion of citizenship, on the other, is hardly a new one. Already at the beginning of the 20th century eminent mathematicians such as Henri Poincaré, Vito Volterra and Emile Borel had called the attention of mathematicians and of the educated elite, on the significant changes implied by the role of probability in science, industry and insurance or daily life in general (Durand & Mazliak, 2011).

Emile Borel and his later followers like Maurice Fréchet (1924), were especially influential in disseminating these ideas and pushing the idea that these questions should imply a reform of mathematics teaching, by the systematic inclusion of statistics and probability as a key subject (Courtebras, 2006, p. 94; 114). For them, it was not just an idea of augmenting the curricula with new subjects, but also reflecting on the value of mathematical knowledge as compared to scientific knowledge in general.

1.3 The principle of this paper: making out some of the reasons for Borel's interest on probability, through a series of quotes

The reasons why Borel, in particular, paid attention to the role of probability in contemporary science, industry and society around the years 1905-6 have been well studied in recent works (Durand & Mazliak, 2011; Mazliak, 2015; Mazliak & Sage, 2014). These studies show, in particular, that this interest is inseparable from Borel's efforts, in association with his wife Camille Marbo, to enlarge his intellectual relations and, by the same token, to promote his first political career through the foundation of a new journal, the *Revue du Mois*. In particular, two articles published by Borel in his own *Revue* (1906, 1908) are highly revealing of what were his key ideas and intentions concerning what he called the practical value of probability theory, on the one hand; and the ground reasons why probability should be made a subject of public interest and culture, and ultimately the subject of an institutionalized teaching, on the other.

In a recent study on the contents and form of argumentation of these articles (Bernard, *forthcoming*), I found out that they had to be understood (among other contexts) against the background of contemporary debates about the foundation and teaching of secular ethics. This subject was in 1905 of great topical importance, due to the passing of the so-called law of separation of Church and State in France. This means, then, that part of the reasons for Borel's interest on probability theory, were related to political and philosophical reasons.

While there is no point at summarizing here the details of this complex story, implying many of Borel's contemporaries in this crucial period both his career and of French political history, I found interesting to render the flavour and style of these debates through an ordered series of quotes taken either from Borel's two papers mentioned above, or from other articles or conferences bearing on questions of secular ethics that were either published in Borel's *Revue* in the same period (1906-1908).

Among the reasons for this choice, is the simple fact that the articles mentioned here are beautifully written and thought-provoking, due to the special attention that the contributors to the *Revue* paid to remaining accessible to their readers (Ehrhardt & Gispert, 2018). Moreover, most of them are inaccessible in English, for even the major book of Borel on Chance (1914) has received no translation. Finally, some of these quotes contain useful reflections that could inspire useful thought for teaching mathematics in a meaningful way[2].

2 Borel's two papers on the practical value and acceptability of probability calculation

The two papers in question (1906, 1908) were of high importance to Borel: this is revealed by the synoptic presentation that Borel made of their contents when he published in 1914 his first "grand public" synthesis on issues of probability. Introducing the third part of his book, he explains the following:

> **Q1**: "We strove, in the first part of this work, to expose how one has succeeded in submitting the laws of chance to calculation. Then, in the second part, we have quickly explained the main practical and scientific applications of the methods thus created. We must now ask ourselves what the practical, scientific and philosophical value of these applications is; should one consider that their value is diminished by whatever mysterious connotation attached to the word "chance", or should one consider that the comprehensive study of laws of chance will best tell us something about the value of any human knowledge?
>
> To deal with these essential questions, it seems necessary to me to reconsider the problem of chance *ab ovo,* so to speak, regarding the previous chapters as a mere introduction aimed at providing precise examples for the forthcoming reflections. In this spirit, we shall first examine (…) the value of probability in practical life and then look for how the individualist sensitivity is often an obstacle for the acceptation by many people of conclusions that nevertheless impose themselves to their reason. We will end up by brief indications on the role that the theory of chance could play in ethics based on solidarity and in the evaluation of the social value of individuals." (Borel 1914, 213-4)

This introduction is directly followed by the quasi word for word reproduction of the two articles in question. We shall point out some key characteristic of the argument with selective quotes.

2.1 The practical value of probability (1906): a dialectical discussion, leading to an original position on probability theory

Borel's 1906 paper begins with a dialectical confrontation, on which his whole argumentation is based. This is illustrated by the introduction:

> **Q2** "*Probability calculus is the study of laws of chance.*
>
> One has already remarked that this definition does explain a contradiction by another contradiction. If one does not understand how one can speak about *calculation* in association with *probability,* one will even less understand that it can be question of *laws* in relation to *chance*. Is not chance precisely what remains outside any rule or law? And does not everyday experience teach us to be wary of laws, to which one pretends to submit fortuitous events? Does it not happen

[2] For more details on my choice of quotes and its rationale, I forward the reader to my paper and the other studies mentioned above.

frequently bizarre coincidences, strange accidents that are apparently contrary to any probability?

'Well, such events do happen actually, would say the mathematician, but less frequently than it appears, and their frequency is itself governed by the laws that their occurrence seems to contradict. One can furthermore provide excellent financial proofs for the rightness of probability calculation: a well-managed insurance company does always make profits; and no roulette game company has ever ruined its manager.'

'But concerning the latter example you therefore hold for impossible, would answer the opponent, that the roulette company might be ruined? It might nevertheless be enough for me to be the only player and to play constantly the number that will appear. It might be very unlikely that I would have such flair, but this is obviously conceivable. What, then, of your principles?'" (Borel 1906, 424)

The originality of Borel's argument lies in the fact that he takes this debate in earnest, insofar as it points out a ground dilemma on the very notion of probability. The first part of his paper takes up and legitimates to some extent the position of the „contradictor"; the second defends the rights of the mathematician to declare impossible in practice, certain events of very small probability[3]. We shall here only give an idea of the epistemological discussion contained in the first part. It is first introduced by small problems of the following kind:

Q3. "Peter exactly possesses one million [francs]; one proposes him to flip a coin for it, against one million (fair game) or even in exchange for one million and fifty francs (theoretically advantageous game). Unless very particular circumstances, it is clear that the interest of Pierre is to refuse.

Jacques is isolated, with no contacts, in a remote country; he is fairly rich and should receive tomorrow a large sum of money; but he has a major interest in taking a boat that leaves within an hour and he has only 300 francs with him, although the boat crossing, to pay paid in advance, costs 400 francs. One proposes to toss a coin for his 300 frances, in exchange of 200 (theoretically disadvantgeous game); his interest is obviously to accept." (Borel 1906, 426)

These and other examples of the same kind, lead Borel to argue that expressions like "mathematical expectation" (*espérance mathématique*) are essentially misleading, because the calculation in itself does not express what one should really *expect* from a game. The calculation might be exact; it still has no practical value, because decision making has other incentives than a mere numerical result, like one's character or economic condition:

Q4. "[one should] renounce to speak about what one calls *mathematical expectation*, or at least to understand this expression has only designating a quantity that it is often advantageous to introduce in probability calculations. But it should not be taken at face value; one gives to Peter a ticket in a lottery in which the unique prize is 100 000 francs within one million tickets; the mathematical expectancy is ... 10 cents. But, if he pessimistic and starving, he will prefer by far to be given two

[3] This second part of the argument leads to arguments akin to Borel's famous evaluation of improbable events by the estimation of the probability that „typewriting monkeys" would reproduce the books of the French national library within one year.

pennies[4] to buy his bread; whereas if he is imaginative, he will be for one day as happy as if one had given him the 100 000 francs. By evaluating his mathematical expectation, one has perhaps made an exact calculation, but with no practical value.” (Borel 1906, 427)

Borel develops further this idea with other, slightly more sophisticated examples ultimately leading to the idea that probability calculation yields no certitude, but has only what he calls a relative value. It only serves to simplify a complex problem into simpler data, and nothing more; it does not provide infallible criteria for decision, but only constitutes one compenent of it. This is what he summarizes throught the following, interesting sentence:

Q5. “The practical value of probability calculation thus appears to us as relative; the practical problem to be solved is simplified in its terms, but not modified in its essence: it keeps being a problem of probability.” (Borel 1906, 430)

By this Borel does apparently not mean a problem in which mathematical probabilities should be calculated, but a problem of decision making through the pondering of various factors which do not depend only on calculation.

Then Borel generalizes these thoughts on any kind of application of mathematical calculation to practical decision and raises one of his main points, about the very possibility of discussing numbers:

Q6 “The intervention of calculation in the decision of practical life gives often way to two of the following, extreme judgments; for some people, it is absurd to mix up calculation to any decision which implies elements that cannot be expressed in figures; for others, figures have a magical power that make infallible anyone who used them according to the rules.

These two opposite tendencies actually correspond in fact to the same state of mind; this is only because figures do appear to them to have an absolute value eliminating any kind of discussion, that certain minds fear about their invention and prefer to do without their help rather than subjecting themselves to their yoke.” (Borel 1906, 431)

Beyond any matter of psychology, Borel immediately points out that the origin of this illusion is related to the „magic of calculation“ itself, meaning that:

Q7. “... one can only do calculation on precise figures; at least, it is quite long and difficult to look for the diverse solution of a problem, that correspond to the diverse values of imprecise data, and one shrinks from this task. One is then led to adopt determinate values for each element of the calculation, even though this element is not known accurately.

On these precise data, one can effectuate precise calculations that lead to precise results. And the longer the calculation is, the more the result might be inexact and the more, nevertheless, we have time to forget that the data were imprecise and to let onself trust in the exactitude inspired by correctly done arithmetical operations.

The same illusion frequently occurs in statistics; one reads every year in the journals that the total production of wheat in France is evaluated, for example, to 115 200 000 quintals[5]. This figure was obtained by the addition of many particular

[4] That is, ten cents in Borel’s time (*deux sous*).
[5] A quintal is a hundred kilograms.

indications, each one of which was inexact; the mass, though, inspires a certain trust and one is ready to make economical deductions from it.

Let us be wary then, about trusting too much in figures; this would be the best way to avoid the contrary excess, which is to deprive ourselves of their help. (...) probability calculation then appears as justified as any other calculation; its practical value is just the same." (Borel 1906, 431-2)

As mentioned above, the end of Borel's paper insist on the fact that, in practice again, very tiny probabilities should be taken as equivalent of strict impossibilities. The ground argument is very similar: if we had to refuse this in daily life, or even in scientific conclusions, then life would become impossible because any event, however improbable, would have to be taken in consideration.

We thus see the very special mixture of epistemological thought on the value of mathematical knowledge at large, and psychological considerations about its use for practical decision. All this is the seed of a more general reflection on why (probability) calculation is acceptable or not to public understanding. Let us now see how these arguments are retrieved and developed in Borel's 1908 paper, in the direction of discussing the social acceptance of probability calculation.

2.2 Borel's defense of probability calculation and "social mathematics", against "individualist sensibility" (1908)

Borel's 1908 paper begins with a very similar argument than the previous one, in the sense that a dialectical opposition is built between the „golden chain" of those famous mathematicians and philosophers who, from Pascal to Poincaré and Bertrand, have developed the theory of probability; and what Borel designates as the ‚sensibility' of people who, though they are ready to accept the conclusions of theories of probability on the authority of such luminaries, are, nevertheless, reluctant to do so:

Q8. "… [for] this is not their reason, but their sensibility that is shocked by the conclusions drawn from probability calculation; or at least, from the manner they understand and interpret it. (…)

While trying to understand in depth the reasons why probability calculation is antipathetic to many minds, I hope to demonstrate that this antipathy mostly relies on misunderstanding; it would be desirable that such misunderstanding would be dispelled; for the vulgarisation of the conclusions, if not of the methods, of this branch of science, would have a great social utility." (Borel 1908, 642)

The notion of "individualist sensibility", on which Borel relies here, is directly taken from the contemporary writer and philosopher George Palante, for which the antinomy of individuals vs society was a key notion of his worldview (Depenne, 2015). Borel quotes an excerpt of a spiritual article published the same year by Palante in the *Mercure de France* (1908), the quote containing an obvious charge against the kind of political opinion and party to which Borel adhered:

Q9. "Pascal said: 'the more spirit one has, the more one finds how many original people there are. Laymen do not see any difference between men'. The social and gregarious sensibility (*sensibilité*) delights itself with the banality of traits; it likes that one is 'like everyone else'. The Christian, humanitarian, solidarist and democratic sensibility would like to erase the differences between egos. Amiel sees in it, with some justice, the sign of a rough intellectuality." (Borel 1908, 642, quoting Palante)

The allusion to the so-called *solidarist* sensibility, as opposed to the individualist one that Palante advocated, goes to a political doctrine of the same name, developed in Borel's time (Hayward, 1961) and to which he adhered very soon, as we shall see later. Borel's point consists in generalizing Palante's remark, which originally applies only to elite minds, to anyone. We all care, says Borel, for our individuality and prefer to think ourselves different from the others – a tendency which is directly opposed to the notion that we might enter anonymously into a statistical account. But Borel immediately adds that this is not the same point of explanation, opposing statistics to probability:

> **Q10**. "Probability calculation does not confound itself indeed with statistics, to which the aforesaid can also be applied; it does not contend itself to count past events, but also pretends, to some extent, to forecast future events: in this, it is a science. This pretention hurts from the outset the psychological feeling of human liberty (the metaphysical value of which is not at stakes here); one asserts that, if there is no exceptional event like war, earthquake etc., there will be for sure more than 1000 marriages in France during the next week. Does it not depend on the fiancés to deny the prediction, by postponing the celebration of their union by one week? The objection does not sustain examination, but it is often implicitly admitted without examination." (Borel 1908, 644)

The point made here is very similar that the one which constituted the point of departure of the 1906 paper. The following discussion, althought based on different and more dramatic examples than those taken in the previous paper (since they bear on the classical discussion of life duration and on the advantage of inoculation campaigns) concludes with a similar point that two years before, about the relative value of probability theory. Having explained indeed, that in certain cases augmenting the global life expectation at the expense of two many losses, although mathematically advantageous, would be unacceptable for common sense, he adds:

> **Q11**. "... but common sense, no more than calculation, does ensure against misfortune; and this will always be little comfort for any individual to think that the probability of misfortune was small, if he is the one who suffers from it. Anyone starving does pay little attention to the rise of average fortune: one should not look into statistics or calculation any argument to comfort those who suffer from social inequalities; but this observation does not diminish in anything the proper value of statistics nor the calculation by which one interprets them.
>
> This is only, indeed, from a particular point of view that the statistician or the mathematician study social phenomena; this study has a limited scope, but it does constitute an exact science, when one does not pretend to extend it beyond its natural limits. One should look in it neither for moral grounds nor for immediate reasons for action: but only, as in physical sciences, for one way to know accurately past events and to forecast with some approximation future events. When one predicts that more than 100 000 Parisians will take the „metro" tomorrow, no one among them is obliged to choose such means of transportation; one only asserts a fact that is confirmed by experience." (Borel 1908, 648)

Borel concludes this part of the argument by recalling the 1906 comparative arguments on the estimation of improbable events, however frightening they might be, when their probability is much smaller than that of risks we take on a daily basis. Lacking the capability of making such comparisons is ironically qualified by Borel as „head in the sand policy“ (*politique de l'autruche*):

> **Q12** "Ignorance might be convenient to those who practice this head-in-the-sand policy; it is never desirable for those who prefer to see clearly and do not let themselves be influenced by the more accurate knowledge of a possible danger, when its probability is much smaller and the unknown dangers to which the most fearful man does expose himself every day. One has nothing to fear from calculation, when one is decided not to align one's conduct on its indications, without first estimating their proper value. To think that individual independence is augmented by ignorance is a peculiar illusion." (Borel 1908, 649)

Borel does not stop his argument at that, but come back at the very end to Palante's passionate defense of radical individualism, but an equally passionate defense of the notion of solidarity, to which mathematics contribute, *insofar as it can be educated*:

> **Q13**. "Is there therefore no ground to the opposition we pointed out between probability calculation and individualism? There is, on the contrary, a very real one insofar as individualism is antisocial and probability calculation is the basis of what one may call *social mathematics*. Its study reminds us indeed that we are living within society and that the social phenomena have real existence and their proper interest. It reminds us that, while men are different from each other in many aspects, they are nevertheless similar in that they are all exposed to accidents, disease, death; and in that they various biological aspects (…) are distributed among regular laws about certain means; and in that, finally, one can establish *laws* that are checked by factual observation and in the statement of which they are considered constitutive elements of the whole to which the law is applied. (…). Such accounts are eminently proper to limit the excesses of the individualist mentality. Such or such a disease causes, on average, a certain number of victims; hail and floods make, on average, a certain amount of damage; one does not know why some are affected and others are spared, but society, as a whole, suffers from a more or less constant damage. The study of such facts can only contribute to develop the notion of solidarity, to recall to everyone that he should not consider himself as independent of the milieu in which he lives and that he should participate in repairing the casual damages that affected his neighbor and could have affected himself. Thus, the study of probability calculation does have a great educational value; one should hope that it would be within reach of all of those who want to take part in governing men and things." (Borel 1908, 649-650)

This passionate defense of Condorcet's ideals is highly revealing of Borel's own political ideals, which were largely shared by the Republicain and Radical-Socialist milieu to which he belonged. This now leads us to expose some contextual elements.

3 The political and philosophical background of Borel's arguments

We here give some elements of background that might help to understand better how Borel's political and epistemological position, both as an author and as the chief editor of the *Revue du Mois*, might help to understand his argumentation and his final goal, which was to promote the knowledge and teaching of probability calculus.

3.1 Alfred Croiset on the teaching of secular ethics (1906): the role of the State in education

The second paper published in the first issue of the *Revue du Mois*, after Vito Volterra's translation of his inaugural lecture on the role of mathematics in biological and social sciences, is written by Alfred Croiset and touches on issues related to the teaching of secular ethics (*morale laïque*).

Alfed Croiset was an important figure in the Republican and Academic intelligentsia by Borel's time. Croiset was an eminent philologist and by then the Dean of the faculty of humanities in Sorbonne. He was also at the head of the "ethics division" (section morale) of the *Ecole des Hautes Etudes Sociales* founded in 1900 by Dick May (Prochasson, 1985). In this function, he organised with much talent several series of conferences, many of them having to do with the kind of "ethics" adapted to the scientific age. The conference, of which Croiset's paper in the *Revue du Mois* is a transcription, precisely introduces to one of these series of conferences. The subject was a great topical actuality in 1905, since it bore on the teaching of secular ethics. The latter had been instituted some 20 years before in the public school by one of the most famous laws passed under the third Republic (Loeffel, 2010).

Croiset, more precisely, summarizes the basis of the problem posed to teachers by this (by then) relatively new school subject. Delivered in the framework of a religiously neutral State, it has lost the dogmatic ground that was naturally attached to confessional ethics. Recalling then the contemporary tendency to provide a *scientific* ground to modern ethics, he opposes the following, which is a reflect of the kind of 'scientific uneasiness" expressed by contemporaries like H. Poincaré:

> **Q14**. "For [teachers] have often heard that the most distinguished scholars that science was not an immutable and definitive work, that it was forever provisory in some sense, and it changed with all the rest. So then, in this flow of everything, what of the absolute character of duty, which seems to be the indispensable imperative of any ethics?" (Croiset 1906, 22)

This is contrasted with the urgency attached to action, thus defining the key problem he addresses to his audience:

> **Q15**. "All these facts make it therefore particularly difficult, today, the task of organizing a secular teaching of ethics. But it also makes it more urgent as ever: for we must live, and since life is action, we need reasons to act. Reality does not wait for us to get out of our ignorance or doubts." (*ibid.*)

Croiset's next turn is to oppose any idea that the ethics in question could be qualified as "scientific", in the sense proposed by his contemporaries Durkheim and Levy-Bruhl, to whom he explicitly refers. The point is that the secular State cannot have any agenda of promoting a dogmatic ethical doctrine, even a positivist one. The problem, then, as defined by Croiset in full coherence with the doctrine of the Republican state ideology of that time, is to define the role of Secular State (*l'Etat laïque*) in the teaching or ethics:

> **Q16**. "Can [the State] teach any ethics? To the name of what principles? And can it do this while keeping his proper field, without impeding on that of individual consciousness, without exposing itself to the reproach of tyranny, and without

meanwhile depriving its doctrine of any virtue and strength? Such is the problem on which I call your attention" (Croiset 1906, 24)

Croiset's essential answer to this question is interesting and revealing of the underlying state ideology above mentioned. Having ensured that the secular State is perfectly capable of giving individual an ideal and a rule of conduct, he adds in grandiose terms:

Q17. "This end is *the development of social life*, which is for us the highest form of human life, for it constitutes the necessary framework of any individual life; it is, at the same time, the result of all past progresses and the condition for the future progresses of humanity. The State … is the *real, concrete instrument of collective life,* without which there is no civilization. (Croiset 1906, 25, our emphasis)

Living out the details of Croiset's own theory justifying this idea, based on his historical understanding of the grounding of common ethics on the sedimented past, let us turn to the underlying political agenda.

3.2 The political background: the solidarism of Bourgeois, Croiset, Bouglé.

For this, let us turn to another cycle of conference organised three years earlier by Croiset together with Leon Bourgeois, an interesting and important political figure of the third Republic (Croiset & Bourgeois, 1902). Bourgeois had synthetised in a coherent doctrine, soon called *solidarism*, a social and political doctrine which defined, for proponents, a kind of mid term between individualism (attached to economic liberalism) and socialism (attached to Marxism). This influential doctrine soon became the „official doctrine" of the Third Republic and the ground for important reform defining a wellfare state „à la Française" (Hayward, 1961).

Croiset, in his interesting and nuanced introduction, recalls the reason why people of his generation felt attracted to the notion of solidarity:

Q18. "If individuals are, so to speak, the cells of society, the word by which biologist express the interdependence of cells, is the one that should from now designate the interdependence of individuals. (…) Our modern generation, so thirsty of positive and objective science, needed such a word that would express the scientific character of moral law. The term "solidarity", borrowed from biology, marvellously echoed this obscure and deep need." (Croiset & Bourgeois 1902, x)

He then explains that the solidarist spirit that thus emerged from the progressive shift from a specialized term to a political one conveyed…

Q19 "… a kind of scientific character that pleases the spirit of our time: we love this regularity of natural and social laws, that exclude the whim of individual wills and the incertitude of feelings" (*ibid* xii)

Croiset finally counterbalances these ideal views with the claims for respecting the rights of individuality:

Q20. "One should avoid going too far along this way. The best things have their dangers. During our talks, one has come out in favor of individualism understood in the most elevated sense, against the absorption of individuals in the whole. (…)

All intelligent solidarist people are perfectly convinced that the love of humans for each other, as well as the complete development of individuals in social harmony, are indispensable elements of true solidarity." (*ibid*. xii-xiii)

We thus retrieve key ideas later developed in the 1905 conference published by Borel in 1906, that we commented above. These ideas are developed in another article published in April 1906 by Célestin Bouglé on the scientific grounds of solidarism (Bouglé, 1906). Bouglé was one of the first members of the early team of scholars who, around Durkheim, launched the *Année Sociologique,* which was the banner of the newly founded sociology (Vogt, 1979). Bouglé was also a militant scholar strongly advocating solidarism (Bouglé, 1907). In a much more sophisticated manner than Croiset, because of the highly learned references to contemporary thinkers he adds to the discussion, Bouglé develops the same ideas, grounded in the same paradox: necessity of grounding the solidarist ethics on a scientific and positivist ground, impossibility to do so without risking violating the right of individual freedom and consciousness.

Through an epistemological reflection on what exactly should be understood under the name "scientific attitude", Bouglé demonstrates that there is in fact no contradiction between the latter and the thirst for justice, fraternity, or social compensation. Having adopted this kind of compromise, he then argues that solidarism still retires something of its proximity to modern science, both on the levels of the means of action, as of its final ends. The key idea is that, although science cannot pretend to *create* ethical feeling (like that of justice, that pre-exist any social experience), it can still *modify* them. For this purpose, ethical education plays a crucial role:

> **Q21**. "To shape the consciousness of the young in our schools, we actually apply the method advocated by solidarism: we fill them with science. By the history of inventions and institutions, we give them the feeling of what human progress is about and what it owes to universal cooperation. We thus inspire them the desire to fulfill their duties, to pay their share, to do their part (…) They are disposed to more altruistic effort through the very fact that they conceive themselves, as Condorcet would say, as 'cooperating in eternal work'. The expansion of intelligence, obtained through the knowledge of facts, thus results in a dilation of the heart." (Bouglé 1906, 450-451)

3.3 The editorial background

As mentioned above, the various papers discussed above, by Borel, Croiset and Bouglé, were all published in the *Revue du Mois*, edited by Emile and Marguerite Borel. More than that, their contents - with the special emphasis on science in practical life and society - were in line with the declared purpose of the *Revue*, which was announced in the following way:

> **Q22**. "The number and the importance of problems that can be treated adopting scientific methods grows every day. It seemed possible to us to imagine a journal which focused on these methods, not as a specialist publication but rather by aiming at the general development of ideas, and the exposition and critical appraisal of the advances in Knowledge and the resultant spread of ideas.
>
> The Revue du Mois attempts to be this journal. It claims, above all, to be a journal containing free discussion, allowing the free unhampered expression of opinions based on science. The titles of the articles that follow this statement testify to the breadth of its scope; the names of the authors are an assurance of the seriousness with which its remit shall be fulfilled." (prospectus translated in Durand and Mazliak, 2011, 314)

The issue of how science and scientific method could help develop general ideas and opinions was thus very much at the centre of the Revue's ambition. The notion of science was understood in a very wide and varied way going far beyond the tradition fields of mathematics of experimental science: it included indeed the nascent "social sciences" like sociology, psychology, economy or human geography.

The study of Borel's editorial correspondence around the publication of Croiset's paper shows that he intended it to become the point of departure of a debate on the teaching of secular ethics. He thus solicited two former friends of him, the physicist Bernard Brunhes and his brother Jean, a geographer, to find a Christian response to Croiset's paper. The Brunhes were indeed known to belong to the modernist movement, also related to "social Christianism". We cannot enter here into any detail on the detail of this exchange of letters; it is enough to say, that Borel was not only interested to have a Christian response, which in fact he never obtained because "modernist Christians" soon met with serious problems through the Pope's condemnation of their opinions (through the encyclical *Pascendi dominici gregis* issued in 1907).

But Borel was visibly convinced by the exchange to seek for another article than Croiset of a more elevated kind, of the level of Durkheim of Levy-Bruhl, and this was probably the reason why he ultimately asked Bouglé to contribute to the subject. Second, and more important, the correspondence also shows that he felt incited to contribute himself to the debate in his own way, and this might be one explanation for the particular tone and structure of his two papers of 1906 and 1908.

3.4 The epistemolocial and didactic background: Borel's ideas on mathematics and their teaching

Another element of explanation for Borel's special positions on the value of probability calculation and of its teaching, surfaces through the papers analysed and quoted above. For we saw that he considered the reflection on probability and on the laws of chance, as part of a more general reflection of wider scope, on the value and applicability of mathematical knowledge in general. This appears, within his argument, through the comparison between the problems raised by the interpretation of probability calculation for real life, with the problems raised by any train of mathematical calculation based on approximate knowledge.

But this point was hardly secondary to Borel, as is shown through his very first public intervention on question of mathematics education, through his 1904 conference held in "Musée Pédagogique" on the role of practical exercises in the teaching of mathematics (Gispert, 2002). In it, Borel defended a kind of approach to teaching making a compromise between theory and practice. The end of the conference is worth quoting, because it shows again, that Borel had very early in mind, that this proposal had potentially both an epistemological and political signification:

> **Q23**. "This new orientation towards the teaching of mathematics in our *lycées* and *collèges*, of which we just outlined, would exert the most beneficial influence on the philosophical ideas of the educated classes, these ideas governing in reality the evolution of the country. (…)
>
> A mathematical education, both theoretical and practical, as the one we tried to conceive, can exert the most beneficial influence on the training of the mind. We can thus hope to shape men having faith in reason and knowing that one should not try

to escape when facing a correct reasoning: one has just to accept it. They will have understood, on multiple examples, the determinism attached to natural phenomena and will be ready to understand the notion of physical law. But, on the other hand, they will be wary about any ungrounded reasoning (*raisonnement en l'air*) with no basis in reality, referring to badly defined words and to calculations effectuated on abstract numbers, the concrete signification of which is not made clear. They will seek to see the tangible object beyond the symbol." (Borel 1904, quoted in Gispert 2002)

4 Conclusions

Having thus immersed ourselves into this literature aimed at the educated public in Borel's time, we can then come back to the contemporary issues raised in the first place and ask what is and might be the interest of these reflections for nowadays teaching.

We propose three kinds of answers that are inspired by the work done in the research group, of which the present paper is an outcome.

4.1 Questionning the value of probability concepts for common understanding

We have seen how Borel's paper are structured around the idea of taking in earnest an imaginary debate between competent mathematicians and „contradictors", that is, laymen having their own intuitions, beliefs and reasoning on the day-to-day events and decisions, for which probability calculation pretends to provide help and insight. Transposed to the teacher and student situation, we retrieve a problem that is well known to middle school mathematics teachers, namely the conflict between the meaning of mathematical concepts such as „chance", „probability", „events", „certain" or „impossible" events, „expectation", and so on. This problem of making sense of the mathematical concepts, through a dialogue with the pupils own conceptions and understanding of the underlying situation, is much at the center of the French formal curriculum in probability.

What we find here, therefore, is an argument that gives room and legitimacy to the conflict between mathematical calculation, concepts, and models, on the one hand and, on the other, the matters of day-to-day life and decision making, the complexity of which cannot be reduced to the straightforward and passive interpretation of results. Many teachers are not at ease with these limits and with the questions of the complex relation of calculation with the reality it only helps to interpret. Getting into these arguments amounts to retrieving the historial fact that the application of probability theory to reality has been a matter of dispute for a long time, and, in fact, remains so.

4.2 Looking for "thought provoking" problems

As we have seen, part of the basic argumentative apparatus in Borel is made of small problems, that do not pose very deep difficulties in terms of calculation, but calls for reflecting on the results and their value. These problems were not Borel's invention but the result of an already long tradition, well represented by the seminal treatise on probability of Joseph Bertrand (Bertrand, 1888). This treatise, which became in Poincaré and Borel's time a basic reference, is entirely made of problems, instead of axioms or formal definitions as in Kolmogorov or other authors that took inspiration from him. Moreover, Bertrand treatise is preceded by a long preface that introduces the reader to the epistemological problems raised by the application of probability to real and especially social events (Bru, 2006).

There is much emphasis, today, on the need for problems that conciliate the teaching of mathematics to the education to citizenship and especially to critical thinking. While many such problems are available, it is important to be conscious they are not new or the natural outcome of the contemporary concerns for the "overflow" of data around us. The historical tradition of probability writing constantly refers to famous dilemmas, like the problem of inoculation that constituted a famous controversy between Bernoulli and d'Alembert in the 18th century. Keeping alive this culture of controversy and problems, is certainly an important challenge.

4.3 Situating probabilistic thought in its political context

Finally, what Borel's papers reveal, when understood against their political and philosophical background, is precisely the fact that their complete understanding depends on the latter. In France, the events surrounding the birth of the third Republic, the eductional laws that were passed under this regime, and in particular the 1905 law of separation of Church and State to which Croiset directly refers, are of huge importance and part of a living tradition: thus, issues like those of *secularism* and the attached notion of *confessional neutrality*, are key concepts of our political culture.

What is less known[6] is the fact that the doctrine of the French state politics was inspired by such doctrines like *solidarism*, although it is at the ground basis of the idea of "social state". There is much room, therefore, for interdisciplinary activities and training, showing how probabilistic thought is rooted in this political culture: the emphasis on the debate between the individual and society, in inseparable from the rise of probability and statistics, as a subject of public concern.

REFERENCES

Bernard, A. (forthcoming). Un aspect des débats sur les fondements scientifiques de la morale laïque au début du 20ème siècle : du solidarisme aux probabilités. *Revue d'Histoire des Sciences Humaines*.

Bertrand, J. (1888). *Calcul des probabilités*. Paris, France: Gauthier-Villars et Fils.

Borel, É. (1906). La valeur pratique du calcul des probabilités. *La Revue du Mois*, vol.1 pp. 424-437.

Borel, É. (1908). Le calcul des probabilités et la mentalité individualiste. *La Revue du Mois*, pp. 641-650.

Borel, É. (1914). *Le hasard*. Paris, France: F. Alcan.

Bouglé, C. (1906). La «Banqueroute de la science» et la morale solidarité. *La Revue du Mois*, pp. 438-455.

Bouglé, C. (1907). *Le solidarisme*. Paris: V. Giard & E. Brière.

Bru, B. (2006). Les leçons de calcul des probabilités de Joseph Bertrand. «Les lois du hasard». *Electronic Journ@l for History of Probability and Statistics*, *2*(2). http://www.jehps.net/Decembre2006/Bru.pdf

Courtebras, B. (2006). *A l'école des probabilités: une histoire de l'enseignement français du calcul des probabilités*. Besançon: Presses universitaires de Franche-Comté, DL 2006.

Croiset, A. (1906). L'enseignement laïque de la morale. *La Revue du Mois*, vol.1 p. 21-36.

Croiset, A., & Bourgeois, L. (Éd.). (1902). *Essai d'une philosophie de la solidarité: conférences et discussions*. Paris: F. Alcan.

[6] Probably because the radical or radical socialist parties, that were so powerful and central with the third Republic, are no more central political forces.

Depenne, D. (2015). 3. Georges Palante contre Émile Durkheim : individualisme et sociologie. In P. Corcuff, C. Le Bart, & F. De Singly (Éds.), *L'individu aujourd'hui : Débats sociologiques et contrepoints philosophiques* (pp. 49-65). Rennes: Presses universitaires de Rennes. Consulté à l'adresse http://books.openedition.org/pur/13614

Durand, A., & Mazliak, L. (2011). Revisiting the Sources of Borel's Interest in Probability: Continued Fractions, Social Involvement, Volterra's Prolusione. *Centaurus*, *53*(4), 306-332. https://doi.org/10.1111/j.1600-0498.2011.00238.x

Dutarte, P. (2011). Évolution de la pratique statistique dans l'enseignement du second degré en France. *Statistique et Enseignement*, *2*(1). http://publications-sfds.math.cnrs.fr/index.php/StatEns/article/view/72/61 Dutarte, P., Delzongle, F., Maati, H., Cardinal, J.-P., Coupry, A., & Dherissard, S. (2007). Statistiques et citoyenneté, le citoyen face au chiffre. IREM de Paris Nord. http://irem.statistiques.free.fr/telechargement/brochure135.pdf

Ehrhardt, C., & Gispert, H. (2018). La création de la Revue du mois : fabrique d'un projet éditorial à la Belle Époque. *Philosophia Scientiæ*, *22*(1), 99-118.

Fréchet, M., & Halbwachs, M. (1924). *Le calcul des probabilités à la portée de tous*. Paris: Dunod, 1924.

Gispert, H. (2002). Les exercices pratiques de mathématiques dans l'enseignement secondaire. Conférence faite le 3 mars 1904 au musée pédagogique. *La Gazette des mathématiciens*, *93,* 47-64.

Hayward, J. E. S. (1961). The Official Social Philosophy of the French Third Republic: Léon Bourgeois and Solidarism. *International Review of Social History*, *6*(1), 19-48. https://doi.org/10.1017/S0020859000001759

Loeffel, L. (2010). Education et morale. In *Une histoire de l'école: anthologie de l'éducation et de l'enseignement en France, XVIIIe-XXe siècle* (pp. 79-85). Paris: Retz.

Mazliak, L. (2015). Les mathématiques du hasard,, outil de l'expérience citoyenne. In É. Barbin & J.-P. Cléro (Éd.), *Les mathématiques et l'expérience: ce qu'en ont dit les philosophes et les mathématiciens* (pp. 281-298). Paris: Hermann, impr. 2015.

Mazliak, L., & Sage, M. (2014). Au-delà des réels : Émile Borel et l'approche probabiliste de la réalité. *Revue d'histoire des sciences*, *67*(2), 331-357.

Palante, G. (1908). La sensibilité individualiste. *Mercure de France*, pp. 577-592.

Prochasson, C. (1985). Sur l'environnement intellectuel de Georges Sorel : l'École des hautes études sociales (1899-1911). *Cahiers Georges Sorel*, *3*(1), 16-38. https://doi.org/10.3406/mcm.1985.900

Vogt, W. P. (1979). Un durkheimien ambivalent : Célestin Bouglé, 1870-1940. *Revue française de sociologie*, *20*(1), 123-139. https://doi.org/10.2307/3321267

THE MAGIC OF THE EAST - FROM THE ALHAMBRA TO THE SAMMEZZANO CASTLE

Symmetries in mathematics, nature and art

Denise CALANDRINO, Michele CECCHI, Attilio FERRINI, Laura ISOLANI, Vilma NATALI, Chiara TOGNACCINI
Istituto Comprensivo di Reggello Via M. Guerri Reggello (Firenze) Italy
michelececchi@diluvio.it attiliofe@virgilio.it lauraiso@hotmail.com
vilma.natali@istruzione.it chiara.tognaccini@libero.it

ABSTRACT
The lack passion some students have for mathematics is known. Understanding that the basis for further mathematical skill development is determined in the primary schools, we chose a project involving 9 to 14 year olds from the Reggello school (Florence, Italy). The project is part of a multi-year course titled "The Pleasure of the Beautiful" with the idea of appreciating the beauty of mathematics and its relationship with other disciplines (art, history, geology, …). In this project we chose to talk about symmetry as a key reading idea to understand art and architectural works, considering it is one of the central ideas of mathematics of the twentieth century. The opportunity was offered by the presence in our landscape of the Sammezzano Castle, a beautiful Moorish style building, built in 1605 by the Ximenes D'Aragona family and restored by Ferdinando Panciatici between 1853 and 1889 according to Orientalist fashion, then in vogue in Florence. The castle is rich in geometric decorations and tiles with reference to the Alhambra mosaics.

1 Introduction

At the beginning of school year, I used to get my first-class pupils to write a composition "Maths and Me" asking them to express openly their opinions. Here are some of the things they said: Elisa (first class) "I don't like maths at all, when I was in primary school, I was one of the best pupils, the same in the first class of middle school. In the second class I began to have difficulties understanding the teacher's explanations.....I would find myself shaking every time the teacher gave back tests". Gabriele (first Class) "Maths and me – like a cat and dog". Giulia "Now I'm in the secondary shool, and I think that I must live another five years with maths, I feel really unhappy. One thing is for sure. When I finish school I will never have anything more to do with maths".

In the light of such comments, our task was not easy, how could we change the opinions of our students about maths and let them appreciate the beauty (the actual beauty) of a subject which people consider arid and lacking in poetry? For this reason we worked on a project, lasting four years, which involved some classes of Primary and Secondary school. This project is called "Il piacere del bello" (The pleasure of beauty). Why have we titled this project "The pleasure of beauty"? Perhaps because we live in Tuscany, where the Renaissance and the Medici have left testimonies of incomparable beauty, as the architect Renzo Piano wrote to the inauguration of Columbias University Art School. Beauty is not only that of consolidated art, it is emotion, research, knowledge, discovery; it is also political gesture, it gives strength to desires and is one of the human emotions capable of competing with those more dangerous: money, power and conquest.

The mathematician's patterns, like the patterns of poet or painter must be beautiful – the ideas like the colors or the words, must fit together in a harmonious way. Beauty is the first test, wrote G. H. Hardy in his book *A Mathematician's Apology*, 1940. It is impossible to be a mathematician without being a poet in soul. (Sofia Kovalevskaya, *Recollection of* *my chilhood*, 1885). Mathematics is perceived as a new relationship between the lightness of the ideas and the heaviness of the world. (Italo Calvino, *American lessons*, 1985). In our opinion, one reason (not the only one), for studying mathematics is to educate the eye and the ear of mind to be able to see it or to feel this beauty. So, why mathematics appears beautiful this reason is necessary, although it is not measured, in case it is dampened. Creative thinking needs time, space, serenity, curiosity and enthusiasm: it must be able to conduct researches without restrictions, excessive fears or anxiety.

2 Interdisciplinarty of the project

The continuity between different orders of school and interdisciplinarity is the key to unify the different disciplines. “The supremacy of a fragmented knowledge, according to the discipline, must be replaced with knowledge capable of grasping mutual relations and mutual influnces between parts and a whole complex world” (Edgar Morin, *Manifesto pour changer l’education*, 2004).

Our project with the recovery of manual dexterity, carried out through the laboratories, the involvement of the students, who themselves become teachers to the other students and promoted initiatives towards adults with the realization of the works carried out with the involvement of the territory and of the family, make students the protagonists of its own knowledge. With the laboratories, learning recuperates its formative value, we recover the place where we search, we experiment, we give meaning to the students to work in harmony with their curiosity.

3 Italian school structure

Education in Italy is compulsory from 6 - 16, and is divided into five stages:

Pre-School (scuola dell’infanzia / not compulsory / 3-5 years), Primary School (scuola primaria / 6-10 years), Lower Junior High School (scuola secondaria / 11-13 years), High School (scuola superiore / 13-16 years), University (università).

Italy has a both public and private education system. In Italy a state-born school system has existed since 1859, when the Legge Casati (Casati Act) was published.

Our school is for students from age 3 to 13 years. It includes three stages: Pre-School: 301 students, Primary School: 532 students; Lower Junior High School: 315 students

4 Methods and teaching strategies

The 8 key competences of European Union are: Communication in the mother tongue; Communication in foreign languages; Maths competence and basic competences in Science and Technology; Digital competence; Learning to learn; Social and civic competences; Cultural awareness and expression; Sense of initiative and entrepreneurship. In our school we have five labs: music, ceramic, physical education, science and maths.

5 The territory of the project

Our territory is located in Italy, in the central part of the peninsula. The area is about one hundred kilometers from the Tirrenian sea. We lie in the shelter of the Appenines mountains. This mountain chain passes through the entire peninsula like a spinal column. We are in Tuscany, near Florence, in a district named Upper Valdarno because of the location in the upper part of the Arno river valley.

In our valley the Arno river flows through its central part. The river goes then to Florence and on to the sea. Sammezzano Castle is located in this valley. On the east side we have the ridge (about one thousand and six hundred meters above the sea level); on the west side we can see the Chianti mountains (about nine hundred meters above the sea level). The lower part of the valley is about one hundred meters above the sea level. Here the valley is like a trough. This hilly area has a very peculiar aspect. It is very interesting for many aspects: first of all the geological one, but also naturalistic and environmental ones. The name of this area is “Balze park”. “Balze” is the name of particular rock outcrops that stretches for many kilometers across Upper Valdarno. A possible translation could be cliffs.

6 History of the Sammezzano castle and Ferdinando Panciatichi Ximenes d’Aragona

The castle of Sammezzano got its current appearance thanks to the work of one man: Ferdinando Panciatichi Ximenes d'Aragona (Florence, 1813 - Sammezzano, 1897) who renovated and expanded the pre-existing building during the period between 1843 and 1889. The history of the place however is significantly older: it is possible to trace it back to the Roman period, as well as to the subsequent centuries. In the *History of Florence* written by the great historian Davidsohn, it is reported that in 780 Charlemagne may have passed by the place on his way back from Rome, where he went to have his son baptized by the Pope. Ferdinando was owner and builder at the same time; although he had no university degree, he had the abilities of an engineer, an architect and a geologist. This allowed him to design, plan and finance the castle by using local products and employing mostly native manpower. The cultural wave of “orientalism” spread around Europe at the beginning of the nineteenth century and Florence was one of the main centers. Influenced by this movement, Ferdinando started to modify the existing Sammezzano structure and realized new halls.

As an expert and passionate botanist Ferdinando planned a vast area surrounding the castle of approximately 65 hectares, the so-called historical park. Around the ancient “Ragnaia”, which was characterized by a great forest of evergreen oaks, he recreated the habitat for numerous rare and exotic plants. His intent was to prepare step-by-step visitors and guests to the magnificence of the “Moorish” style of the castle.

In the late 90’s, after several events, the property passed into the ownership of a group of English and Italian societies. The society plan was to perform an important renovation and re-open Sammezzano for tourism. However, no actions have been realized and the castle remained unused until today.

7 Geology lab. Upper Valdarno geological evolution

The Castle location in the “Balze area” was the occasion for an interdisciplinary work comprising not only mathematics, art and history, but also geology. The students have discovered the geological history of the area using books, internet, magazines, the direct knowledge of the places where they live, and the observations of rock outcrops. For example, the students have drawn a section of their territory, starting from the bottom of the valley to the top of the mountain. They noted the presence of clay, silt, sand, gravel, pebbles and huge blocks of rock: deposits usually carried by the rivers. They understood these materials had been eroded, transported and deposited by torrents. Then they noticed that the clasts' size increases from the valley to the mountain. They answered: "This is because we are closer to the place where the rocks have been eroded." They noted that the Valdarno consists of a kind of erosion plateau, so they confirmed what they had already heard in their childhood: in this area there was an ancient lake. This lake was filled with debris coming from mountain erosion. Then the materials that filled the lake were in turn eroded and gave rise to the Balze.

Through research on the Internet they also found more detailed explanations that we can observe in some slides taken from a work of the University of Florence, Department of Earth Sciences, in which we see the evolution of the area. The students then carried out research on the fossil record. In fact, the bodies of many vertebrate mammals, that then became fossilized, were deposited in this lake during Pliocene and Pleistocene (Paleontological Museum of Florence and Montevarchi). The suggestion of the "Balze" has most probably inspired Leonardo in the landscape of the Gioconda painting. According to some scholars, this is not true. However, we have proof that Leonardo studied the Valdarno area from his writings. For example, in the Hammer Code, in which the scholar talks about the Valdarno and the Pratomagno.

8 The project in the primary school

The Hand touches, the Brain says (Rita Levi Montalcini, Nobel Prise in Medicine, 1986).

The 4th class of the primary school in Cascia (near Reggello) composed of 16 9 year-old pupils, took part in the project. The central idea of our work is GEOMETRY that we have developed in various disciplines such as art, environmental education and history of our territory. The themes have been developed through manipulation and the construction with the use of hands. The manipulative aspects are in fact very important, in this phase of growth. Nowadays the children are "overwhelmed" by technology which, although important, has taken away from their childhood and much of their practical experience in play, movement, manual realization and the practical creativity that should distinguish this phase of age.

So our work has been based on reproduction of some geometric shapes of the Castle of Sammezzano and the Alahambra, using the square and the ruler, and on the construction of a mandala through the goniometer. The word "Mandala" means "Circle", and it is from this geometric shape that we created an infinite number of symmetrical and harmonious forms. Drawing, coloring and building a Mandala is also a satisfying and extremely relaxing activity, as well as being very effective to acquire the knowledge of angle and geometric shapes.

The steps of our program are: visit to castle of Sammezzano, the geometric transformations, reproduction of Sammezzano tiles, creation of our tiles using goniometer, ruler and especially our imagination. Finally the older pupils taught Mandala's constructions to the younger ones.

9 The project in the secondary school

In the secondary school the same topics are gradually studied in more detail by developing the use of a more symbolic language, more specific to mathematics, but continuing to develop practical skills still using ruler, paper, pen, protractor, compass, glossy paper, mirrors, multimedia board and computer. Here we have a more systematic and analytical analysis of the same topics concerning the symmetries faced by primary school pupils. However, we tried to leave space for students' imagination and creativity without forcing them into too tight schemes. In this way the students have worked for a long time amusing themeselves.

We used the traditional blackboard and also the electronic dashboard. We created some examples from the notebooks concerning the rotations using the glossy paper. We treated the translations analytically using Cartesian coordinates, without forgetting the symmetries of the Castle of Sammezzano!

The students worked with mirrors in axial symmetry, succeeding in doing it thanks to their manual skills, to understand these geometric transformations in more depth. The application of symmetries and tiles was made by the guys using also the free Geogebra software.

The work concerns the tiles of the plane using transformations of central symmetry. The students have tried with triangles, quadrilaterals (convex and concave) and have always managed to fill the plan, deducing that it is possible because the sum of the internal angles is 360 degrees for the quadrilaterals and 180 degrees for the triangles. Then by joining the figures you can form a central angle without empty spaces, while they could not do it with the regular pentagon (internal angles 108°). With the regular pentagons is not possible to tile, empty spaces remain. We have an anecdote about a 8 year old primary school child: during a lesson on tiles, a teacher gave various geometric figures to the pupils (triangles, quadrilaterals, regular hexagons, regular pentagons). All the pupils succeeded except one: he had received the pentagon so he began to cry.

Here we have another example of the lesson on tiles given in the secondary school. This time the students worked in the three-dimensional space. During a lesson the students tried to make a dome using hexagons. They started by observing that with the regular hexagons it is possible to tile the plane. The materials used were: toothpicks, straws, cardboard, scissors and glue. They realized they couldn't get a dome using only the hexagons, that very well tiled the floor instead (they believed that they could apply the same laws to tile either three-dimensional space or the plan) ... the solution was at hand in front of everyone ... the football! Just adding the regular pentagon (maybe you could comfort the child who cried because he could not tessellate the plan with the pentagons!).

This experience was very useful also in the study of solids, in particular for the properties of the solid angles.

The students have applied the concepts learned from the symmetries and tiles to natural sciences that are an integral part of the scientific teaching in the secondary school.

They discovered the microscopic organisms (radiolarian, diatoms) that make up the marine plankton that have a siliceous skeleton that appears under the microscope in wonderful shapes. As the Aulonia exagona, which used hexagons and some pentagons to produce its skeleton in space. Then they again found symmetries in 3D space in the mineral world. They did various researches on this topic and some videos as well.

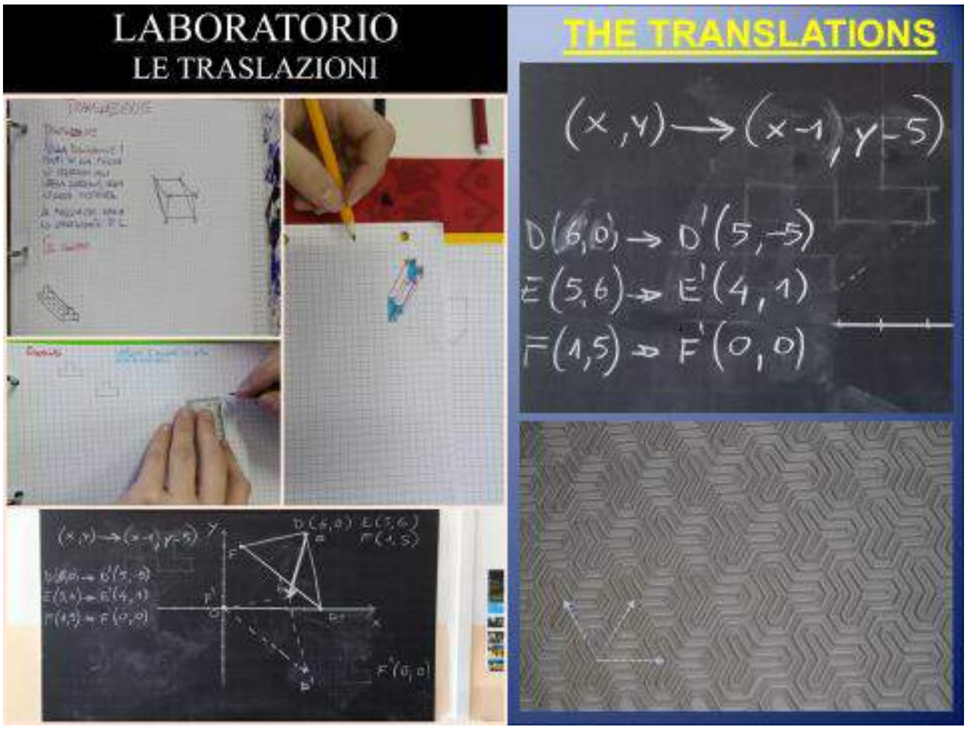
Again large-scale axial symmetries in the mid-ocean ridges, in the oceanic expansion, in particular the very interesting symmetry between normal and inverse magnetic polarization bands with respect to the medioceanic ridge, brought by scientists as a proof of the plate tectonic theory.

Also the music teacher with some students put the attention on the symmetry in music, in particular very interesting the experience of the German Chladni who put into practice the saying of Pythagoras “Geometry is solidified music”. Sand on a smooth metal plate that forms symmetrical and different interference figures according to the frequency of the sound.

The students participated in a ceramic workshop, with a reproduction of a bas-relief of the Castle of Sammezzano.

Finally we organized a trek in the Balze’s area in collaboration with the local environmental association and the involvement of the family.

At the end we realized the conference and the exposition of the students’ works to which the families, the town council, members of the committee, and local associations were invited. The students directly exhibited their works in front of a crowded audience in the town council auditorium.



Some phases of a lab about the translations.

Ceramic lab

Some phases of the construction of a dome

Trekking in the “Balze park”

NOTHING LEFT TO BE DESIRED

The naming of Complex Numbers

Gavin HITCHCOCK
SACEMA, University of Stellenbosch, 19 Jonkershoek, Stellenbosch, South Africa
aghitchcock@gmail.com

ABSTRACT

This paper aims to motivate and exemplify the use of theatre to exhibit the human drama behind mathematics-making, by communicating a sympathetic understanding of mathematicians in their historical context. (In this way, the ESU8 talk served also as an introduction and motivation for a workshop incorporating a play about Niels Abel.) Historical processes of profound transformation in human numerical cognition are illustrated by John Wallis, speaking in the late 17th century on negative numbers, and by Carl Friedrich Gauss and Augustin-Louis Cauchy in dialogue in the 1830s on the re-naming of ‘impossible numbers’ as ‘complex numbers’. Such theatrical devices can help teachers respond to the conceptual difficulties faced by learners. By evoking the intellectual adventure through which notions like ‘irrational’, ‘fictional’, ‘false’, ‘real’, ‘imaginary’, and ‘impossible’, were negotiated, refined, named and re-named, learners may be encouraged to see negatives and complex numbers as wonderful products of a human quest.

1 Introduction

When I first started teaching analysis, beginning with extensions of number systems, I had little idea of the great historical transitions negotiated by the early pioneers in developing and naming the number systems: natural numbers, fractions, negatives, integers, rationals, irrationals, complex numbers. But it soon became clear to me that similar enormousintellectual gulfs have to be crossed in mathematics education today.Our students may repress their anxiety and perplexity, and appear to swallow without painour over-hurried treatment and purely logical development.But we can get a feel for their inevitable struggle by exploring the parallel historical struggle in the evolution of those intriguing names: false, fictional, negative, impossible, imaginary, complex.Appreciating this human intellectual adventure is critical for both history and pedagogy. In this paper I aim to show how the devices of theatre can help to bring alive the colourful story behind the concepts–to show, as vividly as possible, how definitions emerged from passionate debate, concepts were forged and re-forged, names were proposed and argued over and changed –nothing was instantly conceived or distilled ready-made from the intellectual air. Introducing concepts in this way, learners may be drawn into the excitement of the human adventure, and coaxed into welcoming ideas that otherwise appear alien and threatening. This may occur at all levels of education, from primary to tertiary, but the play featured in this talk is especially relevant to the introduction of complex numbers, which may happen informally in secondary (high) schools or colleges (particularly when quadratic equations are encountered), or more formally at university level. The first monologue in the play (by John Wallis) may even be used effectively in primary school when negative numbers are introduced, with suitable explanation or editing of Wallis’ seventeenth century English expressions.

In my talk at ESU8, three other participants kindly agreed to help the audience get inside the heads of three historical characters who were instrumental in negotiating the concept and fixing the name, complex numbers. In the final full acceptance of ‘impossible’ numbers, and their more respectful re-naming as ‘complex numbers’, the authority of Carl Friedrich Gauss (1777–1855) was central. In contrast, the long drawn-out resistance of Augustin-Louis Cauchy (1789–1857) shows that skill in formal manipulation and willingness to make fruitful application are only partial stages on the journey towards embracing mathematical objects in and for themselves. But it was John Wallis (1616–1703), much earlier, who first pointed the way towards representing numbers as relations, grounding his relations in concrete geometrical ideas.The establishment of the negative number system was achieved historically (and is approached today in the classroom) by blending the ideas of geometrical number line and algebraic rules. Similarly, the final step in the establishment of the complex number system, for the pioneers and for numerical cognition of students today, is achieving a fully blended conceptualization of the complex plane and the algebra of complex numbers. This blending is accompanied by a transformation in perception of numbers, from representing objects to representing relations.

Many mathematicians, from Bombelli to Euler, made great use of the so-called impossible or ‘phantom’ numbers without believing fully in them, and even Cauchy struggled, well into the 19th century, to accept them as numbers.The dawning of geometrical representation was crucial, but it came slowly. Opinions differed widely, especially across the English Channel, and communications were bad. But one mathematician, based in Brunswick, Germany, was universally respected. Gauss decided in 1831 to go public with his strong conviction that the so-called impossible numbers were as meaningful as the real numbers. Much earlier hehad made great use of imaginaries in his first proof of the fundamental theorem of algebra. It was in his first paper of 1799, and in correspondence with Bessel in 1811, that he indicated his conviction that these things might be more than just strangely useful phantoms. Finally,two decades later he produced a new and respectful name that would stick, insisting that bad names had been a curse, and asserting that geometry was the handle by which to grasp these things.

It was John Wallis, back in the 17th century, who first proposed a geometrical picture of the square root of minus one as a unit perpendicular to a number line. In his Algebra, Wallis first makes great play with the analogy of the negatives – they can be seen as translations backwards along the accepted positive number line.

2 John Wallis introduces negative quantities

[ENTER Wallis with stick/cylindrical ruler]

WALLIS:[1] It is impossible (everyone agrees) that any quantity can be negativesince it is not possible that any magnitude (or geometric length) can be less than nothing, or any number fewer than none.Yet [...] that supposition (of negative quantities) is not either unuseful or absurd, when rightly understood. And though, as to the bare algebraic notation, it imports a quantity less than nothing, yet, when it comes to a physical application, it denotes as real a quantity as if the sign were ‘plus’ [gestures + in the air with his stick], but to be interpreted in a contrary sense.

As for instance: supposing a man to have advanced or moved forward (from A to B) 5 yards [*begins pacing*]; and then to retreat (from B to C) 2 yards. Suppose it be asked, how

[1] Based closely on (Wallis, 1685), including his language and spelling for historical flavour.

much he had advanced (upon the whole march) when at C, or how many yards he is now forwarder than he was at A. I find (by subducting 2 from 5) that he is advanced 3 yards. (Because plus 5 minus 2 equals plus 3).

But if, having advanced 5 yards to B*[returns to B]*, he thence retreat 8 yards to D*[he paces them out and stands at D]*; and if it be then asked, how much he is advanced when at D, or how much forwarder than when he was at A: I say minus 3 yards! (Because plus 5 minus 8 equals minus 3). That is to say, he is advanced 3 yards less than nothing!

[throughout the rest of his speech, he points with his stick while talking]

Which, in propriety of speech, cannot be (since there cannot be less than nothing). And therefore as to the line AB *forward*, the case is impossible.But if (contrary to the supposition) the line from A be continued *backwards*, we shall find D, 3 yards *behind* A (which was presumed to be before it).

And thus to say, ‘he is advanced minus 3 yards’, is but what we should say (in ordinary form of speech): ‘he is retreated 3 yards’, or ‘he wants 3 yards of being so forward as he was at A’.

[EXIT]

3 From negatives to imaginaries to ‘complex numbers’

It is clear how new and strange Wallis feels the concept of *negative quantity* might be for his readers. Today we introduce young children to‘directed numbers’. Perhaps we can learn from Wallis’ concrete, kinetic style here, in giving one of the first intimations of the doubly infinite number line. He goes on to extend this careful justification of negatives to an analogous representation for imaginaries, as located off the line, in a plane. His aim, as he put it, is ‘to explicate what we commonly call the Imaginary Roots of Quadratick Equations’.

It took a century longer for mathematicians to begin to accept the full planar geometrical representation of what were still called ‘imaginary quantities’.This is not surprising, for even the one dimensional representationof positive and negative numbers was not yet an integral part of mathematicians’ cognitive scaffolding, as Wallis’ painstaking exposition shows. And he could go no further, for the fully developed Cartesian plane was not part of his world.Then, a century later, a stream of others, notably Wessel, Argand, Buée, and Warren, independently saw how to match imaginariesusefully with points in the Cartesian plane, which was then available to them as a cognitive resource. The name ‘imaginary’ survives today, in referring to the ‘real’ and ‘imaginary’ parts of a complex number. Other apparently disrespectful names survive also: negative, irrational! It has been suggested that these names might discourage learners, who could infer that such concepts are fearsome and inaccessible; some have even campaigned to change ‘complex numbers’ to‘composite numbers’, or ‘compound numbers’; we would then call imaginary numbers ‘perpendicular numbers’, and real numbers ‘limit numbers’. However, the rudenames can be usedto great pedagogical advantage by introducing the historical adventure and drama of the making of mathematics.

In the 1830s, Carl Gauss finally grew impatient with the ambivalence and caution of the mathematical community, and announced a new name for the ‘imaginary’ creatures, making a strong claim for their objective existence. In the next section he engages in dialogue with Cauchy, instigator of the theories of complex integration and complex functions,but one who nevertheless took most of his life to accept that the indispensable tools he was using could have realexistence as numbers. The dialogue is the author's invention, but is based on primary sources.[2]

4 Cauchy and Gauss in dialogue

[Cauchy and Gauss walk on in conversation, stopping mid-stage]

CAUCHY: Herr Gauss, I hear you are seriously proposing to give equal rights to impossible numbers!

GAUSS: *Liberté, égalité, fraternité*, eh, Monsieur Cauchy? I would expect the French to be the first to agree with me!–not you, of course – I believe you are no *Révolutionnaire!*

CAUCHY: My father lost his job after the Revolution, and our family was forced to leave Paris when I was a child. But I hope I have been a mathematical revolutionary.

GAUSS*: Oh wirklich,* you have inspired new directions, and a fine rigour in the analysis. But I have not seen much from you lately?

CAUCHY: Cursed kings and wretched revolutionaries – all bent on disrupting or exploiting the creativity of liberal men of science! I have, since the July Revolution, once again been forced into exile from home and from my wife and daughters –

GAUSS: And, I suppose, lost your position and income. That is regrettable. Where have you been?

CAUCHY: It has not been easy …Fribourg, then Turin,and now Prague, where I am tutor to the Duke of Bordeaux, an exceedingly dull boy who will never learn any science. What's more, he insults me incessantly –

GAUSS: Insults you? A teacher is to be respected!

CAUCHY: Ah, I committed the folly of telling him that as an engineer I once repaired the sewers in Paris, and he delights in spreading the lie that M. Cauchy began his career in the sewers!

GAUSS: *Himmel*! I would not tolerate such behaviour from a pupil. Can you not return to Paris where you can be once more at the heart of science, and work in peace?

CAUCHY: *Non* – they would demand that I swear an oath of allegiance to regain my position. At least I now have my family with me in Prague. One day we will return …

GAUSS: And you will get back to inventing revolutionary mathematics!

CAUCHY: Ah, *mon ami*, for that I long! But let's sit down and discuss your own revolutionary mathematics – you can tell me why you seek to elevatecertain mathematical symbols beyond their station. I am told you even propose a new name for them!

[during the narrator's speech they take seats at a café table, each partially facing the audience; the following dialogue may have optional ad-libbed interludes while wine is ordered from a waiter]

NARRATION: So, now our characters have some human background, context and

[2] Mostly based on (Gauss, 1831), with fragments from (Gauss, 1899) and (Gauss, 1811), and fragments from (Cauchy 1821, 1847, 1849).

personality. Having brought them to life,we can now use them to bring to life the story of mathematical ideas – in particular the distressing square root of minus one*[slide appears]*.

$$\sqrt{(-1)}$$

GAUSS: Monsieur, in your illustrious work, you treat these symbols as mere slaves, and indeed they serve you well. But I elevate them to the status of full citizens[3]of the world of number. My paper offers a brief exposition of the principal elements of a new theory of these so-called *imaginäre Mengen*, imaginary quantities. I call them *die komplexen Zahlen*!

CAUCHY: *Des nombres complexes*! You give the impossible square roots of negatives the name of *complex numbers*–and have you thus conjured them into reality? *[shrugs and shakes his head in disbelief]*

GAUSS: *Mein Freund,* you are famous for working freely with *[quotes with mocking gesture] '*impossible numbers' …definite integrals taken between imaginary limits, and a marvellous new type of calculus analogous to the infinitesimal calculus. Why deny them equal rights with real numbers?

CAUCHY: *Bien sûr, naturellement,* I make great use of these symbols, as did our master Euler, but I hold that an imaginary equation is only a symbolic representation of two equations between real quantities. The roots of negative numbers remain impossible! *[shakes his head and pulls a face]* I confess that I harbour a horror of the square root of minus one, even while I write the symbol all over my manuscripts!

GAUSS: *Ach so*, like Euler, you saythe square root of a negative number is an impossible quantity by nature, existing merely in the imagination! It is fortunate nothing prevented him from making use of it in calculation! But between his wonders and yours we should by now have all our painful doubts removed!

CAUCHY: *[shaking his head] Non!* I am still tempted to completely repudiate that horrible symbol, abandoning it without regret, because I do not know what this alleged symbolism signifies nor what meaning to give to it. I only knowhow to make use of it.

GAUSS: *Wissen Sie*, Monsieur Cauchy, the early algebraists likewise fretted over the symbol '*minus*', and called the negative roots of equations *false* roots. And these roots are indeed false when the problem to which they relate has been stated in such a way that the quantity sought allows of no opposite.

CAUCHY: *Oui*! The old objection that there cannot be less than nothing – for how can there be less than no objects? But now we have many applications in mechanics where quantities like extension and time have opposite directions.

GAUSS: *Ganz so*! The acceptance of extensions of numberdepends on the variety of applications being forced upon us. Let us go back in time, and straight to the heart of the matter. M. Cauchy, tell me…in general arithmetic we admit fractions, although there are so many countable things where a fraction has no meaning. Not so?

CAUCHY: *Oui, oui*! And you will now hasten to point out that, just so, we ought not to deny to negative numbers the same rights, simply because innumerable things allow no opposite.

[3] *Bürgerrecht*, in (Gauss 1831, p. 171).

GAUSS: As you say – *Précisément, Monsieur*! The reality of negative numbers is sufficiently justified since in innumerable *other* cases they find an adequate substratum.

CAUCHY: But for impossible numbers there is no adequate foundation, or substratum, as you call it.

GAUSS: If imaginary quantities are to be retained in analysis – which, the versatile M. Cauchy will surely agree, seems better than to abolish them, they must be established on a sufficiently solid foundation. I have long believed it is necessary that they be considered as equally possible with real quantities.[4]

CAUCHY: Equally possible! Equally real? *Non, non, impossible*!

GAUSS: On which account I should prefer to include both real and imaginary quantities under the common designation 'possible quantities'. My recent paper gives a vindication of these names, and a fruitful exposition of the whole matter.[5]

CAUCHY: Some entities are quantities, some are numbers, some are, whatever you say, merely symbols – tools we invent in order to reach real conclusions about real possible quantities.

GAUSS: *Monsieur,* is your work really just an empty play[6] upon symbols, representing impossibilities? Consider the rich contribution which this 'play' has made to the treasure of the relations of real quantities? How can you deny them an adequate foundation?

CAUCHY: My way of treating these equations as purely formal and symbolic spares me the torture of finding out what is represented by that symbol, for which you German geometers simply substitute the letter *i*, and think thus to remove the pain.[7]

GAUSS: *Mein Freund*, I have admired your work for many years, and wondered why you never publicly worried (as the British have a habit of doing) over the question of just what you are talking about.

CAUCHY: These symbols do not have the same sort of existence as real numbers!

GAUSS: I have long considered this highly important part of mathematics from a different point of view, where imaginary quantities can be fully naturalized rather than merely tolerated, and have an objective existence.

CAUCHY: Objective existence! Nonsense!

GAUSS: That's what some of the die-hard English still say of the negatives, and they are being more consistent than you, my friend! Tell me once more, what convinces you that negative numbers exist as objective entities?

CAUCHY: Positive and negative numbers find clear application when the thing being counted – geometrical extension, or time, or velocity – has an opposite, which, when conceived of as united with it, has the effect of destroying it.

GAUSS: Exactly, and this can happen only where the things enumerated are not substances (objects thinkable in themselves), but *relations* between any two objects.

---

[4] Based on (Gauss, 1899).
[5] His intention to do this was signalled 32 years earlier in his 1899 paper.
[6] This is '*inhaltleeres Zeichenspiel'*, in (Gauss, 1831, p. 175).
[7] Based on (Cauchy, 1847), quoted in (Andersen, 1999).

CAUCHY: Are you proposing to define impossible numbers as relations too?

GAUSS: For relations of a single series, plus one and minus one are sufficient to indicate the order of the transition, but for a series of series, i.e., a two-dimensional manifold, an additional pair of units denoting opposition is required, namely *plus i* and *minus i*.

CAUCHY: This is not intuitive – though you may certainly name the vertical unit anything you like.

GAUSS: These relations can be made intuitive only by a representation in space –

CAUCHY: A real number may be identified intuitively with a geometrical quantity, but they are surely not the same thing in themselves!

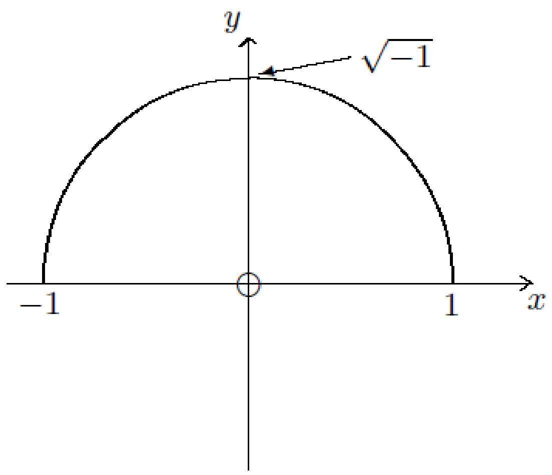
GAUSS: Just as one can think of the entire domain of all real magnitudes as an infinite straight line, so one can make the entire domain of all magnitudes, real and imaginary, meaningful as an infinite plane.[8]

CAUCHY: The directions of positive and negative unity are clearly opposite, but how do we justify assigning their square roots a real direction?

GAUSS: The directions of *plus one* and *plus i* may, in principle, be arbitrarily assigned. But if you consider that, numerically, the *square root of minus one*is a mean proportional between *plus one* and *minus one [slide appears]*

$$-1 = \left(\sqrt{-1}\right)^2 \text{gives } \frac{1}{\sqrt{-1}} = \frac{\sqrt{-1}}{-1}$$

And geometrically *[gestures with his arms throughout this speech, as another slide appears]* the rotation of the direction positive unity through a right angle to the vertical unit *plus i* is a mean proportional between the directions positive and negative unity. Then it becomes entirely natural to let *plus i* correspond to the *square root of minus one.*



CAUCHY: And you claim that an intuitive signification of your symbol *i* has now been fully justified?

GAUSS: Indeed, M. Cauchy, the arithmetic of the complex numbers is provided with *der anschaulichsten Versinnlichung,*[9]and nothing more is necessary to bring this quantity into the domain of objects of arithmetic.

[8] From his letter to Bessel (Gauss, 1811).
[9] Concrete sensory representation. From (Gauss, 1831), in *Werke*, vol. II, 174-175

CAUCHY: Hmm … so you are claiming thatplanar direction, with rotation as the new relation, can give objectivity to the mysterious impossibles …

GAUSS: If people have considered this subject from a false point of view and thereby found a mysterious obscurity, this is largely due to an unsuitable nomenclature. If plus one, minus one and the square root of minus one had not been called *positive, negative*, and *imaginary* (or *impossible*) unity, but perhaps *direct, inverse*, and *lateral* unity, such obscurity could hardly have been suggested.

CAUCHY: It's not all in a name, surely, Herr Gauss! There are serious metaphysical objections to overcome!

GAUSS: *Nein*– by this geometrical device, the effect of the arithmetical operations on the complex quantities becomes capable of sensible representation, such that there is nothing left to be desired. In this way the true metaphysics of the imaginary quantities is placed in a bright new light.

CAUCHY: Hmm … it is not impossible that I may come to grant some sort of existence to these *quantités imaginaires* – pardon, Herr Gauss, I should say, *die komplexen Zahlen*, complex numbers! [*laughs*] I perhaps owe them some dignity for I have made such profitable use of them!

GAUSS: I predict that the architect of the new calculus will, after mature reflection, adopt not only the geometric representation of so-called impossible numbers, but also begin to call them by my far more respectful name.And I prophesy that your new calculus will oneday be called 'complex function theory'. Come M. Cauchy, let us proceed to my house and discuss more mathematics over some good *Bier und Wurst*…

[BOTH RISE]

CAUCHY: Complex numbers! Complex function theory! Fine names, Herr Gauss! But our fears of alien concepts die hard – it reminds me of these scandalous 'non-Euclidean geometries' that some of our colleagues are whispering about –

GAUSS: *Ah, ja*, now there's an interesting conversation we might have…

[EXIT DEEP IN CONVERSATION]

5 Conclusion

And so we end with a hint of the wider context in which strange new algebras and incredible new geometries were breaking through the official boundaries of mathematical authenticity. The battle to grasp the true nature of abstract mathematics raged throughout the nineteenth century. Cauchy himself, as late as 1847, would refer to the torture of finding out what is represented by the symbol for square root of minus one.

[CAUCHY appears again, about 15 years older, shrugging and looking irritated]

CAUCHY: We completely repudiate the symbol, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it.[10]
[EXIT]

Shortly afterwards, still driven by his lifelong crusade for rigour, he describes in more nuanced form his embracing of a geometrical representation. He admits that he has arrived

[10] From (Cauchy, 1847).

at this new position after ‘mature reflections’. He will have owed much to the influence of Gauss, whom we heard deriding Cauchy’s earlier style as an empty play upon symbols.

CAUCHY: In my *Analyse Algebrique*, back in 1821, 1 was content to show that the theory of imaginary expressions and equations could be rendered rigorous by considering these expressions and equations symbolic. But after new and mature reflections the better side to take seems to be to abandon entirely the use of the sign[for] $\sqrt{-1}$ and to replace the theory of imaginary expressions by the theory of quantities which I shall call geometric. Of course I shall have to define very carefully the term ‘geometric quantity’ and further define the different functions of these quantities, especially their sums, their products and their integral powers, by choosing such definitions as agree with those admitted when we are dealing with algebraic quantities alone.[11] *[EXIT]*

Thus, at last the primary architect of complex analysis comes around to a geometric grounding of the complex numbers he has been using so fruitfully for over thirty years! Today, we keep both the symbolic and the geometric in fruitful union, much as Gauss did.Curiously, Gauss did express some doubts, both privately and publicly, in 1834 and 1849, about whether the geometric representation could really capture the true essence of complex numbers.In 1834 he wrote in a letter:

[cameo appearance of GAUSS]

GAUSS: I admit that this *Darstellung* – this geometrical representation, is not really *der Wesen*–the essence – of their being, which is to be grasped by higher faculties in a more general way. But it may be the only completely pure and convincing example of their application.[12]

[EXIT]

Our aim in this paper has been to illustrate, by example, how the devices of theatre, based on extracts from primary sources, can bring out the excitement, the struggle, the sheer achievement, of historical advances in mathematics. Concepts we expect students to swallow unquestioningly are seen to have been fiercely contested by the great mathematicians who gave them birth in a community dialogue. Re-constructing and re-living with the pioneers such debates may help us understand what conditioned prejudices at the time, and what fuelled the journey to new thought paradigms. Use of theatre can bring vividly to life both the people and events behind the abstract concepts of the mathematics curriculum. Such an activity may be mounted with very little preparation, and few theatrical props, as indeed this dialogue was at ESU8 in Oslo. From my own experience introducing complex numbers at university level, motivating concepts in this way, using historical characters in dialogue, is well worth the time taken, as the receptivity of learners is enhanced.

I pose the concluding challenge: Can similar plays add value at various levels of mathematical instruction, for simpler ideas or even more complex ideas? How can we make use of theatre to ease the pain and enhance the joy in learners’ conceptual development?

[11] Based on (Cauchy, 1849).
[12] From (Gauss, 1834).

Acknowledgements

I would like to express my gratitude to the three people who helped to bring life and authenticity to the dialogue by their portrayal of Wallis, Cauchy and Gauss: Peter Ransom, Fred Metin, and Gero Stoffels. In addition, Peter Ransom kindly contributed items of period dress and equipment to enhance the staging.

REFERENCES

Andersen, K. (1999). Wessel's work on complex numbers and its place in history. In B. Branner, & J. Lützen (Eds.), *Caspar Wessel. On the Analytic Representationof Direction. An Attempt Applied Chiefly to Solving Plane andSpherical Polygons*. Copenhagen: C. A. Reitzel. See (Beman, 1987) for more on Wessel, Gauss and Cauchy.

Beman, W. W. (1897). A chapter in the history of mathematics. *Proceedings of the American Association for the Advancement of Science*, *46*, 33–50. Full text available (accessed 19 November 2018) at http://www.archive.org/stream/proceedingsamer23sciegoog/proceedingsamer23sciegoog_djvu.txt

Bottazzini, U. (1986). *The higher calculus: A history of real and complex analysis from Euler to Weierstrass*. Trans. W. Van Egmond. New York: Springer.

Cauchy, A-L. (1821). *Cours d'analyse de l'école royale polytechnique, 1re partie: analyse algébrique*. Paris.

Cauchy, A-L. (1882–1974). *Œuvres complétes d'Augustin Cauchy*, Series 1, 12 vols, Series 2, 15 vols; publieés sous la direction scientifiques de l'Académie des Sciences. Paris: Gauthier-Villars.

Cauchy, A-L. (1847). Quoted and referenced in (Andersen 1999), pp. 91-92. In *Oeuvres complétes II*, vol.14.

Cauchy, A-L. (1849). *Sur les quantités geometriques*. Quoted and referenced in (Andersen, 1999, pp. 91–92). In *Oeuvres complétes II*, vol.14. For more on Cauchy's struggle to understand the nature of complex numbers, see (Bottazzini, 1986).

Gauss, C. F. (1799). *Demonstratio nova theorematis omnem functionem, algebraicam rationale integram unius variabilis in factores reales primi vel secundi gradus resolvi posse*. Helmstedt: C.G. Fleckeisen,

Gauss, C. F. (1811). Letter to Bessel, in *Werke*, Göttingen, 1866, vol. 8, pp. 90–92; quoted in (Katz, 2009, p. 797), in (Andersen, 1999), and in Bottazzini, *ibid.*, p. 156.

Gauss, C. F. (1831). Theoria residuorum biquadraticorum, commentatio secunda, *Göttingische gelehrte Anzeigen*, April 23, 1831. In Gauss, *Werke*, vol. II (see especially pp. 174–175). Quoted in (Beman, 1897).

Gauss, C. F. (1834). Letter, Gauss to Max Drobisch, August 1834, in *Werke*, vol. X, 106. See also Gauss's statement in his *Jubiläumschrift,* 1849, in *Werke*, vol. III, 79.

Smith, D. E. (1959). *A Source Book in Mathematics*. New York: Dover.

Wallis, J. (1685). *De Algebra Tractatus, Historicus et Practicus. (A treatise of algebra both historical and practical, shewing the original, progress, and advancement thereof, from time to time, and by what steps it hath attained to the heighth at which now it is)*. London. Chapters LXVI-LXIX are on '…Negative Squares and their Imaginary Roots in Algebra'.Originally in Wallis, 1673. Chap. LXVI, in Vol. II, p. 286 of the Latin edition, this speech is taken from Wallis' own English translation in 1685. In *Opera mathematica*, 3 vols, 1693–1699. Reprinted in facsimile, 1972, pp. 1–482. Hildesheim: Olms. Also in (Smith, 1959, p. 48).

THE CONCEPT OF SPACE IN THE HISTORY OF MATHEMATICS AND IN THE HISTORY OF PAINTING

Ladislav KVASZ
Institute of Philosophy of the Czech Academy of Sciences in Prague & Pedagogical Faculty, Charles University, Czech Republic
ladislavkvasz@gmail.com

ABSTRACT

In the presentation I will discuss the notion of space as it is used in paintings. I will start with the Renaissance painters Lorenzetti, Masaccio, Ucello, Leonardo, and Duerer, continue with Mannerists Tintoretto, Parmigianino and El Greco, followed by Baroque painters Velazquez, Rembrandt, and Pozzo, then Turner, the Impressionists like Monet and Renoir, post-Impressionist Seurat and Cezanne and will end with Cubists Picasso and Braque and Abstract painters Malevich and Kandinsky.

In analyzing the paintings I will concentrate on the construction of the spatial illusion, the point of view from which the painting is constructed, the way in which a contact with the spectator is introduced and the particular geometric tricks (as the splitting of the horizon in Mona Lisa, the use of a mirror by Velazquez, or deformations by El Greco) used by the painters. It seems that very similar changes as can be seen in painting occurred, on a more abstract level, also in geometry. Thus showing and discussing the particular changes in the history of painting, where they are clearly visually detectable, may contribute to the understanding of these changes in the development of geometry, where the abstract setting sometimes hides the nature of the epistemic shift.

Thus I will discuss the connection between the space used in the Renaissance paintings and projective geometry. In geometry the projection happens between a 3D object and its 2D image and thus is comprehensible in a realistic manner. In projective geometry the 3D object is replaced by its 2D picture and thus the projection is between two images, which is more difficult to comprehend. Similarly I will discuss the relation between Impressionism and the Erlanger Program of Felix Klein. In Impressionism the distinction between the neutral visual basis and the structure that we introduce into it can be directly perceived. In Klein the neutral basis is the projective plane and the structure introduced into it is the metric structure of the particular geometrical system. But in a deep sense they do the same thing - detach the neutral visual basis from the structure. And again it seems that this detachment can be more easily understood in the context of the Impressionist paintings that in the abstract setting of the Erlanger Program. And a similar relation is between Cubism and combinatorial topology. The aim of combinatorial topology, just like of Cubism is to learn to represent objects independently of the surrounding space. Cubists do it by cutting the objects into cubes and rearranging these. Combinatorial topology uses instead decomposition into simplexes, but the basis aim and the basic idea is similar. Thus again the basic idea of liberating geometric objects of their dependence from visual space is common to painting and geometry and painting is technically more accessible.

I use this material in my courses on the history of geometry to illustrate the sometimes rather abstract geometrical theories as non-Euclidean geometry or combinatorial topology. It is important to realize that the very notion of space did not originate in the geometry of

Euclid, but was introduced into geometry from treatises on painting, written by artists like Alberti or Duerer. I have the experience that painting makes it possible to discuss the fundamental ideas of geometry in a non-technical way, what improves comprehension and helps also to remember these deep ideas. Thus one may say that the end goal is a conceptual understanding of geometrical abstraction and building connections between different geometrical theories as projective geometry, algebraic topology and non-Euclidean geometry.

HOW THE COUNTING ROD CONFIGURATION AFFECTS THE PRESENTATION OF THE METHOD OF *FANGCHENG* IN QIN JIUSHAO'S *SHUSHU JIUZHANG*

Chieh-Cheng LIAO
Department of Mathematics, National Taiwan Normal University, Taiwan
jenchengqq@gmail.com

ABSTRACT

Chapter 8 of the *Jiuzhang suanshu* 九章算術 (1st century CE, *Nine Chapters on Mathematical Art*, from here on abbreviated as the *Jiuzhang*) is called *fangcheng* 方程,[1]in which there are17problems with solutions as well as their solving formulas and algorithms. Liu Hui 劉徽 (263 CE) made commentary to the texts upon which the subsequent scholars could base to better understand *Jiuzhang*. After Liu Hui, many Chinese mathematicians also made in-depth researches on *Jiuzhang*. This study is, however, devoted to the *Shushu jiuzhang* 數書九章 (*Mathematical Treatise in Nine Sections*) written by Qin Jiushao 秦九韶in Southern Song Dynasty 南宋. We would try to explain how Qin Jiushao came up with a different viewpoint from Liu Hui in terms of the methodology of *fangcheng shu* 方程術 and *zhengfu shu* 正負術.

Qin Jiushao 秦九韶 (1202 ~ 1261) lived in Puzhou 普州 in Southern Song Dynasty.[2] During Qin's life, Southern Song Dynasty was menaced by north country Mongol. The threat from the Mongols continued until the Mongols conquered the regime of the Southern Song Dynasty and established a new dynasty, Yuan Dynasty 元朝. Qin was in this turbulent era to do research and publish his book *Shushu jiuzhang*. The text, consisting of 18 *juan* 卷 (volumes),[3] is a mathematical treatise which could be compared, in form and content, to *Jiuzhang* at the time. In fact, there are several subjects with which the author deals can be divided into nine chapters: indeterminate analysis (Vol.1, 2), astronomy (Vol.3, 4), surveying (Vol. 5, 6), telemetry (Vol. 7, 8), taxes and levies of service (Vol. 9, 10), storage volumes (Vol. 11, 12), fortifications and buildings (Vol. 13, 14), military affairs (Vol. 15, 16), and commercial affairs (Vol. 17, 18). This paper explores basically the solving of simultaneous linear equations that is applied to the Volume17.

When solving simultaneous linear equations, authors of the *Jiuzhang* provides the methods of *fangchengshu* and *zhengfu shu* which can be regarded to be an algorithm. It is interesting to note that the presentations of the methods of the *Shushu jiuzhang* is different from that of *Jiuzhang*. And it seems that the difference escapes historians' attention.

Therefore, this paper will be devoted to the explanation of the following questions:

(1) What happened to the order of the steps of the procedure in Qin's text?

(2) In contrast to the original version of *Jiuzhang* why did Qin Jiushao give a method with different order of procedure?

(3) Was this different order beneficial to the readers in Qin's contemporaries and the

---

[1] *Fangcheng* means simultaneous linear equation.
[2] Puzhou is located in the province of Sichuan 四川in China today.
[3] There are two volumes in each chapter in *Shushu jiuzhang.*

time that followed?

TECHNOLOGY, RADICAL EDUCATION, AND APPLICATIONS OF MATHEMATICS IN THE PRE-INDUSTRIAL PERIOD IN BACONIAN ENGLAND (1580-1750)

Leo ROGERS
Freelance Researcher
leorogers@appleinter.net

ABSTRACT

The European reformation had considerable effects in England. Competing political and religious views led to regicide, civil wars, parliamentary rule, and the final restoration of monarchy in 1660.Against this background, the founding technologies of early industrial life emerged, and economic, cultural and academic activities flourished.

This paper continues the themes pursued in earlier papers (Rogers, 2012, 2016, 2017) that considered the significance of mathematics in the English social and economic context of early modern science and describes a series of events that prepared the ground for the industrial revolution of the later eighteenth century.

Francis Bacon (1561-1626) was aware that investigating nature could be aided by scientific endeavour founded on new developments in astronomy and mathematics. His *Novum Organon* of 1620 was critical in the development of experimental scientific method employing the 'mechanical arts' as a model for the reform of natural philosophy introducing new ontology and epistemology, thus bringing the natural philosopher and the craftsman together. William Harvey's circulation of the blood (1628) was an early example of applied mechanical philosophy.

The resulting advances in technology were attributable to the accumulation of small, empirical, collaborative and democratic improvements in emergent manufacturing industries, where improving the means of production enhanced technical advances in instrument making, supporting Copernican astronomy, navigation, surveying, architecture, economics, and the prosecution of war (Rogers, 2016).

The idea that the universe was not fixed and immutable grew, that man was able to discover something of God's creation, and that this was a legitimate pursuit aided by new kinds of scientific endeavour and founded on the new developments in astronomy and mathematics. Given this context, the Royal Society was founded in 1660 as a "Colledge for the Promoting of Physico-Mathematicall Experimentall Learning" by men such as Wren, Boyle, Wilkins, Moray and Brounkner.

Hooke's, (1665) "operative knowledge", characterized a relationship with the development of practical mathematics by recognizing theory and practice as a continuum providing potential for new methods, enquiries, and social structures that legitimized tools that 'enhanced our senses' (like the newly improved telescope and microscope)and established new facts. By experimenting with objects, and publicizing results, communities developed professional habits, thus establishing a reflexive relationship between object and operator (Gunter,1624), (Moxon, 1677).

The failure of traditional institutions to adapt to the expansion of activities in commerce, also prompted new social and political theories: Thomas Hobbes, *Leviathan* (1651) on the "Matter, Forme and Power of a Common Wealth" and John Locke, with *Some Thoughts Concerning Education* (1693), offered radical ideas developed by Samuel Hartlib (1600-1662)and William Petty (1690) who advocated teaching 'useful knowledge'. By combining Baconian and Puritan ideas they put forward educational projects including practical 'experimental' subjects to introduce a better, easier, way of teaching, emphasizing that reason requires knowledge, where reasoning skills are necessary for success in any society.

In many educational systems, the history of science and technology is often taught separately as episodic events, or not at all. The relationship between the political and social changes, the development of instruments for measuring and observing the natural world, the consequent increase of scientific knowledge, and the underlying role of mathematics may appear only in the most spectacular results, attributed to one person, where the achievements of many individuals who may have enabled the final result with their 'incremental' improvements have been neglected.

Many young people are unaware of the role of mathematics in the social and scientific developments of today, and turn away from mathematics due to unimaginative curricula and the rote learning and parroting of algorithms. The problem raised on a European scale is ongoing, (Brown, et.al. 2008) (European Commission 2007) and (Kjelsden and Blomholj 2012) have shown how new approaches can benefit students' mathematics education and provide opportunities where historical elements can be used to support students' learning of mathematical concepts, theories, and techniques.

Given a vision of a social-economic context, where mathematics is seen developing in a dual role as generator and also a consequence of cultural, economic and scientific development, we have a considerable range of material here in a context which is less overly technical, to exploit in developing school and college curricula that appeal to a wider audience as well as providing insights into the role of mathematics in society.

PRINCIPAL REFERENCES

Bacon, F. (1620). *Novum Organon*. London.

Boyle, R. (1661) *The Sceptical Chymist*. London.

Brown, M., Brown, P., & Bibby, T. (2008) "I would rather die': reasons given by 16 year olds for not continuing their study of mathematics. *Research in Mathematics Education. 10*(1), March 2008, 3-18.

European Commission. (2007). *Science Education Now: A Renewed Pedagogy for the Future of Europe*. (The Rocard Report.)

Gunter, W.(1624) *The Description and Use of the Cross-Staffe* . London

Harvey, W. (1628). *De Motu Cordis* (On the Motion of the Heart and Blood) Frankfurt.

Hobbes, T. (1651) *Leviathan. Or the Matter, Forme and Power of a Common Wealth Ecclesiasticall and Civil*. London.

Hooke, R. (1665). *Micrographica.* London.

Kjelsden, T., & Blomholj, M. (2012). Beyond motivation: history as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics 80*(3) 327-349

Locke, J. (1693). *Some Thoughts Concerning Education*. London.

Moxon, J. (1677). *Mechanick Exercises or the Doctrine of Handy-Works.* London.

Petty, W. (1690). *Political Arithmetic*. London.

Rogers, L. (2012). Practical Mathematics in 16th Century England: Social-Economic Contexts and Emerging Ideologies in the new Common Wealth. In K. Bjarnadóttir, F. Furringhetti, J. M. Matos, & G. Schubring (Eds.), *"Dig Where You Stand" -2* (pp. 421- 444). Lisbon: UIED.

Rogers, L. (2016) English Mathematical Practitioners and the new Mechanical Philosophy. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.) *Proceedings of the 2016 ICME Satellite Meeting – HPM 2016* (pp. 543-556). Montpellier: IREM de Montpellier.

Rogers, L. (2017) *From the Art of War to the Profession of Gunnery: the consolidation of Custom, Experience, Technical Development and Theory in the Early Industrial Revolution.* Proceedings of the British Society for the History of Mathematics, York Meeting 5-6 April 2017 to appear.

Ward, J. (1719). The Young Mathematicians Guide. London.

SOURCES FROM 16th CENTURY FOR THE TEACHING AND LEARNING OF MATHEMATICS[1]

Fàtima ROMERO VALLHONESTA[2], Mª Rosa MASSA-ESTEVE[3]
UniversitatPolitècnica de Catalunya, Av. Diagonal, 647, 08028 Barcelona, Spain
[2]fatima.romerovallhonesta@gmail.com,[3]m.rosa.massa@upc.edu

ABSTRACT

The use of original texts from History of Mathematics is a way to introduce the sources on which mathematical knowledge is based into the classroom. Choosing historical texts carefully can help students to develop their mathematical reasoning skills and to realize the humanistic aspects of mathematical knowledge through the understanding of the formation process of mathematical thinking. In this paper we present two problems drawn from four mathematical treatises from the 15th and 16th centuries. For the first one, we present three similar wordings drawn from the *Arithmetica* (1484) by Pietro Borghi, the *Coss* (1525) by Christoff Rudolff, and the *Libro Primero* (1552) by Marco Aurel, which concern the hiring of a worker. The second one is a situation that was posed by one student to another in the last book of the *Arithmetica practica y speculative* (1562) by Juan Pérez de Moya. Our aim is to analyse some aspects of the development of algebraic thinking through the mathematical activity of problem solving, implemented in the classroom. The analysis of the different mathematical procedures for solving these historical situations can encourage students to devise their own methods when faced with an unknown problematic situation.

1 Introduction

The history of mathematics shows how mathematics has frequently been used to solve problems concerning human activity, as well as for helping to understand the world that surrounds us. The use of original texts from the history of mathematics is a way to introduce in the classroom the sources on which mathematical knowledge is based.[2]

In this paper we present two historical problems drawn from four mathematical treatises from the 15th and 16th centuries. We have chosen three versions of the first one, drawn from the *Arithmetica* (1484) by Pietro Borghi, the *Coss* (1525) by Christoff Rudolff, and the *Libro Primero* (1552) by Marco Aurel, which deal with the question of hiring a worker. The second one involves a situation that was posed by one student to another in the last book of the *Arithmetica practica y speculative* (1562) by Juan Pérez de Moya. This book is written as a dialogue between students, which allows the author to present different points of view about the usefulness of mathematics and the reasons for the importance of acquiring such knowledge.

Our aim is to analyse some aspects of the development of algebraic thinking through the mathematical activity of solving historical problems, implemented in the classroom (Radford, 2006; Filloy *et al.*, 2008; Broker & Windsor, 2010). The analysis of the different mathematical procedures for solving these historical situations can encourage students to devise their own methods when faced with an unknown problematic situation.

[1] This research is included in the project HAR2016-75871-R.

[2] All this knowledge will undoubtedly enrich the mathematical background and training of teachers. Historians of mathematics have pursued various research lines at an international level that investigate how to use knowledge of the history of mathematics effectively in the classroom in order to improve the teaching and learning of mathematics. For more, see, Barbin, 2000, pp.63-66; Jahnke, 1996; Fauvel & Maanen, 2000; Demattè, 2006; Massa Esteve *et al.*, 2011.

2 History of Mathematics in the Construction of Mathematic Knowledge of Students

The knowledge of the history of mathematics can assist in the enrichment of the construction of students' mathematic knowledge in two ways. First, the use of the history of mathematics in the mathematics classroom can provide students with a conception of mathematics as a useful, dynamic, human, interdisciplinary and heuristic science (see Massa-Esteve, 2003; Romero Vallhonesta *et al.*, 2015) and second, the use of the history of mathematics in the mathematics classroom can provide students with a further relevant feature of mathematics – that it can be understood as a cultural activity (Furinghetti, 1997; Katz, 1997b). History reveals that societies develop as a result of the scientific activity undertaken by successive generations, and that mathematics is a fundamental part of this process. Mathematics can be presented as an intellectual activity for solving problems in each period. Furthermore, the history of mathematics as an implicit and explicit didactic resource can provide tools to enable students to understand mathematical concepts better.[3]

First of all, history can be employed explicitly in the content of compulsory research work undertaken by students in Catalonia in their second year of Baccalaureate (17-year-olds). History situates students in a more general context, since problems are addressed within a global framework of mathematics and within the overall field of science. These research projects not only show the historical evolution of an idea or a concept, but also involve mathematical research enabling students to become familiar with mathematical reasoning from other periods and cultures, as well as in other contexts.[4] The mathematical work that can be undertaken in each of these research fields is highly diverse and range from very simple problems to more complicated proofs.

Secondly, the holding of workshops, centenaries and conferences provides further types of activities in which history can be used explicitly to achieve a more comprehensive learning experience for students. For instance, the workshop devoted to the study of the life and work of René Descartes (1596-1650), held in 1996 at the INS Carles Riba (a high school in Barcelona), afforded students additional education from a mathematical, philosophical, physical and historical perspective.

Finally, the explicit use of significant original sources in the classroom is an activity that can provide students with more valuable means for a better understanding of mathematical concepts, that is the case presented below (Massa-Esteve, 2012; Romero-Vallhonesta & Massa-Esteve, 2015).[5]

[3]The history of mathematics as an implicit resource can be employed by teachers in the design phase by choosing contexts, by preparing activities (problems and auxiliary sources) and also by drawing up the teaching syllabus for a concept or an idea.

[4] The list of titles of research works dealing with the history of mathematics proposed by mathematics teachers is quite extensive, but as examples we may quote the following: Pythagoras and Music; The Golden Mean; On Fermat's Theorem; Pascal's Arithmetic Triangle as a Tool for Resolution; Perspective and its History in the Work of Leonardo da Vinci, Luca Pacioli and Albert Dürer; Women and Science; On Incommensurability: A Mathematical and Philosophical Problem, etc.

[5] Historical texts can be used throughout the different steps in the teaching and learning process; to introduce a mathematical concept; to explore it more deeply; to explain the differences between two contexts; to motivate study of a particular type of problem or to clarify a process of reasoning

3 Catalonian Curriculum 2015. Problem Solving and the History of Mathematics

The history of mathematics has enjoyed an official place in the Catalonian curriculum of mathematics for secondary schools (ESO) since 2007. In the academic year 2007-2008, the Catalan Government Department of Education introduced some compulsory elements of the history of science into the curriculum for secondary education. Specifically, the new mathematics curriculum for secondary schools in Catalonia, published in June 2007, contains notions of the historical genesis of relevant mathematical subjects within the syllabus.[6]

In Decree 187/2015, of August 25th, on the regulation of Catalonian compulsory secondary education, the basic skills in the different areas that students must achieve at this stage are established. These skills are associated with the standards of each area. Therefore, since the last curriculum decree in 2015, the history of mathematics forms part of the contents of the mathematical concepts to be taught.

Also in the last curriculum, problem-solving has played an important role as one of the main methods for teaching and learning mathematics, which goes beyond finding simple solutions or just knowing the basic facts and formulas (Bednarz, Kieran, & Lee, 1996). The best known strategies for solving problems can be found in the work of Pólya (1957), which include, for example, trying special values or cases, working backwards, guessing and checking. By learning how to tackle problem-solving, students acquire confidence in a variety of situations that can help them to use mathematical approaches to solving real-life problems. Problem-solving is a complex mathematical activity, which contributes to the development of algebraic thinking.

The problem-solving standard of mathematics consists of four basic skills:

1. Translate a problem into mathematical language or to a mathematical representation using variables, symbols, diagrams or appropriate models.
2. Use concepts, tools and mathematical strategies for solving problems.
3. Maintain a research attitude towards a problem by testing different strategies.
4. Generate questions of a mathematical nature and raise issues.

In the explanation given in the decree on the problem-solving standard, it is stated that a problem is a proposal to confront an unknown situation that is posed through a set of data within a context, and which initially has no obvious solution and requires reflection, decision-making and strategies. The decree also emphasizes the need to distinguish a problem from an exercise.

From this explanation, it can be inferred that while for some students a proposal may represent a problem because they do not know the algorithm by which it could be solved almost immediately, for others it is the simple application of a technique.

[6] The list of these historical contexts includes: The origins of the numeration system; the introduction of zero and the systems of positional numeration; geometry in ancient civilizations (Egypt, Babylonia); initial approaches to the number π (Egypt, China and Greece); Pythagoras' theorem in Euclid's *Elements* and in China; the origins of symbolic algebra (Arab world, Renaissance); the relationship between geometry and algebra and the introduction of Cartesian coordinates; the geometric resolution of equations (Greece, India, Arab World); the use of geometry to measure the distance Earth - Sun and Earth - Moon (Greece).

4 Setting the scene

We outline a brief historical journey through the development of algebraic equations for setting the scene (Katz, 1997a; Bashmakova & Smirnova, 2000; Massa-Esteve, 2005). While it is possible to deduce an algorithm for solving equations of second degree from Babylonian tablets (1800 BC), Arabian mathematicians were those who took the decisive step in the development of algebra. The mathematician, astronomer and member of the House of Wisdom in Baghdad, Mohamed Ben-Musa al-Khwarizmi (850 AD), is regarded as the creator of the rules of algebra fully rhetorical. In his work *Hisâb al-jabrwal-muqqabala* (813 and 830), he classified equalities (now called equations) up to the second degree according to six different types, as well as explaining rhetorically the method for solving them. It was Leonardo de Pisa, son of Bonacci (1180-1250), better known as Fibonacci, who disseminated, also rhetorically, all this knowledge in the West. Many of the problems addressed in the algebra of the Arabs are to be found in Fibonacci'swork *Liber abaci* (1202), as well as methods for calculating with Indian numerals. The sketchiest period in the development of algebraic equations corresponds to the 13th and 14th centuries, when commercial mathematics flourished with the *Mercantile Arithmetic*, works that are still being explored and analyzed. The knowledge of these mercantile arithmetic and Arab algebra were collected in a work by Luca Pacioli (1447-1517) entitled *Summa de Arithmetica, Geometria, Proportioni & Proportionalità* (1494), which was widely known at that time and very influent in Spanish algebra's authors quoted in the practical activity.[7]

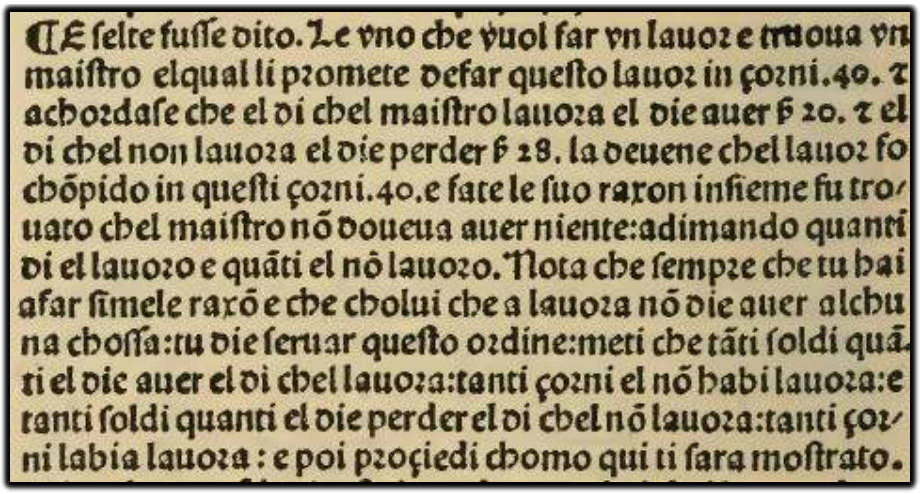
4.1 Practical activity in the classroom

What we now propose are two historical problems from 16th century's sources, period that began the development of resolution of algebraic equations rhetorically. Several versions exist of the first problem, which consists of a problem that can be solved immediately by solving simultaneous equations.

From this problem we present three versions to students; the first by Pietro Borghi, the second by Christoff Rudolff and the third by Marco Aurel. After solving the problems, we discussed with the students the solutions given by the authors themselves, and also explained the characteristics of the mathematics of that time. In this paper we focus on the resolution of the problems.

The first version, from the text by Borghi (Figure 4.1), that contains the wording of the problem and some considerations about the resolution, and the resolution itself (Figure 4.2):

[7] Later, Girolamo Cardano (1501-1576) and Rafael Bombelli (1526-1573), among other algebraists of the *Cinquecento*, also contributed to the solution of cubic and quartic equations with syncopated algebra through their respective works *Artis Magnae sive de Regulis Algebraicis* (1545) and *Algebra* (1572), respectively.



Esel te fusse dito. Le vno che vuol far vn lauor e truoua vn maistro elqual li promete defar questo lauor in çorni.40. e achordase che el di chel maistro lauora el die auer ß 20. e el di chel non lauora el die perder ß 28. la deuene chel lauor fo chõpido in questi çorni.40.e fate le suo raxon insieme fu trouato chel maistro nõ doueua auer niente:adimando quanti di el lauoro e quãti el nõ lauoro. Nota che sempre che tu hai afar simele raxõ e che cholui che a lauora nõ die auer alchuna chossa:tu die seruar questo ordine:meti che tãti soldi quãti el die auer el di chel lauora:tanti çorni el nõ habi lauora:e tanti soldi quanti el die perder el di chel nõ lauora:tanti çorni la bia lauora : e poi prociedi chomo qui ti fara mostrato.

Figure 4.1: [Borghi, 1484, 111v]

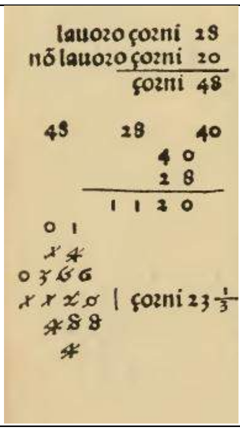
The resolution by Borghi	In current notation
	20 (days not worked) + 28 (days worked)= 48 Let us suppose that in every 48 days, the worker works for 20 days and doesnot work for 28. Therefore, the profit for the worker will be: 28·20 – 20·28 = 0. This would be the solution if the contract was for 48 days, but it was for 40 days. Solving this rule of three: 48 → 28 we obtain 40→ x $x = 23\frac{1}{3}$

Figure 4.2: Theresolution of theproblemabouthiring a worker, by Borghi

We have adapted the situation to a modern context to make it more familiar to the students[8] but maintaining the same quantities:

A girl is looking for a part time job to help in houses doing odd jobs (installing wall sockets, hanging pictures, painting and decorating, etc.) and a woman hires her for 40 days. The woman knows that the girl is not very reliable and the conditions she imposes on the girl are the following: for every day worked, she will receive 20 euros, but if she doesn't show up, she has to pay 28 euros to the owner. After 40 days, the woman doesn't owe anything to the girl. How many days has the girl worked?

The other problem is taken from a book on arithmetic in which various questions are formulated in the form of a dialogue. The first part consists of a dialogue between two students: Sophornio and Antimacho. Sophronio argues for the importance of knowing arithmetic by posing some questions to Antimacho, who at the beginning is doubtful about

[8] Una noia busca feina per fer petits arranjaments a les cases (posar endolls, penjar quadres, pintar alguntros de paret deteriorat…) i una senyora n'hi dóna perquè hi vagi una estona els matins durant 40 dies. La senyoras' assabenta que la noia no ésgaire formal i li proposa pagar-li 20 euros per cada dia treballat i, si algundia no va a treballar, haurà de donar 28 euros a la senyora. Passats els 40 dies, la senyora no li deu res a la noia. Quants dies va treballar i quants dies no va treballar?

this importance, but who in the end has to agree with Sophronio. What we are concerned with here relates to proportionality,which is one of the key concepts of mathematics at all stages of education.

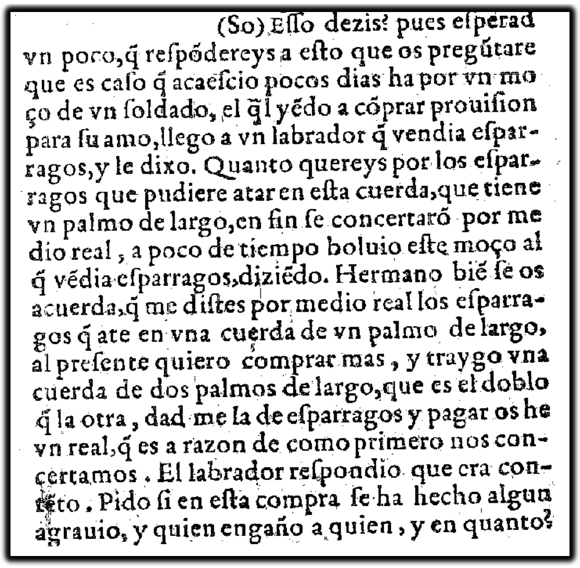

(So) Esso dezis? pues esperad vn poco, q̃ respõdereys a esto que os pregũtare que es caso q̃ acaescio pocos dias ha por vn moço de vn soldado, el q̃l yẽdo a cõprar prouision para su amo, llego a vn labrador q̃ vendia esparragos, y le dixo. Quanto quereys por los esparragos que pudiere atar en esta cuerda, que tiene vn palmo de largo, en fin se concertarõ por medio real, a poco de tiempo boluio este moço al q̃ vẽdia esparragos, diziẽdo. Hermano biẽ se os acuerda, q̃ me distes por medio real los esparragos q̃ ate en vna cuerda de vn palmo de largo, al presente quiero comprar mas, y traygo vna cuerda de dos palmos de largo, que es el doblo q̃ la otra, dad me la de esparragos y pagar os he vn real, q̃ es a razon de como primero nos concertamos. El labrador respondio que era concieto. Pido si en esta compra se ha hecho algun agrauio, y quien engaño a quien, y en quanto?

Figure 4.3: [Pérez de Moya, 1562, 701]

The problem as posed to the students (not adapted in this case, but translated into Catalan[9] from the original 16th century Spanish, with little changes in the two final questions) is as follows:

A soldier went to the market to buy asparagus from a farmer and asked about the price of a bunch of asparagus that could be tied with a string measuring the span of his palm? It was agreed that the soldier would pay half a farthing to the farmer. After a few days, the soldier returned to the same farmer and told him that he wanted to buy a bunch of asparagus that could be tied with a string measuring 2 spans of his palm. For this he offered to pay the farmer double, 1 farthing.
Is it fair for the soldier to pay the farmer 1 farthing?
Is this correct? If you think that this is not right, how much should be paid and why?[10]

In this case, the solution is given as a rhetorical reasoning. The question is posed by Sophronio to Antimacho. Antimacho believes that there is no doubt that the soldier has to pay one farthing for the asparagus tied by a string measuring 1 handspan. Sophronio asks him to take one string and another of double its length and to check himself the quantity of asparagus that can be bound by in each string, in order to see that the quantity of asparagus in the second string will be four times the quantity in the first. Therefore, the

[9] Un soldat va anar al mercat i va anar a comprar a un pagès que venia espàrrecs i li va dir: quant em demaneu pels espàrrecs que puc encerclar amb aquesta corda que té una longitud d'1pam? Van acordar que el soldat pagaria mig ral al pagès. Al cap d'uns dies, el soldatva tornar a anar al mateix pagès i li va recordar que feia uns dies li havia cobrat mig ral pels espàrrecs que es podien encerclar en una corda d'1 pam i que ara en voliamés: totsels que es podienencerclaramb una corda de 2 pams, que és el doble de l'altra. Per tant, posa-m'hi els espàrrecs que hi caben, i jo et pago el doble, 1 ral.
Ésjust el que li volia pagar el soldat? Justifica-ho.
Si creus que no ésjust, digues quant li hauria de pagar i justifica-ho.
[10] We have changed the last question from the original text, in order to be more understandable for the students.

amount of money that should be paid to the farmer is 2 farthings.

These problems were put to 28 3rd-grade students of ESO[11] and to 29 1st-grade students of ESO[12] at the Eugeni Xammar Secondary School [13] in l'Ametlladel Vallès. In secondary schools in Catalonia, students start their secondary education at age twelve and they can continue until they are eighteen, although compulsory education finishes in 4th of ESO when students are sixteen. The 3rd-grade students had already been taught how to solve simultaneous equations, while the 1st-grade students had not.

In the presentation of the activity to students, they were told the following:

1. This activity is part of a research project.
2. It is important that you note down any possible ideas, strategies or reflections about the resolution of the problems, even if you have doubts about the accuracy.

The students were given this advice because, as they were not used to solving problems in which they were required to apply known techniques, we wished to avoid the risk that many of them might not attempt to solve the problems for fear of an incorrect answer.

In the case of the 3rd-grade ESO students, the only correct answers were given by students who solved the problem by using simultaneous equations. None of the students who tried to resolve the problem using a more imaginative procedure managed to arrive at the solution, as shown in the following table:

Resolution of the first problem by 3rd-grade ESO students				
Using simultaneous equations		Without using equations		Unanswered
correct	incorrect	correct	incorrect	
7	4	0	13	4

The following image shows a typical correct resolution by a 3rd-grade ESO student:

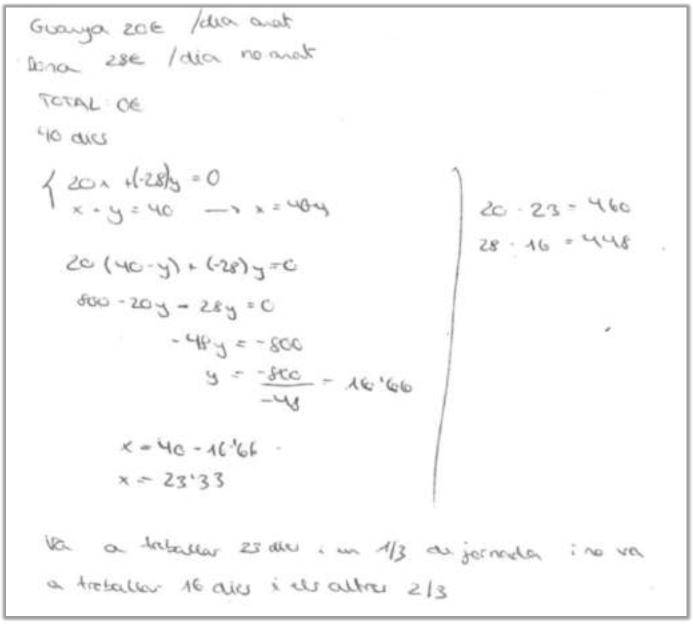


Figure 4.4: Resolution by a 3rd-grade ESO student.

[11] 14-year-old students.
[12] 12-year-old students.
[13] We express our gratitude to the mathematics teaching stafffor their collaboration and effort in implementing this activity.

The types of solutions given by the 1st-grade students were more varied. Most of them attempted to find the solution by looking for common multiples of 20 and 28, with the condition that the numbers by which we have to multiply 20 and 28, added together, gave 40. In this case, the way to solve the problem is more interesting. In the case of the 3rd-grade students who solved the problem by solving simultaneous equations, on arriving at the solution they took it for granted that it was correct. However, some of the 1st-grade students thought that a non-integer solution did not make sense, and since they were unable to find two entire quantities according to the conditions of the wording, they said that the problem had no solution. In these cases, if the procedure adopted was correct, we considered the resolution to be correct.

Resolution of the first problem by 1st-grade ESO students			
Resolve the problem correctly	Use a good method but do not arrive at the solution	Try to solve the problem with an unclear method and do not arrive at any conclusion	Unanswered
8 (6 of whom say that the problem has no solution)	10	5	6

In this regard, we present two interesting resolutions. In the first one, the student says that the circled numbers in the table represent the days on which neither the woman nor the girl are indebted to each other. The sum of these days should be 40, but in none of the cases is this the result. Therefore, the problem has no solution. However, the student probably found it strange that the problem has no solution, and adds: if we read the question carefully, it is stated that the woman does not owe the girl anything, but nothing is said about the possible debt owed by the girl to the woman. We could assume that the girl has not worked at all, and in this case, she would have to pay 1120 euros to the woman, who does not owe the girl anything.

In fact, as the student remarks about the statement of the problem, nothing is said about the debt that the girl owes to the woman, but according to the solution provided by Borghi, we see that the author supposes that the girl owes the woman nothing.

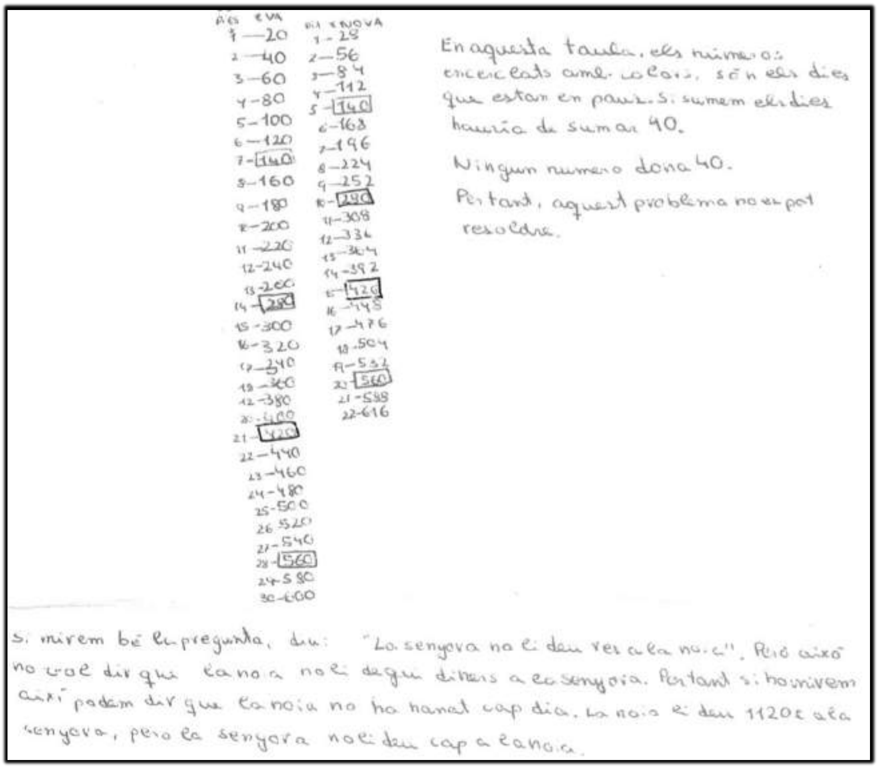
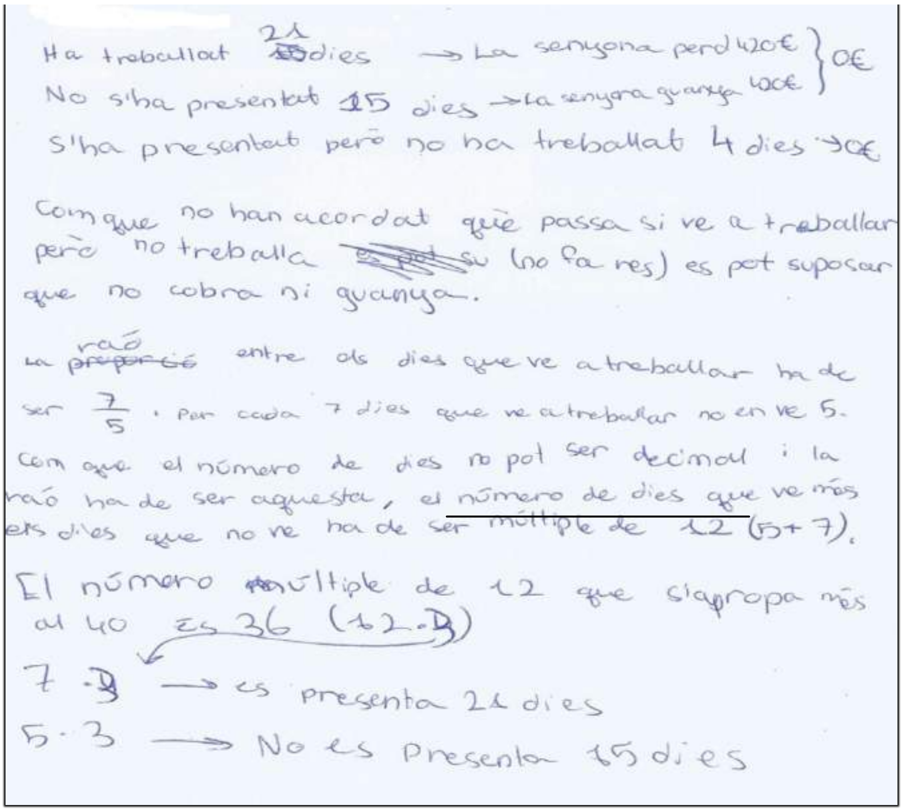


Figure 4.5: Resolution by a 1st-grade ESO student.

In the second case, the student considers that the ratio between the days worked and those not worked must be 7/5, which is the ratio between 28 and 20. For the balance to be 0, then for every set of 12 days the girl would work on 7 days and on 5 days she would fail to show up. As the number of days worked cannot be a decimal (as the student supposes), the total number of days should be a multiple of 12 (7+5). The multiple of 12 nearest to 40 is 36 (12·3). Therefore, the girl goes to work on 7·3=21 days and fails to show up on 5·3=15 days, which leaves 4 days unaccounted for. On considering what may have happened on these days, she believes that since the two parties reached no agreement on what might happen if the girl went to work but did not do anything, then neither would the woman pay the girl nor would the girl owe anything to the woman.


Ha treballat 21 ~~15~~ dies → La senyora perd 420€ } 0€
No s'ha presentat 15 dies → la senyora guanya 420€
S'ha presentat però no ha treballat 4 dies → 0€

Com que no han acordat què passa si ve a treballar però no treballa ~~es pot~~ su (no fa res) es pot suposar que no cobra ni guanya.

La ~~proporció~~ raó entre els dies que ve a treballar ha de ser $\frac{7}{5}$. Per cada 7 dies que ve a treballar no en ve 5.
Com que el número de dies no pot ser decimal i la raó ha de ser aquesta, el número de dies que ve més els dies que no ve ha de ser múltiple de 12 (5+7).
El número múltiple de 12 que s'apropa més al 40 es 36 (12·3)
7·3 → es presenta 21 dies
5·3 → No es presenta 15 dies

Figure 4.6: The resolution of a 1st-grade ESO student.

It is interesting to remark that only the students who solved the problem using their own procedures gave any consideration to the authenticity of the solution. None of those who solved the problem by using simultaneous equations add any comments to the result.

Resolution of the second problem by 3rd-grade ESO students		
State that it is not fair, but fails to justify or do it incorrectly	State that it is fair	Unanswered
9	16	3

None of the students gave the correct solution to the question and none of them used formulas for the perimeter of a circumference and the area of the circle.

In the case of the 1st-grade ESO students:

Resolution of the second problem by 1st-grade ESO students			
Give a good solution	State that it is not fair, but fail to justify or do it incorrectly	State that it is fair	Unanswered
1	2	9	17

For the second problem, we present the only correct answer:

In this case, since the student probably did not know the formulas for the area of a circle and the perimeter of a circumference, he drew a square with each side measuring 1 cm (the area of thissquare being easier to calculate than that of a circle) and another square with each side measuring 2 cm. He realizes that the area of the second is four times the area of the first and concludes that it is not fair to pay 1 farthing to the farmer.

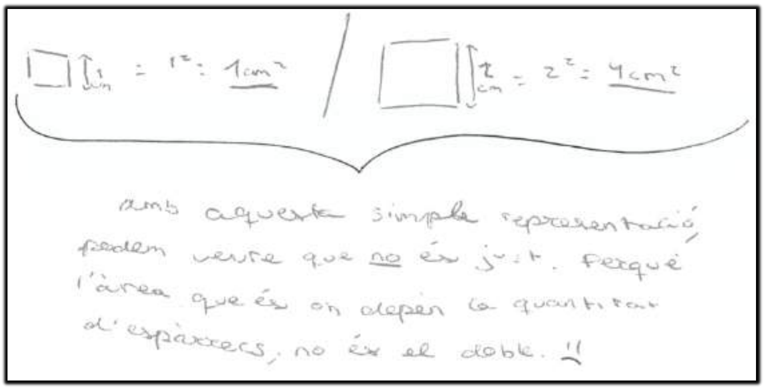


Figure 4.7: Solution by a 1st-grade ESO student.

We might ask ourselves why there were many students who tried to solve the first problem by guessing and checking, whilethe only imaginative solution for the second problem is that shown in Figure 4.7. One of the reasons could be that in the first case it is easy to guess and check using pen and paper, while in the second case it would be more practical to use objects like pieces of thread, and pencils for the asparagus,and the students are less familiar with this method.

5 Practical activity for pre-service teachers

The problems in the case under study were also posed to pre-service teachers, who were divided in six groups of five or six. They were asked to think about how they might solve these problems without using equations and also to explain in detail the four steps of the Pólya's approach to problem-solving:

- Understanding the problem
- Devising a plan
- Carrying out the plan
- Looking back

In the case of the *lazy worker,* for the first step, all the groups recommended identification of the main facts in the wording of the problem and then trying to solve it with some specific values in order to determine what the outcome would be.

Five of the six groups drew a table, two of whom regarded it as part of devising a plan and three of whom considered it as part of carrying the plan out. For this problem, they all thought that a table was a good device for visualizing all the possibilities.

None of the pre-service teachers thought about a non-integer quantity as a solution. On arriving at the conclusion that there was no solution, they proposed a re-reading of the statement of the problem and realized that no reference was made about the debt owed by the girl to the woman, and then put forward different solutions. Except for minor differences, their reasoning is similar to that followed by the student whose resolution is shown in Figure 4.5.

One of the groups analyzed the problem in a way similar to that in the work of the student shown in Figure 4.6; that is, by considering sets of 12 days and concluding that there is no solution. This group also re-read the wording and remarked that nothing is statedabout the debt owed by the girl to the woman and said that is not possible to obtain the right result, but only an approximation.

In the case of the problem of the asparagus, all the groups came to the conclusion that the problem for the students would be that they would fail to take into account that the proportionality between areas and perimeters is not the same.

One of the groups proposed a drawing with GeoGebra (Figure 5.1), for the conception of the plan, as follows:

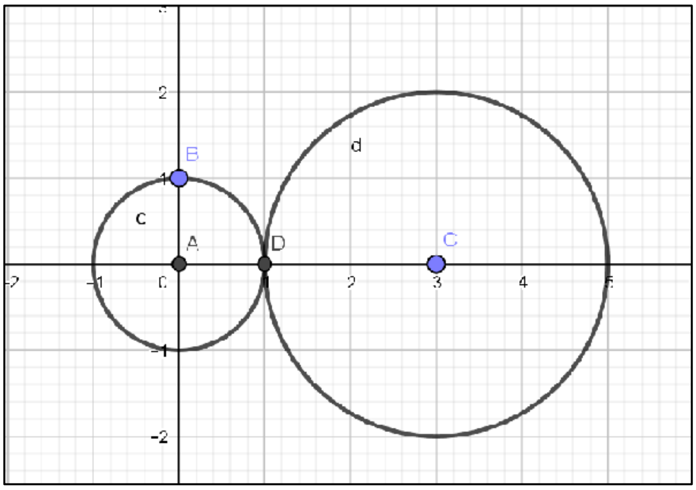


Figure 5.1: Devising a plan by pre-service teachers.

They believed that even though the relationship between the areas of these two circles might not be clear to the students, it seemed obvious that the area of the biggest circle was greater than double the area of the smaller one.

Another group proposed that students use their hand-spans or use threads or similar objects in order to experiment.

The solution given by one student, as shown above in Figure 4.7, was also taken into account by one of the groups (Figure 5.2) with the following representation:

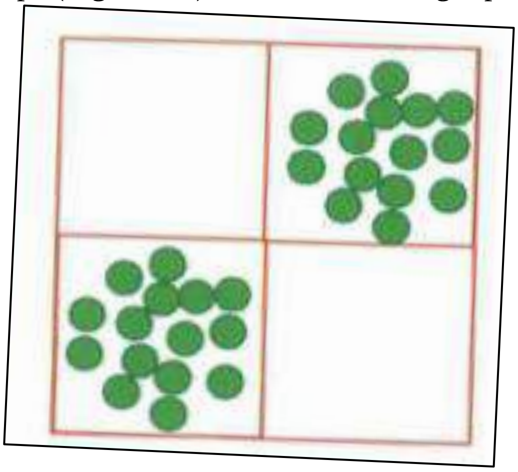


Figure 5.2: Solution by pre-service teachers.

This clearly shows that the soldier’s proposal is not fair for the farmer.

All the groups found the proposal interesting, since it admits different levels of resolution and also enables teachers to follow the reasoning of the students.They also agreed that it should be considered when equations and some algorithms are introduced.

6 Concluding Remarks

The activities based on the analysis of historical texts using original sources as a point of departure contribute to improving students' overall education and provides them with additional knowledge of the social and scientific context of the periods involved.

These activities can be designed to allow for different levels of development and, in some cases, the distribution of tasks among students according to their skills.

We presented to students four texts written in Italian, German and Spanish, three of them about a *lazy worker* and the other related to proportionality. In the case of the problem of the *lazy worker*, which is a typical problem to be solved by simultaneous equations, we have seen different approaches, the most interesting of which were carried out by students who had not been taught about simultaneous equations. These students proved to be more creative, because they were unfamiliar with the standard method of resolution.

When students are told how to solve simultaneous equations and are then required to apply the method to solving a problem, the situation that initially arises could constitute a difficulty for them, since they may find it hard to translate the situation into mathematical language. However, for some students this translation may occur to them almost immediately, and the situation is not considered a problem following the explanation in the curriculum.

Early introduction of the simultaneous equations may restrict students in their exploration of methods that might otherwise be employed. In the problem of the asparagus, the student who gave the best solution was unaware of the relationship of the proportion between the lengths and the areas. He did not remember the formula for calculating the area of the circle and drew two squares, one of which was double the perimeter that the other one. He calculated the areas of these squares and deduced that the relationship between the areas of the circles, on which the number of asparagus depends, should be the same.

When teachers pose problems to students, it is important that they take into account the different backgrounds of the pupils and adapt their approach accordingly, depending on the outcomes desired.

Reading ancient texts enables students to acquire an understanding of mathematics, not as a final product but as a science that has developed on the basis of trying to answer the questions that humanity has been asking about the world throughout history.

Finally, we would like to stress the importance of using different approaches to the curriculum, for a better understanding of some concepts. In this paper, by looking at the solutions of students, using historical problems, we are aware of the difficulties that some of them have, and the strategies of others to cope with non-standard situations.

REFERENCES

Bashmakova, I., & Smirnova, G. (2000). *The Beginnings & Evolution of Algebra*, A. Shenitzer (Trans.), Washington: The Mathematical Association of America.

Barbin, É., Bagni, G. T., Grugnetti, L., Kronfellner, M., Lakoma, E., & Menghini, M. (2000). Integrating history: research perspectives. In J. Fauvel & J. van Maanen, (Eds.): *History in mathematics education: the ICMI study* (pp. 63-90). Dordrecht: Kluwer.

Bednarz, N., Kieran, C.,& Lee, L. (Eds.) (1996). *Approaches to Algebra. Perspectives for Research and Teaching.* Dordrecht: Kluwer.

Borghi, P. (1484). *Arithmetica.*

Dematté, A. (2006). *Fare matematica con i documenti storici. Una raccolta per la scuola scondaria de primo e secondo grado*, Trento: Editore Provincia Autonoma di Trento – IPRASE del Trentino.

Fauvel, J., & van Maanen, J. (Eds.) (2000). *History in mathematics education: the ICMI study*. New ICMI Study Series, vol. 6. Dordrecht: Kluwer.

Filloy, E., Puig L., & Rojano, T. (2008). *Educational Algebra. A theoretical and Empirical Approach.* New York: Springer.

Furinghetti, F. (1997). History of Mathematics, Mathematics Education, School Practice: Case Studies in Linking Different Domains. *For the Learning of Mathematics, 17*(1), 55-61.

Guevara, I., Massa-Esteve, Mª R., & Romero, F. (2008). Enseñar Matemáticas a través de su historia: algunos conceptos trigonométricos. *Epsilon 23* (1&2), 97-107.

Jahnke, H. N., Knoche, N., Otte, M., & Aspray, W. (Eds.) (1996). *History of Mathematics and Education: Ideas and Experiences*, Göttingen: Vandenhoeck und Ruprecht.

Katz, V. J. (1997a). Algebra and its Teaching: An Historical Survey. *Journal of Mathematical Behavior,16*(1), 25-38.

Katz, V. J. (1997b). Some Ideas on the Use of History in the Teaching of Mathematics. *For the Learning of Mathematics, 17*(1), 62-63.

Massa-Esteve, M. R. (2003). Aportacions de la història de la matemàtica a l'ensenyament de la matemática. *Biaix21*, 4–9.

Massa-Esteve, M. R. (2005). Les equacions de segongrau al llarg de la historia. *Biaix 24*, 4–15.

Massa-Esteve, M. R., Guevara-Casanova, I., Romero-Vallhonesta, F., & Puig-Pla, C. (2011). Understanding Mathematics using original sources. Criteria and Conditions. In E. Barbin, M. Kronfellner & C. Tzanakis (Eds): *History and Epistemology in Mathematics Education.Proceedings of the Sixth European Summer University* (pp. 415-428). Vienna: Verlag Holzhausem GmbH.

Massa-Esteve, M. R. (2012). The Role of the History of Mathematics in Teacher Training Using ICT. In O. Bruneau, P. Grapi, P. Heering, S. Laubé, M. R. Massa-Esteve, & Th. De Vittori (Eds.) *Innovative Methods for Science Education: History of Science, ICT and Inquiry Based Science Teaching.* Berlin: Frank & Timme GmbH Verlag für wissenschaftliche Literatur.

Pérez de Moya, J. (1562). *Arithmetica Practica y Speculativa.* Salamanca: Mathias Gast.

Polya, G. (1957). *How to solve it. A new Aspect of Mathematical Method*, 2nd Edition. Princeton and Oxford: Princeton University Press.

Radford, L. (2006). The cultural-epistemological condition of the emergence of algebraic symbolism. In F. Furinguetti, S. Kaijser & C. Tzanakis (Eds.): *Proceedings of HPM 2004 & ESU 4*(pp. 509-524). Iraklion: University of Crete.

Romero-Vallhonesta, F., & Massa-Esteve, M. R. (2015). Història de les matemàtiques per a l'ensenyament de les matemàtiques. Analitzant les fonts. In P. Grapi & M. R. Massa (Eds.): *Actes de la XIII Jornada sobre la Història de la Ciència i l'Ensenyament* (pp. 59-70). Barcelona: SCHST-IEC.

Romero-Vallhonesta, F., Massa-Esteve, M. R., Guevara Casanova, I., Puig-Pla, C., & Roca-Rosell, A. (2015). Teacher Training in the History of Mathematics. In É. Barbin, U. T. Jankvist & T. H. Kjedsen (Eds): *History and Epistemology in Mathematics Education. Proceedings of the Seventh European Summer University* (pp. 113-128). Aarhus: Danish School of Education, Aarhus University.

TIME MEASUREMENT AS AN INTERDISCIPLINARY SUBJECT IN MATHEMATICS EDUCATION

Constantinos TZANAKIS
Department of Education, University of Crete, Rethymnon 74100, Greece
tzanakis@edc.uoc.gr

ABSTRACT

Nowadays measuring time is considered almost instinctively a subject trivially known and well established in our everyday life. However it has been interesting, scientifically fascinating and mathematically nontrivial throughout the ages, constituting a “meeting point” of disciplines epistemologically very remote from each other, based on quite different motivations and having different objectives. It is an exceptionally rich subject that could be beneficial and stimulating in the context of mathematics education at various instructional levels, in a variety of situations, and be approached from many different perspectives. This paper aims at providing evidence to support this point in the context of an appropriate HPM framework, by means of some characteristic examples, which - either explicitly or implicitly - are related to the calendar as we know it today or has been developed historically.

1 Introduction

1.1 The context

Unlike space, time can be conceived and delimited only if represented in terms of symbols, which themselves require and/or are susceptible to different interpretations. As a result, understanding time (more than understanding space) implies the imperative need to rationalize its representation in terms of arithmetical or geometrical symbols of a most unequivocal meaning. Perhaps this is the generic element underlying the relation between the concept of time and its mathematical elaboration (cf. Borst, 1993, pp. 5-6). Therefore, as a fundamental formative category of human thought and perception of the world, conceiving time is indissolubly connected to its quantification through measurement.

On the other hand, inherent to the concept of measurement is the act of comparison; namely, to compare objects with respect to a certain characteristic they possess in common, by agreeing to choose and choosing one among them as a standard of comparison; the unit of measurement of this common characteristic. This is true for any kind of measurement. In the case of time, this means specifying and using time units that are “stable” enough. This, in turn, is realized by focusing on **periodic** (cyclic) phenomena. (Enough) “stability” in this context means the existence of periodic phenomena **compatible** with each other; i.e. **their periods’ ratios do not change** (appreciably) in the course of human life, society’s existence, or generally, during a specified corpus of (individual and/or collective) human experience (cf. Fraser, 1987, ch. 2 pp. 58-59). And it is a fundamental **empirical fact** that phenomena do exist for which this condition is (approximately) fulfilled. Somewhat loosely, they can be called “clocks”.

From a historical point of view two points should be particularly stressed: (i) it was first perceived and understood that astronomical and physical clocks compatible in the above sense do exist. Therefore, they were preferably selected for quantifying (thus, measuring) time; (ii) it was gradually (and slowly) realized that their compatibility was only approximate; that the more human observations and experimentations were becoming finer, the more rough this approximate compatibility was getting. As we shall argue, this development was strongly interrelated with that of the required mathematics.

1.2 The "*three basic clocks*"

It is a fact that since early in history, time measurement has been based on what henceforth is called the "*three basic clocks*" (Borst, 1993, ch. 2, p.9; Whitrow, 1988, ch. 2, pp. 14-17; Richards, 1998, ch. 1, pp. 7-8):

- The *day*: Earth's rotation (around its axis) causing the day-night alternation;
- The *month* (and the week): Moon's revolution (around the earth) causing the (four) lunar phases.
- The *year*: Earth's revolution (around the sun) causing the succession of the seasons and the return of the fixed stars to the same position in the sky;

For later use and for comparison, the **current** values of their periods' ratios are given below, using the **modern** definition of (atomic) *second* in SI units, the currently accepted system of units in physics (International System of Units, n.d.):

$$1\text{ (atomic) sec} := \frac{9{,}192{,}631{,}770}{\nu}$$

where ν is the frequency of radiation emitted due to the transition between the two hyperfine energy levels of the ground state of the Caesium isotope ${}^{133}_{55}\mathrm{Cs}$. In this unit, the "day" is by definition 86,400sec

$$t_{\mathrm{D}} = \frac{\text{day (SI)}}{\text{sec}} := 86{,}400 \tag{1.1}$$

whereas, the three basic clocks' periods are to a reasonably good degree of approximation (Richards, 2013, §15.1.3; for the terms "tropical" and "synodic" see §2.3).

$$t_{\mathrm{Y}} = \frac{\text{year}}{\text{day}} := \frac{\text{(mean) }\textbf{tropical}\text{ year}}{\text{day (SI)}} = 365.2421897 = 365^{\mathrm{d}}\,5^{\mathrm{h}}\,48'45'',19 \tag{1.2}$$

$$t_{\mathrm{M}} = \frac{\text{month}}{\text{day}} := \frac{\text{lunar (mean) }\textbf{synodic}\text{ period}}{\text{day (SI)}} = 29.5305885 = 29^{\mathrm{d}}\,12^{\mathrm{h}}\,44'2'',88 \tag{1.3}$$

$$t'_{\mathrm{D}} = \frac{\text{day}}{\text{day (SI)}} = 86{,}400.003 \tag{1.4}$$

We note that (i) the expression of the above periods as **decimal** fractions of the **day** and **sexagesimal** fractions of the **hour** already underlies the complexity of the measurement problem[1]; (ii) the *second* is not a concept as simple as one may think, but is founded on the current theory of **microphysics**, far beyond any "naïve" approach to the measurement of the periods of the basic clocks (Auerbach, 1995; Nelson *et al*, 2001).

1.3 The *social* context and the fundamental *mathematical* constraint

These "innocent-looking" and taken for granted to be simple notions (day, month, year), will be found to be very complicated both conceptually and technically.

Firstly, their implementation is strongly delimited by the **social context**: For several political, religious and economic reasons social life is requested to be based on "simple" temporal cycles; i.e. **integral** periods of appropriate periodic phenomena.

Secondly, it is a **mathematical fact** that there are no simple **integer** relations among the periods of the **empirically** determined three basic clocks; i.e. their periods' ratios

[1] E.g. that sun's annual revolution in the sky is not 12 lunar cycles, or 365 days as originally thought. Due to the complexity of these periodic phenomena it took a long time in history to realize that these are not good approximations for long time intervals, and to find better approximations of their relative periods.

cannot be expressed as rational numbers sufficiently simple to be practical for civil purposes in a straightforward manner. Evidently, given any prescribed degree of accuracy (determined by social or scientific criteria) and (implicitly) conceiving (real) numbers as a continuum, such rational relations may in principle result by choosing sufficiently small time units. However, this presupposes an ever increasing accuracy of the corresponding measurement processes.

To summarize, all this indirectly points to the (in principle) importance of specific mathematics both conceptually and technically in relation to the measurement of time by means of the three basic clocks. Therefore, this paper is organized as follows. Section 2 gives evidence that measuring time has been a multifaceted, interdisciplinary and intercultural issue throughout its historical development, based on mathematical knowledge available at the time, possibly stimulating its further development as well. It is also argued that aspects of this development can be beneficial for mathematics education. In section 3 a general framework for integrating history in mathematics education is outlined. Finally, in section 4 this main point is exemplified by five specific examples, at the same time commenting on their placement in the general framework of section 3.

2 Motivation

2.1 The basic questions

Two basic questions have been implicit so far:
(1) **Why** was/is it **important** to determine accurate and stable **time units**?
(2) **How** was/is it **possible** to determine accurate and stable **time-keeping** in terms of such **time units**?
In other words: In the context of which domains of human activity and knowledge, and in relation to which questions and problems was this important?

These two interrelated questions are related to mathematical issues, either elementary or advanced when judged by current standards. In fact, measuring time is closely related, touches upon and addresses interconnected issues in the context of several distinct disciplines, leading to questions and problems that finally led to important developments both in mathematics and in these disciplines. However, because time (as a quantified concept) is so deeply rooted into our (modern) civilization, its accurate measurement seems to be an elementary subject, at least conceptually. As a consequence, its emergence (hence, the awareness of the time concept itself; see §1.1) and the way it has been interrelated deeply with many intellectual, practical, political and religious aspects of human history is hardly appreciated in general, and in education in particular (for a concise overview see Whitrow, 1972a).

Therefore, in relation to question (1), it is helpful to comment briefly on some of the reasons **why** accurate time-keeping and time units have been important:

History: Already in antiquity it was realized that it is important to have trustful means to reckon historical facts (i.e. some sort of accurate chronology) by carefully considering the appearance of each historical fact in relation to others' (Smyntyna, 2009; Borst, 1993, ch. 2); e.g. Herodotus' and Thucydides' historical accounts in ancient Greece, Varro's chronology in the Roman republic period starting "from the founding of the City [Rome]" (AUC; *Ab Urbe Condita*), Flavius Josephus' account of Jewish history (1st century AD), Bede's "Ecclesiastical History of the English People" (8th century AD) etc.

Politics: Early in history it became necessary to have a measure of the duration of the term of persons and collective bodies holding public positions (e.g. Athenian Archons, Roman Consuls etc), or/and to reckon events by means of regnal periods or eras. This was achieved by using regularly occurring or important and exceptional events; e.g. the Olympiads in ancient Greece; counting years from the founding of Rome (AUC) in ancient Rome; the Diocletian era (or the era of the Martyrs – *Anno Diocletiani*) used by the Alexandria's early Christian church and starting on the first regnal year of the Roman emperor Diocletian (284AD); the AD (*Anno Domini*) era introduced in the 6th century by Dionysius Exiguus replacing the Diocletian era and used in the Christian world since then; the *Hegira* era in the Islamic world and starting with Mohammed's migration to Mecca (622AD); the republican era of the French Revolution (starting on 1792 AD), etc (Hannah, 2005, chs. 3, 5; Richards, 1998, ch. 6, pp. 104-109; Fraser, 1987, pp. 91-95).

Economy: For economic reasons, temporal cycles were specified already in antiquity; e.g. the division of the month by the Romans into *calends*, *nones* and *ides*[2], the calends indicating both a month's first day (originally starting with a new moon) and the day on which debts should be paid according to the accounting books called *kalendaria* (Hannah 2005, ch. 5; Holford-Strevens, 2005, pp. 28-31; Richards, 1998, p. 210; Whitrow, 1988, p. 68); or, longer temporal cycles like the Roman 15-year *indictions* - a fiscal period for the agricultural or land taxes' reassessment adopted by the Byzantines, used in medieval Europe and kept until late in modern times (Whitrow, 1988, p. 67; Richards, 1998, p. 101). Moreover, from the late Middle Ages to the industrial revolution, the gradual rise of a "money economy" led to the need of measuring and paying for the human work and wages, which led to the conception of time's uniformity, thus requesting its measurement in terms of appropriate, well-defined, "unchanging" units (Whitrow, 1988, pp. 108-110)[3]. It is interesting that already *The Treviso Arithmetic* (the earliest known printed arithmetic book; 1478) concerning commercial arithmetic for the general public, includes calendrical calculations for finding the date of Easter Sunday (Swetz, 1987, pp. 164-168). Though looking strange nowadays, these were important for merchants because civil holidays and religious feast days (greatly determined by the celebration of Christian Easter; see below) put constraints on activities related to trade and human labour (Swetz, 1987, pp. 248-253).

Theology: Since early Christianity – especially after it was no longer persecuted in the roman empire from the early 4th century AD onwards – the variety of habits and rituals adopted by different early Christian groups in relation to their worship led to an imperative request to establish a **uniform** religious canon, to be followed by the clergy all over the Christendom. Such a canon presupposed the ability of a sufficiently accurate determination of the time when each religious activity is done. In this connection developing methods for the indisputable determination of the Christian Easter day was of immense importance; a theological problem that acted as a catalyst in the development of the western civilization's concept of time, its measurement, and effective methods for doing complicated numerical calculations. It is worth noting that "computation" comes from the Latin "*Computus*", which - since the *Computus Paschalis* of Cassiodorus' disciples in the mid 6th century AD - signified the calculation method to determine the calendar date of the Christian Easter

[2] In Latin: *Kalendae*, *Nonae*, *Ides*, the modern word *calendar* coming from the first.

[3] Except the astronomers in the Hellenistic period, dividing the day into 24 **equal** hours stems from this, contrary to the unequal hours in earlier periods (12 *temporal* hours for the daylight period, varying with the season and geographical latitude; 12 *equinoctial* hours for the night period; the vague *canonical* hours of the Christian monasticism for religious and other duties (Richards, 1998, p. 44, Whitrow, 1988, pp. 28, 108).

(Borst, 1993, pp. 28-29; Swetz, 1987, p. 33). Ever since in the Middle Ages, finding Easter Sunday's date on any given year, and the AD time reckoning (introduced by Cassiodorus' contemporary, Dionysius Exiguus) were interlinked (Borst, 1993, ch. 4; Duncan, 1998, ch. 5). Along the same lines *St Benedict's Rule* by Benedict of Nursia (early 6^{th} century AD) contains precepts for his monks on their daily occupations, duties and worship, which gradually were spread out beyond the clergy to the entire society. By dividing the day into canonical hours, "Benedictine monasticism in the early Middle Ages formed the basis of the modern European measurement and discipline of time"[4] (Borst, 1993, p. 3; see also Richards, 1998, ch. 30; Duncan, 1998, ch. 5).

Geography and ***Navigation***: Although finding the geographic latitude of a place on earth is simple (being the inclination of the celestial pole(s) to the place's horizon), the accurate determination of geographic longitude requires a sufficiently accurate determination of time in order to be able to compare the geographic longitude of two different places by observing the position of the sun (or other celestial bodies) at a given moment[5]. This was very important for explorers, especially during the great expeditions in the Renaissance. It greatly motivated and stimulated technical methods and their theoretical background for developing high accuracy time-keeping devices; especially, marine chronometers for determining the position at sea (because of the unavoidable movement of any object on a ship, accurate marine chronometers required much more elaborate techniques; Whitrow, 1988, ch. 9; Newton, 2004, chs. 4, 5).

Similarly, several disciplines are involved in relation to question (2) (**how** accurate time keeping and time units have been determined):

Astronomy: Through the systematic study of the periodic motion of celestial bodies and their periods' determination as accurately as possible (cf. §1.2).

Physics: Through the study of specific periodic phenomena and the determination of the physical laws governing them. These range from mechanical systems of great mathematical and historical interest (like Galileo's *simple pendulum*, or Huygens *cycloidal pendulum*; §4.5), to modern crystal and atomic clocks based on understanding microscopic periodic phenomena (oscillations of atoms and nuclei).

Technology: Through the construction of devices operating accurately as artificial periodic phenomena, mutually compatible in the sense of §1.1; from devices based on macroscopic phenomena like water clocks, sundials and mechanical clocks (that use springs or pendulums and some type of "escapement mechanism"; Appendix B), to modern high-precision clocks based on microscopic vibrations of crystals and nuclei like those of SiO_4, ${}^{133}_{55}Cs$, ${}^{1}_{1}H$, ${}^{87}_{37}Rb$ (Fraser, 1987, ch. 2 pp. 45-75; Whitrow, 1972b, ch. 4).

This outline of the social and scientific domains in which accurate time measurement has been important, at least indirectly suggests that educationally it can also be beneficial.

2.2 Examples: A short list

From an educational perspective, the above outline of the social and scientific domains in which accurate time measurement has been important, suggests that the

[4] In this context, an hour indicated not a fixed period of time in today's sense, but less precisely specified parts of the day devoted to religious and other duties (Whitrow, 1988, p. 108). It is worth noting that modern *siesta* comes from Benedict's *sexta hora* that included a midday break for rest (Borst, 1993, pp. 26-27).

[5] A 4' time difference corresponds to 1^o difference in longitude; about 111km along the equator. Finding the geographic longitude in this way is a mathematically nontrivial astronomical problem (Smart, 1971, ch. XIII).

study of its development in history and in different cultures, constitutes a multifaceted, strongly interdisciplinary area touching upon a variety of subjects. Or, aspects of it provide insightful examples that in elaborated form could illuminate and reveal the crucial role of (nowadays considered classical, elementary, or even trivial) mathematics in addressing and tackling problems in several different disciplines and shaping man's ever-changing view of the world. An indicative list of interrelated examples and their not always obvious relation to (often deep) mathematical issues is:

(a) Measuring the **compatibility** of the three "basic clocks" (§1.1) requires the use of rational numbers. Since this involves increasingly more accurate measurements, hence successive approximations, this problem relates to fractions and their decimal[6] expansions.

(b) **Temporal cycles**, i.e. regularities among the three "basic clocks", require looking for their common multiples. This may involve deeper mathematics, e.g. congruences in Number Theory (§4.1).

(c) The accurate determination of the **periods** of the three "basic clocks" and their ratios, especially for periods very long compared to the duration of an individual's life, is related to the search for a **calendar** both physically correct and computationally simple enough to be understood and used by the laymen, hence convenient for civil purposes. This may involve much of the theory of continued fractions (§4.2).

(d) Finding the **week day on a given date** (especially **Easter Sunday**) involves clever **tabulation and treatment** of **data**. This is greatly facilitated by data parameterization using algebraic representations, symbolism and operations, ranging from elementary school algebra manipulations, to more sophisticated algebraic modelizations appropriate for algorithms to be used by modern computers (§§4.3, 4.4).

(e) Specifying and constructing **accurate** (mechanical) **clocks** greatly stimulated the development of important parts of mathematics. Seen in a modern context, it involves a lot of mathematics: from Calculus and differential equations (e.g. Galileo's simple pendulum), to the geometry of plane curves (e.g. Huygens' cycloidal pendulum); §4.5.

In all these cases (except (e)), the *leitmotiv* is the existence and acquaintance with the **positional number system**, automatically taken for granted nowadays, though it is not so either historically or didactically!

2.3 Temporal cycles: A mixture of astronomical facts & social conventions

Before outlining an HPM framework and considering in its context these examples, it helps to present the *temporal cycles* involved, thus giving hints into why and how mathematical issues are also involved. Determining such adequate cycles consists of searching for integer common multiples of the basic clocks' periods, and difficulties result because there are no simple rational relations among the *tropical year*[7], the *lunar synodic month* (or *lunation*)[8] and the *civil month* (from 28 to 31 days), when measured in *days*!

2.3.1 The *Metonic* cycle

Geminus' (*Introduction to the Phenomena* - Elementa Astronomiae) and Ptolemy

[6] Or using other bases; e.g. sexagesimal expansions were used in medieval astronomical calculations (§4.2).

[7] The time between two successive passages of the sun from the *vernal equinox*, which is the intersection point of the celestial equator and the *ecliptic* (sun's (apparent) annual orbit around the earth) when the sun passes from the southern to the northern celestial hemisphere (Smart, 1971, §86).

[8] The time moon takes to return to the same position relative to the earth-sun line; i.e. lunar phases to be repeated (Smart, 1971, §83).

(*Almageste* - Syntaxis Mathematica) mention that earlier astronomers in Athens had observed that to a very good approximation lunar phases are repeated on the same date every 19 years; i.e. 19 tropical years t_{Y} are nearly equal to 235 lunations t_{M}, or approximately 6940 days (though doubtful, this is attributed to Meton, 5$^{\text{th}}$ century BC; Heath, 1991, pp. xvii, 140-142). With t_{Y} the value of the Julian year (t_{Y}=365.25=365+1/4) established as the official duration of the civil year later by Julius Caesar (§4.2) and t_{M} in two decimals (t_{M}=29.53=29+1/2+1/33; §1.2), we get 19t_{Y}=6939.75 and 235t_{M}=6939.55. With [x] the integer part of x, this means

19-year ***Metonic* (*lunar*) *cycle***: $[19t_{\mathrm{Y}}] = [235t_{\mathrm{M}}] = 6940^{\mathrm{d}}$

The tiny discrepancy of about $0^{\mathrm{d}}.2$ per cycle was taken into account later by considering longer periods. Callipus of Cyzicus (4$^{\text{th}}$ century BC) noticed that a better approximation results if one day is omitted every four lunar cycles (4×19=76 years) because $\frac{6940}{19} = 365 + \frac{5}{19} = 365 + \frac{1}{4} + \frac{1}{4\times 19}$ (Heath, 1991, *op.cit*; Hannah, 2005, pp. 55-58; Richards, 1998, pp. 33, 96, 198; Whitrow, 1988, pp. 45, 189).

It is insightful to consider this from a different perspective using *continued fractions*, a historically much later concept closely related to Euclid's algorithm for the greatest common divisor of two integers (Khinchin 1964; Vinogradov 1954, §I.4): By (1.2), (1.3)

$$\frac{t_{\mathrm{Y}}}{t_{\mathrm{M}}} = 12.36827 = 12 + 0.36827$$

to five decimals. Developing the decimal part as a continued fraction

$$0.36827 = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{17 + \cfrac{1}{287/218}}}}}}}$$

gives its *convergents* as decimal and fractional approximations (Table 2.1)

Convergents of 0.36827	1/2	1/3	3/8	4/11	7/19	123/334
$t_{\mathrm{Y}}/t_{\mathrm{M}}$	25/2	37/3	**99/8**	136/11	**235/19**	4131/334
Decimal approximation	12.5	12.333	12.375	12.364	12.3684	12.3682
			Octaeteris		**Metonic** cycle	

Table 2.1

including not only the Metonic cycle, but also the earlier less accurate *Octaeteris* mentioned by Geminus and introduced by Cleostratus of Tenedos c.6$^{\text{th}}$ century BC (Heath, 1991, pp. xvi, 137-138; Hannah, 2005, pp. 35-39; Richards, 1998, pp. 94-95; Whitrow, 1988, p. 45); an 8-year cycle used by the Greeks to reckon time in terms of Olympiads (50 lunar months for one Olympiad and 49 for the next).

2.3.2 The *Solar* cycle

Because 365 = 52×7+1 ≡ 1 mod7, in a 365-day year divided into 7-day weeks each date on a given year moves forward by 1 weekday from one year to the next. Therefore, if there were no leap years, calendars would repeat every 7 years. But because there is one leap year every 4 years and 28 is the least common multiple of 7, 4, this happens every 28 years; i.e. every 28 years each date of the year falls on the same weekday. This is the *Solar cycle* (valid for the Julian calendar; §4.2) with the convention, that cycles **start** on a **leap year** with **January 1** being a **Monday** (Richards, 1998, p. 303).

2.3.3 The *Indiction* cycle

As mentioned in §2.1, an *Indiction* (in Latin: *Indictio*, meaning "declaration", "statement", "tax") was a fiscal period for reassessing agricultural or land taxes', originally introduced in Roman Egypt as a 5-year cycle. Constantine the Great extended it into a 15-year period for imposing taxes, starting on 313AD. Since 313AD was the first year of an indiction, any AD year Y is the year of an indiction cycle given by

$$\text{Indiction Year of } Y \equiv (Y+2) \bmod 15 + 1 \qquad (2.1)$$

3 Measuring Time as a subject of/in Mathematics Education: An *HPM* framework

In the previous sections enough evidence was given to support that time measurement is a multifaceted interdisciplinary subject, whose detailed treatment certainly touches upon mathematical issues presupposing a great deal of what is considered to be elementary or trivial mathematical knowledge nowadays. Identifying those aspects of this subject that could be beneficial in the context of mathematics education is facilitated by considering it in the context of an appropriate educational framework. In this section this is done with reference to an HPM framework presented in more detail in previous work and which refers to six interrelated aspects of integrating historical issues in mathematics education (Tzanakis, 2016, §3; Clark *et al*, 2016, §2.3; Clark *et al*, 2018, §1.3).

More specifically, this framework is structured along the following questions: In the context of mathematics education, **which history** is suitable, pertinent and relevant? Having **which role** and **objective**? Serving in **which way**? By following **which approach** and implementing **which methodological scheme(s)**?

Although these questions point to the key issues to be addressed while integrating history in mathematics education and provide the spectrum of possible relevant aspects to be considered, no unique answer is to be expected in each particular case, because approaches may vary in size and scope according to the specific didactical aim, the subject matter, the level and orientation of the learners, the available didactical time, and external constraints (curriculum regulations, number of learners in a classroom etc). With this in mind, these questions are considered in relation to time measurement and possible answers are outlined to be further detailed by considering specific examples in section 4.

3.1 Which *history*?

The question of ***which history** is suitable, pertinent, and relevant for didactical purposes* has been a permanently debated issue among historians and educators with no easy answer (Fried, 2001; Clark *et al*, 2018 §1.3.1; Clark *et al*, 2019 §4.2). Grattan-Guinness' proposed distinction of what he calls *history* and *heritage* was an important step towards clarifying conflicts and tensions among mathematicians', educators' and historians' view of mathematical knowledge. In brief, a *history* perspective focuses on what happened in the past and why did it happen (or did not happen), whereas a *heritage* perspective focuses on what impact (new) knowledge had on later work and the ways it was embodied in later contexts. These are equally important perspectives for understanding the development of mathematics comprehensively. However, they are complementary to each other in the sense that although both are legitimate, they are incompatible because muddling them is not permissible since this may lead to a distorted view of the past (Grattan-Guinness, 2004a, b; Tzanakis, 2016, §3.1, Tzanakis & Thomaidis, 2012, §12.2).

In this context, though a *history* perspective is certainly possible for problems related to time measurement (e.g. by elaborating on the mathematics underlying many of the points outlined in section 2 placed in the appropriate historical context), adopting a *heritage* perspective seems to be more suitable for didactical purposes; e.g. when exploring the development of the calendar or the clock, to stress the fact that modern life is almost unthinkable without them and raise the question whether they "were always there?", aiming to help learners get aware of "why and how did we get to the present situation".

3.2 With which *role* and *objective*?

(a) The question of ***which role*** *the history of mathematics can play in mathematics education* has been discussed and analyzed considerably on the basis of a priori theoretical and epistemological arguments and empirical research. Nowadays there is consensus that generally, history can play one or more of three mutually complementary roles or functions (Barbin, 1997; Furinghetti *et al*, 2006, pp. 1286-1287; Jahnke *et al*, 2000, §9.1; Jankvist, 2013, §7; Clark *et al*, 2019 §4.3):

Replacement: To replace mathematics as usually understood (a corpus of results consisting of finished and polished intellectual products), by something richer (not only such intellectual products, but also a vivid intellectual activity including the mental processes leading to these products).

Reorientation: To look at what is familiar and taken for granted, from a different perspective as something that has not always been existing in its currently established form; hence to make it appear less familiar. Thus modifying the conventional perception of mathematical knowledge as something "time-independent", into the deeper awareness that mathematics is an evolving human intellectual activity and that mathematical knowledge is potentially subject to changes; i.e. *historicity* is one of its ontological characteristics.

Cultural: To help appreciating mathematical knowledge as an integral part of human intellectual history in the development of society; hence, perceiving mathematics from perspectives that lie beyond its currently established boundaries as a discipline.

Though in principle all three roles may be relevant in the context of problems related to time measurement (e.g. by appreciating the significance of using a positional number system, which is considered as something instinctively familiar to us today), the cultural role can be dominant: Time measurement problems in historical perspective can help to appreciate that facts and customs taken for granted nowadays, emerged via mathematics (often) simple by modern standards, under the strong influence of factors, problems and questions of social, political or religious origin and focus.

(b) In connection with the ***objective*** of integrating the history of mathematics in mathematics education, there are five main areas in which this could be beneficial:

(i) The learning of mathematics;

(ii) The development of views on the nature of mathematics and mathematical activity;

(iii) Teachers' didactical background and pedagogical repertoire;
(iv) The affective predisposition towards mathematics;
(v) The appreciation of mathematics as a cultural-human endeavour.

Each of them can be analyzed into finer objectives, providing in this way a more detailed description of history's role(s) in the educational process (Tzanakis *et al*, 2000, §7.2; Tzanakis & Thomaidis, 2012, §12.3; Clark *et al*, 2018 §4.3).

Implicit to the discussion in section 2 (to become clearer in section 4), is that time measurement problems in historical perspective could be beneficial

- for (iii); by interrelating mathematics with other disciplines, providing interesting, non-trivial recreational problems, and enriching the teaching of mathematics with historically important questions from other domains;
- and for (v); e.g. by considering the historical background for the emergence of the calendar and the mathematics needed and/or developed for this purpose; the non-mathematical (but socially important theological) problem of finding Easter Sunday; the problem of determining geographic longitude; the mathematics underlying the construction and operation of accurate clocks and their importance etc.

3.3 In which *way*?

In relation to the ***way*** history could serve in mathematics education, Jankvist (2009) made an important distinction between history serving (i) as a *tool* for assisting the actual learning and teaching of mathematics; and (ii) as a *goal* in itself for the teaching and learning of the historical development of mathematics (a similar distinction was made by Furinghetti, 2004; 2019, §5). These are complementary ways to integrate history in mathematics education, in the sense that they may co-exist, though not necessarily being of equal weight, depending on the other factors analyzed in this section. In case (i) history functions as a motivational, cognitive or affective tool to assist and support learning mathematics. In case (ii) history poses questions and suggests answers about the development of mathematics, identifies and explores the (intrinsic or extrinsic to mathematics) driving forces of this development and its cultural and societal aspects.

In a historical perspective of time measurement problems, history serving as a *goal* is expected to be dominant while considering particular cases related to the two basic questions of §2.1; e.g. the questions why it was important and how it was possible to construct accurate clocks as a source of stimulation for developing the mathematics required for their answer (§4.5). Or, by elaborating on the religious and political importance of an accurate calendar and the difficulties encountered without the (nowadays) "simple" mathematics; the positional number system and the algorithms of arithmetic operations, the algebraic symbolism and its elementary use, basic concepts and methods in number theory (like congruences and their properties), etc.

3.4 Following which *approach* and implementing which *methodological scheme*?

(a) Following Tzanakis *et al* (2000, §7.3) for the classification of the approaches to integrate history in mathematics education, there are three broad possibilities that may be combined (thus complementing each other), each one putting emphasis on a different issue: (i) *providing direct historical information*, with emphasis on learning history; (ii) *implementing* *a teaching approach* (explicitly or implicitly) *inspired by history*, with emphasis on learning mathematics; (iii) *focusing on mathematics as a discipline and the cultural and social context* in which it has been evolving, with emphasis on developing awareness of its evolutionary character, epistemological characteristics, relation to other disciplines and the influence exerted by factors both intrinsic and extrinsic to it.

Though for time measurement problems all three approaches are possible (for (ii) see e.g. Anderson n.d.), it is more direct to follow a combination of (i) and (iii). For example, to discuss the search for an accurate calendar and elaborate on different proposals in different historical periods and cultures and their relative advantages or disadvantages either mathematical or non-mathematical; e.g. the Julian calendar and its amended successors like the Gregorian calendar and its long forgotten but more accurate rival proposed by Omar Khayyam several centuries earlier (§4.2).

(b) The methodological schemes to be employed for integrating history in mathematics education, are classified by Jankvist (2009, §6; cf. Clark *et al*, 2018, §1.3.3) into three broad categories: (i) *Illumination approaches* in which teaching and learning is supplemented by historical information of varying size and emphasis; (ii) *Module approaches* in the form of instructional units devoted to history, often based on specific cases; (iii) *History-based approaches* in which history shapes the sequence and the way of presentation, often without history appearing explicitly, but rather being integrated into teaching.

As mentioned at the beginning of this section, implementing any particular approach and methodological scheme in a specific case depends on several additional factors. Having this in mind and the interdisciplinary and multifaceted character of time measurement problems, (i) and (ii) seems better suited to these problems. E.g. indicative examples of a *module approach* could be: (1) The calendar in the western world: Its history and mathematical background (§4.4; cf. Anderson, n.d.); (2) The clock, its history and the underlying physico-mathematical basis (cf. §4.5). Similarly, indicative examples of an *illumination approach* could be: (1) To find the week day of a given date, focusing on presenting various methods and their history, or/and emphasizing their modelisation using the algebra of congruences in number theory (§4.3); (2) The astronomers' Julian date, its historical origin and number-theoretic basis (§4.1).

In section 4, the above framework is illustrated by means of specific examples.

4 Examples

Below five examples illustrate in more detail what has been presented in sections 1 to 3.

4.1 The astronomers' Julian date: Its origin, Gauss & the Chinese remainder theorem in Number Theory

4.1.1 The historical issue

The *Julian date* (JD) is a time reckoning, measuring in days the time elapsed since 1 January 4713BC. It was introduced in 1583 by the French classical scholar Joseph Justus Scaliger and is still indispensable in Astronomy (Smart, 1971, §90; Richards, 2013, §15.1.10; IAU, 2017, §2.3). Scaliger considered this date as the beginning of a very long temporal cycle of 7980 years! These mysterious at first glance numbers were not chosen arbitrarily, however. On the contrary, their justification provides an example rich in interrelations among number theory, astronomy, the history of

Chinese mathematics, and historical-theological considerations. Here we touch upon some of them briefly.

One year after the Gregorian reform of the calendar, Scaliger in his *Opus novum de emendatione temporum* (*New treatise on amending time* [chronology]), he made elaborate calendrical calculations (which he called *Computi Annales*) and by starting to count sufficiently backwards in time, he introduced a new long temporal cycle aiming to disentangle chronology from the absoluteness of religious creeds and unreliable records, and to avoid the difficulties of reconciling the 3 "clocks"[9] (Borst, 1993, pp. 104-106). The argumentation had theological motivation and was based on placing the year 1AD within the three temporal cycles of §2.3:

(i) Dionysius Exiguus had introduced the AD era in 525AD (§2.1). Knowing that there was a new moon on 23/3/323AD, he readily concluded by simple counting that there was a new moon on 1/1/325AD as well. He considered this as an important theological coincidence, because this was the year of the *Council of Nicaea* which created the (first part of the) Nicene Creed and the Christian Easter celebration canon. Because of this, he considered 323AD as the 1st year of a Metonic cycle. But $323 \equiv 0$ mod19, which implies that "year zero", that is 1BC[10] was also the start of a Metonic cycle (Richards, 1998, pp. 350-351), hence

1AD is a 2nd year of a Metonic cycle

(ii) Moreover, by eq(2.1),

1AD is a 4th year of an Indiction cycle

(iii) It is known that in the Julian calendar (§4.2), the year of the *Council of Nicaea* started on Friday; i.e. 1/1/325AD was a Friday. Therefore, 1/1/328AD was a Monday of a leap year, hence by the definition of the 28-year solar cycle (§2.3.2), this was a 1st year of a solar cycle. Since $328 \equiv 20$ mod28

1AD is a 10th year of a Solar cycle

4.1.2 The mathematical problem

Instead of going into Scaliger's early elaborate approach, we outline Gauss' treatment in his *Disquisitiones Arithmeticae* (Gauss, 1801, Part II §36; Ore, 1988, pp. 245, 247).

The mathematical problem consists of finding the year x (1AD) which is the 2nd year of a Metonic cycle, the 4th year of an Indiction cycle, and the 10th year of a Solar cycle; i.e.

$$x \equiv 10 \bmod 28, \quad x \equiv 2 \bmod 19, \quad x \equiv 4 \bmod 15 \tag{4.1}$$

Since (28,19,15) are pair-wise relatively prime, by the *Chinese Remainder Theorem* (Dence & Dence, 1999, §4.5; Ireland & Rosen, 1982, §3.4; Vinogradov, 1954, §IV.3) a solution x exists, **unique** modulo 28×19×15=7980

$$x \equiv (10x_1 + 2x_2 + 4x_3) \bmod 7980 \tag{4.2}$$

where (x_1, x_2, x_3) is the solution of the auxiliary system of congruences

$$x_1 = 19\times15y_1 \equiv 1 \bmod 28,\ x_2 = 28\times15y_2 \equiv 1 \bmod 19,\ x_3 = 19\times28y_3 \equiv 1 \bmod 15 \tag{4.3}$$

This can be solved easily to get $(y_1, y_2, y_3) = (45, 10, 28)$, and therefore

---

[9] Reiner in Paderborn Germany (12th century), making one of the first uses of Arabic numerals and the decimal system, had argued already that all time reckoning methods introduced significant errors over long time periods (Borst, 1993, pp. 73-74).

[10] There is no zero AD. In astronomy years BC are given by non-positive integers, with 0 for 1BC; Richards, 2013, §15.1.9.

$$x \equiv 196234 \bmod 7980 = 4714 \quad (4.4)$$

as the **unique** solution in a long temporal cycle of 7980 years, starting on 4713BC; the temporal cycle on which the JD is based.

This is a rich example that can be extended in several directions in a *heritage*-like perspective (§3.1), ranging from insights into the history of Chinese mathematics[11], the appearance of similar problems in Fibonacci's *Liber Abbaci* (Sigler, 2002, pp. 402-403) and the importance of this book for the emergence of arithmetical concepts and methods, or the significance of Gauss' *Disquisitiones Arithmeticae*[12], to subjects less central to mathematics like the origin and significance of the temporal cycles of §2.3, etc. Here, history has a cultural role that helps both to appreciate the cultural aspects of (otherwise abstract) mathematical problems and to provide teachers' with resourceful material (§3.2). Clearly, in this example, the historical development (both extrinsic and intrinsic to mathematics) is the main goal to a large extent (§3.3) by focusing on the cultural and societal aspects of the problem and implementing an illumination approach, e.g. in the context of a course in number theory and its applications (§3.4).

4.2 The optimal *leap year rule* & continued fractions: Julius Caesar, Omar Khayyam & Pope Gregory XIII

4.2.1 The problem in historical perspective

This example concerns the historically long struggle to reconcile two of the basic clocks (the year and the day), in a way applicable for civil purposes and easy enough to be understood by people. This has been a difficult problem for several reasons:

(i) There is **no unique definition** of the three basic "clocks" because of the relative motions of the sun, moon and earth, and because these motions are quite complicated as a result of the complicated interaction dynamics among these bodies and the other planets. This gives rise to several periodic phenomena with slightly different periods. However, these differences cannot be ignored over long time intervals and/or if high precision is required, leading to many refinements of the basic clocks. For the *year*: sidereal, (mean) tropical, anomalistic, lunar; for the *month*: sidereal, (mean) synodic, anomalistic, nodical; for the *day*: sidereal, apparent solar, mean solar, day (SI). Conventionally, the *tropical year* t_Y (for the seasons), the *synodic month* t_M (for the lunar phases) and the *day* (SI) t_D (§§1.2, 2.3) are used (for details see e.g. Smart, 1971, ch.VI and §§24, 28, 81-83, 86).

(ii) For social reasons, **both** the day & the (mean) tropical year are important.

(iii) The ratio *tropical year/day* is **not** an **integer**, eq(1.2). This required introducing additional days, done in diverse ways in different historical periods and/or cultures (Richards, 1998, part II). E.g. the calendar in ancient Egypt (util the 1st century BC) consisted of 12 30-day months plus 5 additional (or "intercalation") days, yielding a year of 365 days; i.e. $[t_Y]$, the lowest-order approximation to t_Y (Richards, 2013, §15.2.1).

(lowest approximation) ancient Egyptian calendar: $t_Y = 365^d = 12\times 30^d + 5^d$ (intercalation)

During the Roman republic a more complicated and considerably less symmetrical variant was used (Richards, 1998, ch.16). However, since t_Y is longer by about ¼ of a day (eq(1.2)), the seasons' periodicity lagged behind the civil year by almost 1 month within

[11] For the history of the Chinese remainder theorem see Katz, 1998, pp.197-199; Dauben, 2007, p. 302.
[12] E.g. close to the context of Goldstein *et al*, 2007, especially chs. I.1, I.2.

one century! On the advice of the Alexandrian astronomer Sosigenes, Julius Caesar introduced in 46BC a better approximation: 4-year cycles consisting of 3 common years and one leap year (the Julian calendar)

(first approximation) Julian year (JY): $t_Y = 365^d.25$,

Julian calendar (4-year cycles): $4\times365.25^d = 3\times365^d$ (common year) $+1\times366^d$ (leap year)

(iv) Table 4.1 shows that this was again an approximation, amounting to a discrepancy of about 10′/year between the JY and the tropical year according to the Alfonsine tables[13] (late middle Ages). This amounts to the loss of 1 day in about every 134 years.

t_Y year current value	t_{JY} Julian year	t_M month (mean synodic) current value	t'_Y Alfonsine tables (1252/1492)	t_D day current value
$365^d.2422$	$\mathbf{365^d.25}$	$29^d.53059$	$\mathbf{365^d.242546}$	86,400 sec(SI)
$365^d\ 5^h\ 48'46''$	$365^d\ 6^h$	$29^d\ 12^h\ 44'2''$	$365^d\ 5^h\ 49'16''$	
$\mathbf{t_{JY} - t'_Y = 10'\ 4'' = 0.0074537}$ days/year, or **1 day** lost every **134.16 years**				

Table 4.1

This tiny **measurement errors** due to the limited accuracy of observations produced **observable** effects only over **long** time intervals (centuries). Nevertheless this was important for several reasons: (1) *religious*: to have regular and strict celebration of festivities (especially the Christian Easter); (2) *historical*: to reckon correctly facts in different epochs at different places; (3) *political*: to coherently realize seasonal activities related to society, economy, agriculture etc.

Many attempts for corrections were made in the middle Ages, greatly enhanced by the gradual establishment of the Arabic numerals and the positional number system in the Renaissance. Thus the French cardinal P. d'Ailly (1412) proposed to omit 1 day every 134 years; more importantly, the Italian astronomer and mathematician P. Pitatus (1568) proposed to omit 3 days every 400 years, because

$$\frac{3}{400} = \frac{1}{133 + \frac{1}{3}} \cong \frac{1}{134}$$

(Dutka, 1988, p. 60). Probably influenced by this proposal and upon request of Pope Gregory XIII, a group of scholars with central figures the Italian physician and astronomer A. Lilius and the Jesuit mathematician and papal astronomer C. Clavius proposed the currently used rule; namely, Pitatus' rule omitting from the leap years the centurial years not divisible by 4, thus keeping only 97 leap years in 400 years. This was the Gregorian reform of the calendar officially declared and implemented in 1582 (Richards, 1998, ch. 19; 2013, pp. 598-599; Dutka, 1988; Duncan, 1998, ch. 13).

Later, in his *Introductio in analysin infinitorum* Euler developed t_Y as a continued fraction. This approach, understood in a *heritage*-like perspective (§3.1) with history playing a strong *cultural* role (§3.2), provides interesting insights into the underlying mathematics and its cultural significance. This enriches the teachers' didactical background by interrelating mathematics with other non-mathematically oriented domains

[13] In honor of Alfonso X of Castille (1252-1284), who supported the conduction of accurate astronomical tables during his regnal period (Richards, 1998, pp. 38-39).

and deepens the awareness of mathematics as an intellectual endeavour having strong permanent bonds with culture and society. We proceed to reveal this point more clearly.

4.2.2 Continued fractions

In view of the historical outline above, two questions naturally arise (Rickey, 1985):

- Since there is always a difference between the tropical and the civil year, **is there any better leap-year-rule**?
- Can the Julian and Gregorian calendar rules be mathematically described **uniformly?**

Here, continued fractions enter the scene; a historically and mathematically interesting and rich concept introduced by Bombelli, Huygens, Wallis, Euler and others (cf. §2.3.1): Any (rational) number a is represented by a (finite) continued fraction by subtracting the integral part from it, taking the reciprocal, recording the integral part and subtracting it, and repeating the procedure:

$$a = [a] + \frac{1}{\frac{1}{a-[a]}} = [a] + \frac{1}{[a'] + \frac{1}{\frac{1}{a'-[a']}}} = \cdots \text{etc}, \quad a' = \frac{1}{a-[a]} \tag{4.5}$$

This leads to the continued fraction representation of the rational number. If truncated at the n^{th} step, it gives its *convergents* $\frac{A_n}{B_n}$ obeying recursive relations already proved by Wallis (1655) and Euler (1737) (Hairer & Wanner, 1996, §I.6, theorem 6.1):

$$a = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{..+\cfrac{1}{..+\cfrac{1}{b_n}}}}}} = \frac{A_n}{B_n} \tag{4.6}$$

with

$$A_n = b_n A_{n-1} + A_{n-2}\ ,\ B_n = b_n B_{n-1} + B_{n-2} \tag{4.7a}$$

$$A_{-1} = 1,\ A_0 = b_0 = [a],\ A_1 = b_1 b_0 + 1\ ,\ B_{-1} = 0, B_0 = 1, B_1 = b_1 \tag{4.7b}$$

With t_Y -[t_Y] = 0.2422 to four decimals (Table 4.1) and b_0=0, b_1=4, we get the continued fraction expansion (cf. Rickey, 1985; Eisenbrand, 2012; Grabovsky, n.d.)

$$0.2422 = \frac{2422}{10000} = \frac{1211}{5000} = \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}}}}}} \tag{4.8a}$$

and its convergents giving possible calendar options shown in Table 4.2.

$$\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{31}{128}, \frac{132}{545}, \frac{163}{673}, \frac{295}{1218}, \frac{458}{1891}, \frac{1211}{5000} \tag{4.8b}$$

Table 4.2 includes both the JY (1st row) and another proposal by the 11th century Persian scholar and poet Omar Khayyam (2nd row), largely forgotten but more accurate than the Gregorian one (3rd row), which is shown for comparison even though it does not correspond to any convergent of t_Y -[t_Y] (Rickey, 1985):

- 2nd row (*Omar Khayyam*): **7** consecutive 4-year cycles with 1 leap year per cycle, followed by one 5-year cycle having 1 leap year

But there are other even more accurate options:

- 4th row: 32 consecutive 4-year Julian cycles with no leap year in the last one;
- 5th row: the Gregorian rule applied to centurial years divisible by 5 (i.e. 96+25=121 leap years in 5 centuries), with the 5000th year not being a leap year (Rickey, 1985).

convergent	Proposal	Value	temporal cycles of JY (1 leap year per cycle)
1st	Julian calendar (46 BC)	**1**/**4** = 0.25	**4**y = 1×4y leap years: **1**
3rd	Omar Khayyam (c. 1076 AD)	**8**/**33** = 0.2424…	**33**y = 7×4y + 1×5y leap years: **8** = 7+1
	Gregorian Calendar (1582 AD)	**97**/**400** = 0,2425	**400**y = 3×(25×4y-1d) + 1×(25×4y) leap years: **97** = 3×(25-1) + 1×25
4th	Other possibilities	**31**/**128**= 0.24219	**128**y = 31×4y + 1×4y-1d leap years: **31** = 32-1
9th		**1211**/**5000**= 0.2422	**5000**y = 10×[4×(25×4y-1d)+1×(25×4y)] + 1d leap years: **1211** = 10× [4×(25-1)+1×25] + 1

Table 4.2

Non-mathematical arguments of a social or practical nature for or against these alternatives can be a stimulating part of an interdisciplinary *teaching module* on this problem, with emphasis on historical information about the mathematics required, used, or underlying it and the cultural and social issues that called for its solution and stressed its significance (§3.4). In this context, while exploring the driving forces behind this problem, historical issues are dominant, thus serving mainly as a *goal* in themselves (§3.3).

4.3 The weekday on a given date: From old *dominical letters* to modern computer algorithms

4.3.1 Outline of the problem and its history

Finding the weekday on a given date has attracted considerable attention throughout history for social, political and religious reasons (§§2.1, 2.2(d); e.g. it is crucial for finding the date of Easter Sunday; §4.4). Its treatment involves three temporal cycles of different origin: the *year*, of astronomical origin, the civil *month*, and the *week*, both being determined by a mixture of astronomical and civil factors[14]. Though mathematically elementary (simple counting is sufficient in principle), it is a non-trivial problem if a sufficiently quick method is requested, and/or dates with a long time separation are considered. This implies that an as much as possible formalized procedure is desirable or necessary. Therefore, this problem has considerable recreational value (e.g. Kraitchik, 1953, ch.5; Ore, 1967, ch.8; Beveridge, n.d.), reinforced by its "asymmetric" aspects that allow for **no simple** solution because of various reasons of diverse character:

(1) Since 365≡1(mod7) and 366≡2(mod7), each date moves forward by 1 or 2

[14] For historical issues see Richards, 1998, chs. 15-17, 21; 2013 §§15.1.6, 15.3.2, 15.3.3; Duncan, 1999, ch. 2; Whitrow, 1988 pp. 32, 55, 68-70.

weekdays per common or leap year respectively;
(2) There is 1 leap year every 4 years (Julian calendar), plus the correction for centurial years (Gregorian calendar). This is due to the lack of a simple rational relation between the year and the day, and the obvious request that socially important temporal cycles should be expressed by relations involving integer numbers only;
(3) For historical or even incidental reasons, the number of days is distributed irregularly among civil months; 4 having 30 days, 7 having 31, 1 having 28 or 29.

Due to the interest it arose throughout history, it has been considered by different people (including many eminent mathematicians) in various ways, several of which are interesting from a *heritage-like* educational perspective (§3.1). No comprehensive presentation will be given. We simply note that the problem can be tackled and solved in different, progressively formalized ways roughly reflecting the historical development:
(i) In the early middle Ages, due to the lack of sufficiently developed mathematical background, the problem was treated by simple counting, possibly using the four *arithmetical operations* and **extensive** *data tabulation*. In this connection the 28-year solar cycle was central (§2.3.2), which is not directly applicable to the Gregorian calendar however, because some centurial years are common (§4.2.2).
(ii) With the development of (elementary modern) mathematics - the *arabic numerals*, the *positional number system*, and the emergence of *elementary algebraic symbolism* and *methods* - it became possible to use these tools to parameterize dates and weekdays, thus reducing the use of *tabulated data*.
(iii) Further development of mathematics led to the invention of more compact methods and their algebraic formulation that minimized the need for data tabulation, and by a progressively more *elaborate formalization* the development of numerical algorithms admitting computer implementation nowadays.

The key historical steps in this development could form the basis of a *teaching module*, focusing on the *religious* and *social* need to obtain a sufficiently simple and practical solution (§3.4). Identifying and elaborating on these steps means that history will serve mainly as a *goal* in itself (§3.3), helping to realize both the *cultural significance* and the *evolving character* even of the most elementary mathematical knowledge that today is often considered as “something that was always there” (e.g. the positional number system, or the existence of algebraic symbolism; §3.2(a)). On the other hand, since there are several different **equivalent** formulations and solutions of the problem (Appendix A), it is mathematically interesting and insightful to check and prove their equivalence. This ranges from simple computational exercises, to more sophisticated mathematical elaborations, thus helping to get acquainted with specific pieces of mathematics and enriching the teachers’ didactical background (§3.2(b)).

4.3.2 A semi-formalized method of solution

To illustrate some points of §4.3.1, a “hybrid” solution method is outlined; a semi-formalized one (close to Richards’ 1998, ch.24) minimizing the need of data tabulation.

(1) Each method uses some mapping of the weekdays to an arithmetical set. Here and in Appendix A we use the following numbering and notation:

Sun	**Mon**	**Tue**	**Wed**	**Thu**	**Fri**	**Sat**	*Weekday number W*
1	2	3	4	5	6	7	

Table 4.3

Year AD		Month	Day of month	date	*Weekday number*
Y	Y=100c+y	m	d	d/m/Y	W

Table 4.4

(2) The dates of the year 1/1, 2/1 …, 31/12 are numbered consecutively from 1 to 365, assigning to each one its remainder when divided by 7, its so-called *calendar number*; that is, the calendar numbers are given by the

$$\text{canonical mapping: } \{1/1, 2/1 \ldots 31/12\} \leftrightarrow \{1, 2, 3, \ldots, 365\} \rightarrow \mathbb{Z}_7 \tag{4.9}$$

with the important **convention** that 29/2 and 1/3 are both mapped to 60 ≡ 4(mod7) (a year's 60th day is 1st March (common year), or 29th February (leap year)).
To avoid confusion with the weekday numbers, identify calendar numbers with calendar letters:

$$\{1, 2, 3, 4, 5, 6, 0{\equiv}7\} \equiv \{A, B, C, D, E, F, G\}$$

(3) Though obvious, it is important to note that **each weekday** in a (common) year has the **same** calendar **number/letter** throughout the year (i.e. the same image in $\mathbb{Z}_7$). For a leap year, each weekday has two calendar letters; one for January and February and one for the other months, this one being identical with the calendar letter of that day in the next year.

(4) Therefore, define the *dominical letter/number* of the year Y as

$$N_Y = \textit{calendar letter/number} \text{ of Sundays of year Y} \tag{4.10}$$

Little inspection shows that N_Y determines the **weekday of 1/1/Y**. By (3), there are 14 alternatives for N_Y, shown in Table 4.5, which readily implies that

$$W_{1/1/Y} + N_Y \equiv 2 \bmod 7 \tag{4.11}$$

		Dominical Number/Letter				
Week day, 1/1	W	N (Common Year)		Leap Year (N \ N -1)		Forward in time
Sunday	1	1	A	1\7	A\G	
Saturday	7	2	B	2\1	B\A	
Friday	6	3	C	3\2	C\B	
Thursday	5	4	D	4\3	D\C	
Wednesday	4	5	E	5\4	E\D	
Tuesday	3	6	F	6\5	F\E	
Monday	2	7	G	7\6	G\F	
		For the **whole** year		Jan, Feb\ Mar-Dec		

Table 4.5

common year	leap year
$W_{1/1/Y+1} \equiv (W_{1/1/Y} + 1)$ mod7	$W_{1/1/Y+1} \equiv (W_{1/1/Y} + 2)$ mod7
$N_{Y+1} = N_Y - 1$	$N_{Y+1} = N_Y - 2$

Table 4.6

Therefore if N_Y is known, this relation gives the week day of 1/1/Y (Table 4.6). This in turn gives the weekday of any date in the year Y, as shown below. This is one of the reasons for which dominical letters played an important role in the past (especially before the development of algebraic symbolism and operations) and were extensively tabulated (Richards, 1998, ch.24; Dominical Letter, n.d.); e.g. Table 4.5 immediately gives the succession of years in the 28-year solar cycle, thus providing a constructive proof that every 28 years, **each** date of the year – and for all dates - falls on the same weekday, as shown in Table 4.7 (solar cycles start on a leap year with 1/1 being Monday; §2.3.2).

Year of the cycle	**N**	**Year of the cycle**	**N**
1	G/F	15	C
2	E	16	B
3	D	17	A/G
4	C	18	F
5	B/A	19	E
6	G	20	D
7	F	21	C/B
8	E	22	A
9	D/C	23	G
10	B	24	F
11	A	25	E/D
12	G	26	C
13	F/E	27	B
14	D	28	A

Table 4.7

(5) Define the *regular of month m*, R_m by

$$R_m = (\textit{calendar number} \text{ of } 1/m) \bmod 7 \tag{4.12}$$

By (4.9), R_m is **independent** of the year (common or leap) and given in Table 4.8

m	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
$\boldsymbol{R_m}$	1	4	4	0	2	5	0	3	6	1	4	6

Table 4.8

It is readily seen that $W_{1/m} \equiv (W_{1/1} - 1 + R_m)$ mod7; hence, for any day d of that month:

$$W_{d/m} \equiv (W_{1/1} - 1 + R_m + d - 1) \bmod 7 \tag{4.13}$$

Therefore, the problem reduces to determining $W_{1/1/Y}$. Given that passing from Y to Y+1 each date moves forward by 1 or 2 weekdays per common or leap year respectively, it is readily seen that for the Gregorian calendar and for a common year there have been Y-1 +

[Y/4] – [Y/100] + [Y/400] shifts forward from 1/1/1 to 1/1/Y[15]. For a leap year, 1 should be subtracted from the r.h.s. for $m \leq 2$ (i.e. for January, February) since its leap day has already been added in the above sum. 1AD can be replaced by any other reference year Y_0, adding a constant depending on $Y-Y_0$ and Y_0 being a common or leap year. Choosing Y_0 to be a common year for which $W_{1/1/Y_0}$ is known and combining the above number of shifts with (4.13) gives the weekday number for any date

$$W_{d/m/Y} \equiv (d - 1 + R_m + Y - 1 + [Y/4] - [Y/100] + [Y/400] + a) \bmod 7$$

with a specified by Y_0; e.g. 1/1/2018 was Monday, hence $W_{1/1/2018} = 2$ (Table 4.3). Therefore $W_{1/1/2018} = 2 \equiv (2507+a) \bmod 7 \equiv (1+a) \bmod 7 \Rightarrow a = 1$ and $W_{d/m/Y}$ is:

$$W_{d/m/Y} \equiv (d + R_m + Y - 1 + [Y/4] - [Y/100] + [Y/400] - \delta) \bmod 7 \qquad (4.14a)$$

$$\delta = \begin{cases} 1\ ; \text{ for Y leap year \& } m \leq 2 \\ 0\ ; \text{ otherwise} \end{cases} \qquad (4.14b)$$

Though what precedes gives only the basics of the method, it is clearly susceptible to considerable elaboration in various directions; discussion on the role of algebraic/symbolic modelization vs. using exclusively tabulation methods (the only thing medieval scholars could do); comparison with other methods (e.g. those in Appendix A); finding other equivalent formal expressions and developing algorithms appropriate for computer implementation; further discussion on calendars different from the Gregorian, and conversion procedures from one to another (e.g. Richards, 1998, part III) etc.

4.4 *Computus* & Easter Sunday: The struggle for reconciling astronomical and civil temporal cycles

Finding the date of Christian Easter Sunday has been central for Christendom since the early middle Ages, closely related to finding the weekday on any given date (§4.3). Though by modern standards the problem in principle requires only very basic mathematics (the positional number system, the four arithmetical operations, the decimal representation of fractions, and an elementary algebraic symbolism and modelization), it is difficult to be dealt without the mathematical sophistication of §4.3. In fact, it is even more difficult because **four** temporal cycles are involved instead of three: two of the "basic clocks" (the (tropical) year and the lunar (synodic) period), and two of social origin (the civil month and the week), all measured in days (the third "basic clock"). The complications inherent to this problem for astronomical and civil reasons, and the lack of sufficiently adequate mathematical background prohibited an accurate - hence definite – solution. This made it a subject of ceaseless interest since the early Christian era, greatly influencing social life, theological debates and religious customs, and calling for more and more accurate astronomical observations and ever increasing mathematical sophistication. The interference of nonmathematical (basically theological) issues as the motive force, with the treatment of complicated astronomical data by means of mathematical techniques developed for that purpose, led to the medieval *computus*; the particular art of calculating Easter Sunday's date developed by specialized scholars and clergymen, and whose basics had to be familiar to the clergy in medieval Europe (cf. §2.1; for historical details see

[15] The number of years that passed since 1AD, plus the additional days because of the leap years, minus the centurial years that are common (for the Julian calendar, the last term [Y/400] is missing).

Borst, 1993, chs. 4-10; Duncan, 1998, chs. 5-7, 11; Whitrow, 1988, Appendix 3).

Being a meeting point of ancient Greek astronomy, time reckoning in the Roman Empire, and the Jewish lunar calendar and festivities, it is a historically fascinating and strongly intercultural and interdisciplinary subject related to elementary mathematics. Therefore, from an educational point of view, it is ideal for perceiving pieces of elementary mathematics from a non-mathematical perspective (§3.2(a)) by revealing the cultural factors that influenced the development of mathematics and the social problems solved in this way (§3.2(b)). In a *heritage*-like perspective (§3.1) the emphasis is on learning about the historical origin and development of some nowadays very basic mathematics (§3.4(a)). By following a *history-based* approach, the key steps of this development will shape the presentation (§3.4(b)) in the form of the questions and problems that were the driving forces for exploring and solving the problem (§3.3).

Some issues related to this problem were discussed in the previous sections. Therefore, only certain basic points are presented with reference to the literature for further details.

4.4.1 The historical milieu

Since its original appearance as a historical religion, Christianity considered Easter's celebration as the most important part of its worship. Originally, this was dependent on the Jewish *Passover* because according to the Gospels, Jesus' Crucifixion took place one day after the Passover feast on the 14th day of Nisan, the first month of the religious Jewish year, starting near the vernal equinox (Richards, 1998, ch.17). Given that the Jewish calendar is lunar, but the Julian calendar (hence, the Gregorian as well) is solar, Christian Easter could not be celebrated on a fixed day of the year. Therefore, after Constantine the Great allowed Christians to follow their faith without oppression in 313AD, he convened the first *Ecumenical Council* in Nicaea, Asia Minor, in 325AD, which – among other things (§4.1.1) – set out the framework for Easter's celebration by requesting: (i) independence from the Jewish calendar, as it was done by the Alexandria's church (namely that it must be celebrated on a Sunday necessarily after the vernal equinox); and (ii) celebration on the same date everywhere. After a transient period in which the Alexandrine computational method was stabilized into its final form, this led to the following convention used as a normative rule throughout Christendom since long ago[16]:

- *Christian Easter* is celebrated on the *first Sunday after the first full moon that coincides with or follows the 21st of March* (the ecclesiastical vernal equinox).

It is here assumed that the full moon occurs on the 14th day after the new moon (counting new moon's occurrence as the first day). The lunar cycle in the above rule is called the *paschal moon* and its full moon is the *paschal full moon.*

4.4.2 The astronomical background and the resulting complications

(a) Easter's celebration rule clearly involves the combination of three temporal cycles: The (Julian) *civil year* via the vernal equinox; the *lunar synodic period* (or lunation; §2.3) via the occurrence of a full moon; and the *7-day week* via the request of Easter's celebration being on a Sunday. Therefore, its actual implementation meets several difficulties for the following reasons (cf. §1.2):

(1) The (Julian) civil year and the tropical year differ and the difference accumulates, even if one leap year is inserted every 4 years. Moreover, leap years introduce further

---

[16] This is an oversimplified picture of the complicated historical development (Richards, 1998, ch.24).

computational complications similar to those in §4.3;
(2) There exist no simple rational relations among the above three temporal cycles;
(3) Civil months (the 4th cycle involved) differ considerable from the lunar synodic period;
(4) Because of (2), the full moon does not occur exactly 14 days after the new moon;
(5) The vernal equinox does not occur exactly on March 21st (March equinox, n.d.).

Hence, the result of applying the above rule for Easter's celebration deviates from the occurrence of the astronomical phenomena it is supposed to describe. It corresponds rather to a notional picture close but not coincident with physical reality. This gets clearer below.

(b) The Alexandrians employed the 19-year Metonic cycle (§2.3.1) to determine Easter Sunday. Based on this, Victorius of Acquitaine (5th century AD) realized that the dates of Easter Sunday repeat every 532 years which was named after him (*Victorian cycle*). But this cycle is the least common multiple of 19, 28 and it is illuminating to note in Table 4.9 its interrelation with other cycles (§2.3).

cycle	**Solar**	**Metonic**	**Callipic**	**Victorian**
duration (years)	28×19 =	$(19 \times 4) \times 7$	= 76×7 =	532
origin	Astronomical & civil conventions	Astronomical		Astronomical & civil conventions

Table 4.9

By combining the Victorian cycle with the AD era introduced in 525AD (§§2.1, 4.1.1), tables with the Easter days were produced for two Victorian cycles (i.e. until 1064AD), originally by Isidore of Seville (early 6th century AD) and then by Bede in his influential *De temporum ratione* (725AD); Borst, 1993, ch. 5, Richards, 1998 ch. 28. But because of the reasons in (a) above, these calculations led to results incompatible with the physical phenomena over long time periods: The use of (i) the **Julian year** ($365^d.25$) instead of the slightly shorter tropical year $t_Y = 365^d.2422$, eq(1.2), and (ii) the **notional** lunation obtained as the mean value of the lunar synodic month in one Metonic cycle (§2.3.1) i.e. $6940^d/235 = 29^d.5319$, instead of $t_M = 29^d.5306$, eq(1.3), produced a cumulative effect: the vernal equinox was falling progressively **earlier** than 21 March, and astronomical new moons appeared progressively **earlier** than the calendar dates of notional new moons.

It is obvious how difficult it was to deal with this problem without the nowadays elementary mathematical knowledge of the positional number system, the arithmetic operations and fractions' decimal representation. It also gives hint why the *computus* (based on simple counting and extensive use of tables) required special training and was accessible to a limited circle of educated people (Fraser, 1987, p. 81).

At the same time, it is illuminating to consider the problem from a modern perspective and comment briefly on why a possible solution was never implemented. Table 4.10 gives the values of the physical and notional quantities involved (approximated to four decimals; cf. §§1.2, 2.3). Imposing the condition that the civil year should have 12 (civil) months, yields

Metonic cycle = (19×12) = 228 civil months = 235 notional lunations = 6940^d

t_Y current value	t_{JY} Julian year	t_M lunar synodic period
$365^d.2422$	$365^d.25$	$29^d < 29^d.5306 < 30^d$
$19t_Y = 6939^d.60$	$19t_{JY} = 6939^d.75$	$235t_M = 6939^d.69$
rounded to **6940$^{\mathbf{d}}$** (**Metonic** cycle)		
	physical fact	$29^d < t_M < 30^d$ [17]
		$12\, t_M < t_{JY} < 13\, t_M$
never used	civil (common) year 365^d = 12 months $= 7 \times 30^d + 5 \times 31^d$	$t_{JY} = 29^d.5 \times 12 + 11^d.25 = 354^d + 11^d.25$
original (roman)	$= 5 \times 30^d + 6 \times 31^d + 1 \times 29^d$	
final (current)	$= 4 \times 30^d + 7 \times 31^d + 1 \times 28^d$	

Table 4.10

If it is further required that notional lunations contain an integer number of days (29^d or 30^d), then the Metonic cycle admits a unique partition into notional lunations; namely

$$\left.\begin{array}{l} 29x + 30y = 6940 \\ x + y = 235 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = 110 \\ y = 125 \end{array}\right.$$

However, this solution is incompatible with the additional requirement of civil years having 12 months, in the sense that no such convenient division exists!

4.4.3 The adopted solution and a modern formulation

Because of the many "asymmetric" features of the problem (partly due to social constraints or historical incidents) the solution finally accepted is as follows (for details see Richards, 1998, ch. 29):

(a) The Metonic cycle is used, consisting of 235 notional lunations, or "months" m of either 30^d or 29^d.

(b) These lunations are distributed among the 19 solar years as follows: 12 years are *common* in the sense that they contain 6 lunations of 30^d and another 6 of 29^d. The remaining 7 are *embolismic* in the sense that they contain a 13th (embolismic) lunation, which has 30^d for the six of these year and 29^d for the last one. Finally the remaining 5^d were added by inserting 1^d every 4 years into whichever lunation contains 24 February. Or, in symbols

$$\begin{aligned} 235\text{m} &= \mathbf{12}\text{y} \times \mathbf{12}\text{m/y} + \mathbf{7}\text{y} \times \mathbf{13}\text{m/y} = \\ &= \mathbf{12}\text{y} \times \mathbf{6}\text{m/y} \times (30\text{d/m} + 29\text{d/m}) + \mathbf{6}\text{y} \times [\mathbf{7}\text{m/y} \times 30\text{d/m} + \mathbf{6}\text{m/y} \times 29\text{d/m}] + \\ &\quad + \mathbf{1}\text{y} \times [\mathbf{6}\text{m/y} \times 30\text{d/m} + \mathbf{7}\text{m/y} \times 29\text{d/m}] + \mathbf{5}\text{d} = 6940\text{d} \end{aligned}$$

(c) The paschal full moon is on the14th day of the notional lunation **on** or **after** 21/3 and this notional lunation should always have 29^d. This can be accounted for by arranging lunations so that the first one has 30^d for all years of the Metonic cycle. This means that the earliest is on 21/3, and the latest on 18/4 (full moon on 20/3 + 29d). Therefore, Easter Sunday falls from 22/3 to 25/4 (paschal full moon on Sunday 18/4).

Clearly, this is not a mathematically elegant solution. For centuries it was necessary to

[17] A purely lunar calendar of 12 lunar synodic periods equals 354.3672^d. It can be approximated by six 30-day and six 29-day months corresponds to a lunar year of 354 days, with some years having one additional day to account for the resulting discrepancies. The Islamic calendar is of this kind (Richards, 1998, ch. 18).

construct appropriate tables satisfying the above conditions and in some way depicting the notional lunations in the calendar of the 12 civil months. To this end further parameters were defined for each year; the *golden number* and the *epact* (see below). With the development of algebraic symbolism and operations it became possible to formalize the solution further, so that nowadays this formalization is complete admitting computer implementation. This procedure is nontrivial and can constitute a nice project in discrete mathematics, computer algorithms etc. No details are given, but we define the additional concepts needed and simply state the final result (Richards, 1998, ch. 29).

(d) We define:

(i) The *Golden Number* G_Y of the AD year Y (the place of Y in the Metonic cycle; §4.1.1):

$$G_Y \equiv 1+Y \bmod 19 \tag{4.14}$$

(ii) The *Epact*[18] of Y: The age of the notional moon on 1/1/Y (ranging from 0 for a new moon, to 28).

(iii) Having in mind (c) above, we define the mapping

$$\{21/3, 22/3 \ldots 25/4\} \rightarrow R = \{21, 22, \ldots, 56\} \tag{4.15}$$

and consider the Easter Sunday number, $\mathbf{S}_Y$ and the next day of the Paschal full moon, r_Y. By (c) above, $\mathbf{S}_Y, r_Y \in R$ and it can be shown that the following relations hold:

The *calendar number* C of r_Y (§4.3.1) is

$$C \equiv (r_Y + 3) \bmod 7 \tag{4.16}$$

Moreover,

$$r_Y \equiv (75 - E_Y) \bmod 30 \tag{4.17}$$

$$E_Y \equiv (11\, G_Y - 3) \bmod 30 \tag{4.18}$$

Writing (4.11) in the equivalent form

$$N_Y \equiv (9 - W_{1/1/Y}) \bmod 7 \tag{4.19}$$

it can be proved that the date of Easter Sunday for the year Y is given by

$$\mathbf{S}_Y \equiv r_Y + (7 + N_Y - C) \bmod 7 \tag{4.20}$$

Clearly the above relations can be combined to give several other equivalent forms of the final result (4.20). The important thing to note here is that by means of some careful algebraic modelization, any reference to tabulated data has been dispensed with. Following in detail the procedure leading to the above result provides a nice opportunity of getting insight into the way an appropriate mathematical formalization replaces more empirical and common sense methods to deal with a problem of non-mathematical origin.

4.5 Accurate clocks, the escapement mechanism & the underlying physico-mathematical theories: Galileo & Huygens

This is a vast subject in which interesting mathematics, (partly) advanced physics and complicated technology interact in a multifarious way. The discussion will be confined to comments on Galileo's and especially Huygens' contribution.

According to the previous examples there was a gradually increasing need for more and more accurate time-keeping, further enhanced by the nontrivial problem of determining geographic longitude (especially at sea) during and after the great geographical expeditions (Whitrow, 1988, ch. 9; Newton, 2004, chs. 5 & pp. 59-61). To this end, devices were invented based on continuous processes (e.g. water clocks), or oscillatory processes (e.g. mechanical clocks), with a gradual shift from the first to the second

[18] Introduced by the Alexandrian church in the 3rd century AD (Holford-Strevens, 2005, ch. 4 & Appendix B).

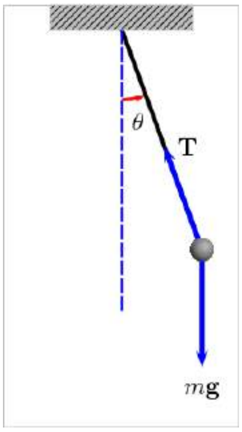
because of the invention of the *verge escapement* (Appendix B). Originally these were inaccurate because of swinging across wide angles (at least ~50^0). Without going into details, we comment on two historically important cases in which technology and mathematics are interconnected:

-The *simple pendulum clock*: Although Galileo's discovery of the simple pendulum isochronism (c.1602; Drake, 1995, pp. 68-70, 72-73) motivated him in his later years to think of it as the basis of a possible time-keeping mechanism described posthumously (1659) by his biographer Viviani (Fig. A.1; Drake, 1995, pp. 378-379, 399, 419-421), such a clock was constructed by Coster in 1657 only after Huygens' theoretical investigations (Fig. A.2; van Helden, 1995; Whitrow, 1988, pp. 122-124; Richards, 1998, p. 58).

-The *Cycloidal pendulum*: Also invented by Huygens in 1659 and described in his influential *Horologium Oscillatorium* (1673/2013, Part I), which is an ingenious construction based on the mathematical properties of the cycloid that he also proved (Fig. 4.5; Whitrow, 1988, p. 123; Horologium Oscillatorium, n.d.).

4.5.1 The simple pendulum clock

The first sufficiently accurate clocks (17th century) were based on the isochronism of the simple pendulum[19]. Though discovered empirically (Newton, 2004, pp. 87-88; cf. Drake, 1995, p.397), it can be deduced as a simple application of Newtonian mechanics (see e.g. Sommerfeld, 1964, §III.15): Since the pendulum bob of mass m moves under its own weight mg and the tension $\mathbf{T}$ along the massless rod or string of length l (g being the gravity acceleration), applying Newton's second law gives the equation of motion in terms of the oscillation angle, accents denoting differentiation with respect to time t (Figure 4.1):



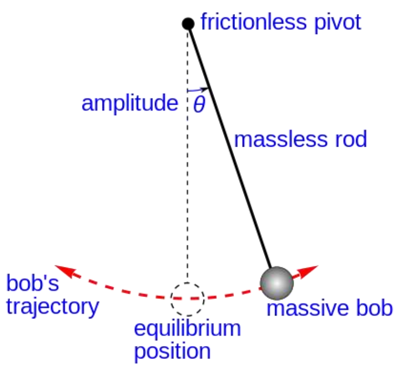


Figure 4.1: The physics of the simple pendulum

$$\theta''(t) = \frac{g}{l}\sin\theta \tag{4.21}$$

This is a non-linear 1st-order differential equation, whose solution can be obtained in terms of elliptic integrals (Sommerfeld, 1964, *ibid*). It is well-known that for very small oscillation amplitudes, i.e. to lowest order in the oscillation amplitude (maximum of $|\theta|$), (4.21) becomes linear and its solution is a simple periodic function independent of this amplitude with period T given by (4.21').

$$\theta''(t) \approx \frac{g}{l}\theta, \quad \Rightarrow T = 2\pi\sqrt{\frac{l}{g}} \tag{4.21'}$$

As an interesting application of elliptic functions and integrals, it is insightful to solve

[19] For earlier constructions see Newton, 2004, ch.3; Whitrow, 1988 pp. 120-122; 1972, ch.4; Clock, n.d.

(4.21) and get T in (4.21') as the lowest order approximation of the elliptic integral's modulus as a function of the pendulum's parameter $(l/g)^{1/2}$.

4.5.2 The cycloidal pendulum clock

(a) The next very important development both theoretically and technically, was Huygens cycloidal pendulum and the underlying mathematical properties and physical theory. Huygens sought and conceived (more) accurate clocks (Huygens, 1673/2013). The key steps of his approach are schematically
- The study of the *cycloid*;
- The introduction of the novel **geometrical** concept of the *evolute* (and implicitly its dual concept, the *involute*) of a curve (Evolute, n.d.);
- The proof of the cycloid's important properties: its *tautochrone* (oscillations along it are isochronous) and its *self-involuteness* (its *involutes* are again *cycloids identical* to it);
- The design of in principle more accurate clocks[20].

Though Huygens' approach is geometrical (in the Euclidean tradition of that period), its outline below is analytical (based on Newton's laws), and it is insightful to be compared in detail with Huygens' approach and proofs, in a *heritage*-like perspective (§3.1). This comparison points to the evolutionary character of the same mathematical results, by looking at them from a different, less familiar perspective (§3.2(a)), thus learning specific pieces of mathematics and developing a wider and richer view of mathematics (§3.2(b)).

(b) A cycloid is the trajectory of the point of contact of a circle and a straight line, as the circle is rolling along the line without slipping (Fig. 4.2).

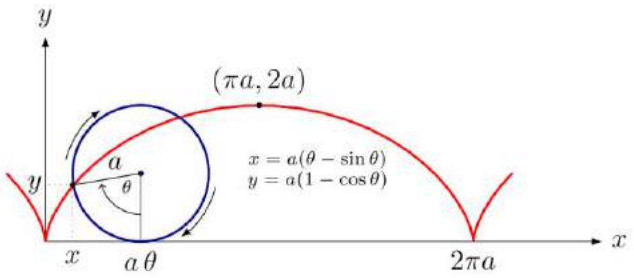


Figure 4.2: The geometry of the cycloid

By Newtonian mechanics, a point mass moving along a cycloid under its weight obeys the equation $\frac{\mathrm{d}v}{\mathrm{d}t} = g\frac{\mathrm{d}y}{\mathrm{d}s}$; v being its tangential speed and s the arc length along the cycloid. Expressing s in terms of x, y and noting that $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, recasts this equation as a linear differential equation in $\cos(\theta/2)$ **formally identical** to (4.21') (Sommerfeld, 1964, §III.17).

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos\frac{\theta}{2} = -\frac{g}{4a}\cos\frac{\theta}{2} \tag{4.22}$$

Therefore, the period of oscillation is independent of the point's initial position, given by

$$T = 4\pi\sqrt{\frac{a}{g}} \tag{4.23}$$

i.e. strict isochronism holds **irrespective** of the **oscillation amplitude**. This is the cycloid's *tautochrone* property (Huygens, 1673/2013, Part II, proposition XXV).

(c) In his investigation of the cycloid, Huygens introduced the geometrical concepts of the *evolute* (Latin: *evolutione*) and *involute* (or *evolvent*) of a curve (without naming the

---

[20] Huygens' design was based on the *verge escapement*, soon surpassed by the much better *anchor escapement* (Figure A.3; Whitrow, 1988, p. 123; Escapement, n.d.).

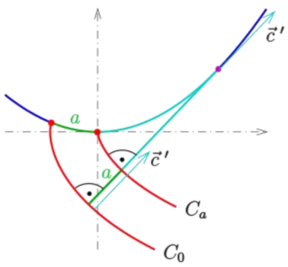
latter explicitly; Huygens, 1673/2013, Part III, Definitions I & III). They can be described in **two mathematically equivalent** but **conceptually different** ways (one geometrical and one physical) and they are dual to each other in the sense below:

- Physical description: An *involute* of a curve c_E is the locus c_I of the free end of a taut string attached to a point of the curve c_E as the string is wound along c_E.[21]
- Geometrical description: The *evolute* of a curve c_I is the locus c_E of the center of curvature of its points.

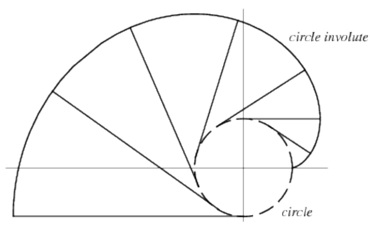
Clearly, each curve has infinitely many involutes (depending on the initial point chosen for the string's fixed end), but only one evolute. It can be shown that an involute of a curve is orthogonal to the curve's tangents (Huygens, 1673/2013, Part III, Proposition I; Stillwell, 1989, §16.2; Evolute, n.d.; Involute, n.d.) implying that these concepts are **dual** to each other in the sense that **the evolute of an involute of a curve is the curve itself**, hence the use of the same symbols in the above definition (Table 4.11; Figure 4.3).

curve c_I	curve c_E
Locus of **free end** of a taut **string** attached to a point of curve c_E as the string is wound along c_E	**Locus** of **center of curvature** of the points of curve c_I
Involute of c_E is c_I	**Evolute** of c_I is c_E
c_I orthogonal to the tangents to c_E	

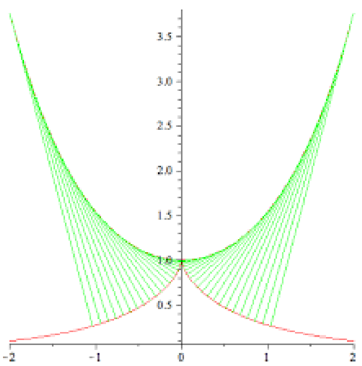
Table 4.11: The involute-evolute dual concepts



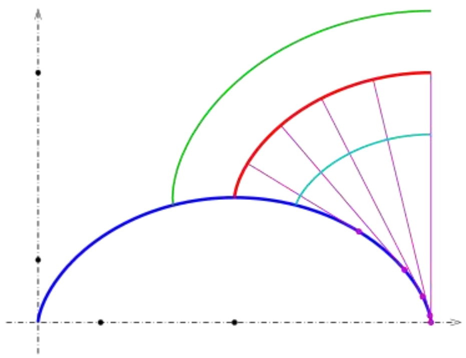
(a) The involutes c_I are orthogonal to the tangents to the evolute c_E



(b) The involutes of a circle c_E are spirals c_I; The evolute of a spiral c_I is a circle c_E



(c)The involutes of a catenary c_E are tractrices c_I; The evolute of a tractrix c_I is a catenary c_E



(d) The involutes of a cycloid c_E are cycloids c_I identical to it; the evolute of a cycloid c_I is an identical cycloid c_E

Figure 4.3: Examples of the involute-evolute dual concepts

[21] This formulation appears already in Huygens' (1673/2013) proof of Proposition I, Part III.

The important result here is the "*self-involuteness***"** of the cycloid (Figure 4.3(d)), proved by Huygens (1673/2013, Part III, proposition VI).

(d) The above two **mathematical** properties of the cycloid were combined by Huygens to give an appropriate clock based on a cycloidal pendulum; i.e. a pendulum constrained to oscillate between two cycloidal cheeks: Because of the cycloid's *self-involuteness*, a point mass (pendulum's heavy bob) suspended by a weightless (inelastic) string, constrained to move under its own weight between two cycloidal cheeks, moves along a cycloid. Therefore, by the *tautochrone* property, its oscillation period is amplitude-independent; that is, it takes the same time for all pendulums to arrive at the lowest point, irrespective of their initial position, i.e. the oscillation amplitude (Figure 4.4).

Fig. 4.5(a) is Huygens' own, showing both the cycloidal cheeks constraining the pendulum's motion and the "verge escapement" converting the pendulum's oscillations into pulses measuring time (cf. Fig. A.3(a)). Fig. 4.5(b) is a modern reconstruction.

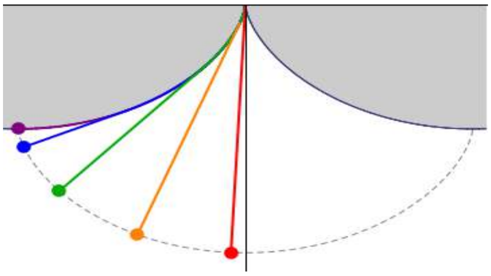
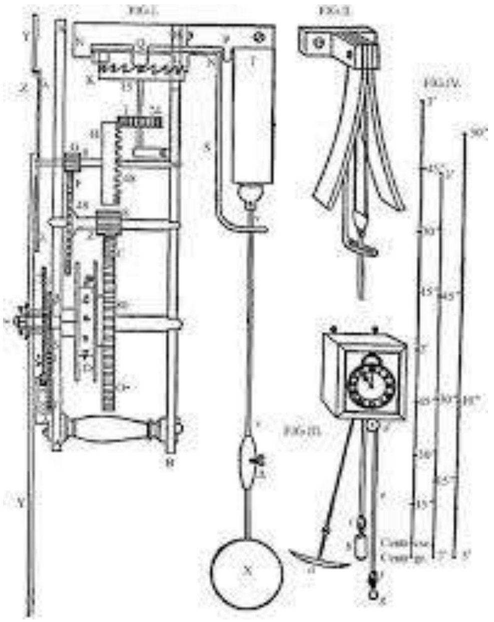
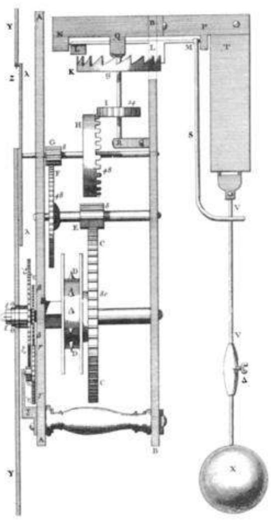


Figure 4.4: Schematic drawing and function of the cycloidal pendulum.



(a) Huygens' cycloidal pendulum clock (Huygens, 1673/2013, p.4)



(b) A modern drawing showing the *verge escapement* (Pendulum, n.d.)

Figure 4.5

The above outline of Huygens achievements related to the cycloidal pendulum, could be the basis of a teaching sequence inspired by history, either as an illumination approach, or as a complete teaching module (§3.4), depending on the time available, the specific didactical objective and other external constraints. History will serve mainly as a tool when following an illumination approach (e.g. in a course on mechanics, calculus, differential geometry, or their combination). On the other hand, in a teaching module devoted to this problem, historical issues (e.g. extensive use of original excerpts) can be treated in more detail and shape teaching explicitly (§3.3). The above presentation could also be used in an interdisciplinary module with emphasis on the mathematical basis of the technological achievements underlying clocks' construction; e.g. by elaborating on the various kinds of escapement mechanisms (see Appendix B), or the industrial applications of the geometrical concept of the involute etc (Involute, n.d.). This will emphasize the social context and its influence on the development of mathematics (§3.4(a)), thus helping to appreciate this development as the result of a cultural-human endeavour (§3.2(b)).

5 Concluding remarks

This paper aimed to provide evidence in support of the following main point: The multifarious aspects of the "*time measurement issue*" constitute an interdisciplinary resource for mathematics education, fruitful and insightful at quite different levels and purposes both for the learners and their teachers. The historical and epistemological facts were considered in the context of a theoretical framework for integrating history in mathematics education and the main point above was illustrated by means of five examples of varying mathematical content and social significance. Their presentation is far from being exhaustive, but was intended to stress the richness of the "*time measurement issue*": its mathematical content per se; its relation with other disciplines; and the social significance of the mathematics involved. A detailed presentation of these examples as teaching sequences and/or resource material will be given elsewhere.

APPENDIX A: Formalized methods to find the weekday on a given date

Examples of such methods are given below without proof (in the notation of §4.4) but with reference to the literature. The interested reader can try to check and possibly simplify them, explore conversions between them etc.

A1. C. F. Gauss

In an unpublished note, Gauss gave a formula for calculating the weekday number $W_{1/1/\mathrm{Y}}$ of year Y, from which a formula for $W_{d/m/\mathrm{Y}}$ results (for the original, details and proof see Schwerdtfeger, 2010; see also Richards, 1998, pp. 376-378):

$$W_{1/1/\mathrm{Y}} = 2 + (5\mathrm{Y} \bmod 4 + 4\mathrm{Y} \bmod 100 + 6\mathrm{Y} \bmod 400) \bmod 7 \qquad \text{(A.1a)}$$

From this, numbering months from March (m=1) to February (m=12), and using Y-1 instead of Y for January and February, a formula for $W_{d/m/\mathrm{Y}}$ results:

$$W_{d/m/\mathrm{Y}} = 1 + (d + [(13m-1)/5] + y + [y/4] + [c/4] - 2c) \bmod \mathbf{7} \qquad \text{(A.1b)}$$

(A.1b) is a variant of the formula in Weekday, n.d.

A2. C. Zeller

In 1883, the German mathematician C. Zeller gave another formula for $W_{d/m/\mathrm{Y}}$ where months are numbered as in (A.1b) (Zeller's congruence, n.d.; Richards, 1998, chs.23-24);

$$W_{d/m/\mathrm{Y}} = (d + [13(m'+1)/5] + y + [y/4] + [c/4] - 2c) \bmod 7, \qquad m'=m+2 \qquad \text{(A.2)}$$

A3. A. De Morgan

In 1845, De Morgan gave a verbally expressed algorithm for finding the *dominical letter* of a given year (§4.3.2; also Richards, 1998, p.377; Dominical Letter, n.d.). It can be adapted to give $W_{1/1/\mathrm{Y}}$ for any year Y (then, $W_{d/m/\mathrm{Y}}$ is found by the method of §4.3.2)

I. Add 1 to the given year.
II. Take the quotient found by dividing the given year by 4 (neglecting the remainder).
III. Take 16 from the centurial figures of the given year if that can be done.
IV. Take the quotient of III divided by 4 (neglecting the remainder).
V. From the sum of I, II and IV, subtract III.
VI. Find the remainder of V divided by 7, and subtract from 7: this is the weekday number of 1/1.

Or, as a formula:

$$W_{1/1/\mathrm{Y}} = 7 - (1 + Y + [Y/4] + [(Y\text{-}1600)/400] - [(Y\text{-}1600)/100]) \bmod 7 \qquad \text{(A.3)}$$

A4. C. L. Dodgson (Lewis Carroll)

In 1887, Lewis Carroll published another verbally expressed algorithm (Richards, 1998, ch.24; Weekday, n.d.):

"Take the given date in **4 portions**, viz. the number of centuries, the number of years over, the month, the day of the month. **Compute the following 4 items**, adding each, when found, to the total of the previous items. When an item or total exceeds 7, divide by 7, and keep the remainder only.
- *Century-item*: For '*Old Style*' (which ended 2 September 1752[22]) subtract from 18. For '*New Style*' (which began 14 September 1752) divide by 4, take overplus from 3, multiply remainder by 2.
- *Year-item*: Add together the number of dozens, the overplus, and the number of 4s in the overplus.
- *Month-item*: If it begins or ends with a vowel, subtract the number, denoting its place in the year, from 10. This, plus its number of days, gives the item for the following month. The item for January is '0'; for February or March (the 3rd month), '3'; for December (the 12th month), '12'.
- *Day-item*: The total, thus reached, must be corrected, by deducting '1' (first adding 7, if the total be '0'), if the date be January or February in a leap year, remembering that every year, divisible by 4, is a Leap Year, excepting only the century-years, in 'New Style', when the number of centuries is not so divisible (e.g. 1800).
The final result gives the day of the week, '0' meaning Sunday, '1' Monday, and so on."

A5. Modern methods

It is remarkable that the problem is still attracting modern researchers' interest. E.g. a much simpler algorithmic method was published recently for finding $W_{1/1/\mathrm{Y}}$[23]; the so-called "odd-plus-11 method", where a very limited tabulation of data is also needed (Fong & Walters, 2011; Dominical Letter, n.d.):

$$W_{1/1/\mathrm{Y}} = \left(7 - \left[\frac{y+11(y \bmod 2)}{2} + 11\left(\frac{y+11(y \bmod 2)}{2} \bmod 2\right)\right] \bmod 7\right) + W_{1/1/100\mathrm{c}} + 1 - \delta \qquad \text{(A.4)}$$

where $W_{1/1/100\mathrm{c}}$ is the (tabulated) weekday number (Table 4.3) of 1/1 of the century to which Y belongs, and $\delta = 1$ for leap years and 0 otherwise.

Though (A.4) appears complicated, the quantity in parentheses is verbally expressed easily: If and only if y is odd, then add 11 to y. Divide the result by 2. If and only if the

[22] The year Great Britain adopted the Gregorian calendar.
[23] The algorithm (slightly adapted to the present context) was invented for finding the so-called "doomsday" of the year (Doomsday, n.d.).

result is odd then add 11. Compute the result modulo 7 and subtract from 7.

APPENDIX B: Basic characteristics of clocks

Some more information is given below about the general characteristics of a clock's mechanism, with emphasis on mechanical clocks (§4.5). Further elaboration could lead to an interdisciplinary module with emphasis on the mathematical basis of the technological achievements underlying clocks' construction (cf. §4.5.2(d)).

Figure A.1 shows Galileo's conception of a time-keeping device based on a simple pendulum's oscillations (§4.5). It is essentially an escapement mechanism, since no dials or drive are shown (Drake, 1995, pp. 419-420). Figure A.2 shows Huygens' original pendulum clock, with the *verge escapement* shown on the right. The improved cycloidal pendulum clock is shown in figure 4.5. For more details see Pendulum clock, n.d.

(a) Galileo's pendulum clock drawn by Viviani (Pendulum clock, n.d)

(b) A 19th century reconstruction[24]

Figure A.1

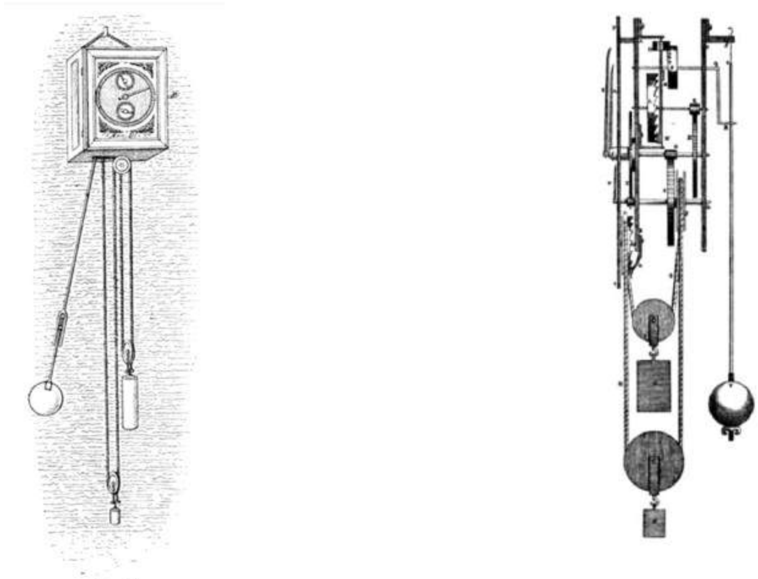


Figure A.2: Huygens' first pendulum clock (Gerland & Traumüller, 1899)

[24] Retrieved from https://cosmolearning.org/images/galileos-pendulum-clock-c-1642/ (11/11/2018).

A clock consists of three main parts (Richards, 1998, pp. 58-60; Clock, n.d.)

I. The *energy source* e.g.

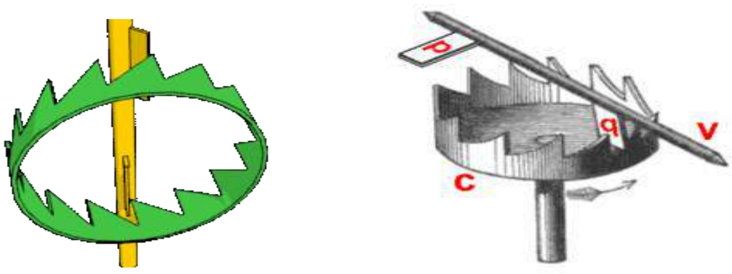
Type of clock	Energy source
Water clocks	water
Mechanical clocks	weight springs (elastic potential energy)
Quartz clocks	electricity

II. The *regulator*; i.e. the time-keeping element (oscillator). It is very important that the regulator possesses a natural frequency to resist vibration at other frequencies and in this way to minimize the effect of external disturbances; e.g.

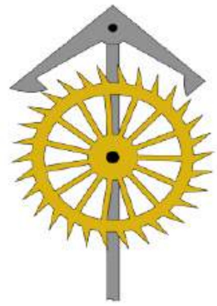
	Regulator	Type of clock
No natural frequency	periodic filling of scoops with water	water clocks
Possess **natural frequency** (resistance to vibration at other frequencies)	balance wheel (foliot) pendulum	mechanical clocks
	quartz crystal	quartz clocks
	vibrating atom	atomic clock

III. The *Escapement mechanism* (control energy release). The decisive step for improving the accuracy of clocks in measuring time was the invention of the *escapement mechanism* (Newton, 2004, ch. 3; Whitrow, 1972a, ch. 4, p. 60; 1988, chs. 7, 8; Fraser, 1987, pp. 49-58; Richards, 1998, ch.3; Verge escapement, n.d.). In general, this is a device that controls the energy release during the oscillatory process, by transferring energy to the oscillator to replace its energy loss due to friction, and allowing its oscillations to be counted; thus measuring time in this way (Escapement, n.d.). The invention of the *verge escapement* was a decisive step, originally combined with a balance wheel (not giving a very accurate clock), then with a spring. Verge escapements are shown in figures 4.5, A.2 of Huygens' clocks, as well as in figure A.3(a), (c), (d) below. It was finally superseded by the *anchor escapement* (figure A.3(b)) in the last quarter of the 18^{th} century (see Verge escapement, n.d.; Anchor escapement, n.d.; Clock, n.d.; Pendulum clock, n.d.)

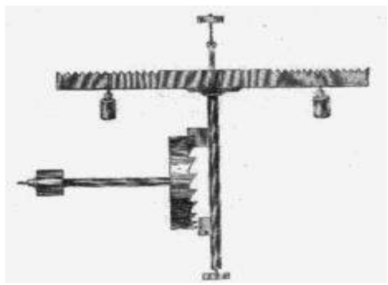
Escapement mechanism	Type of clock
Verge escapement Anchor escapement	mechanical clocks
Electronic oscillator circuit	electronic clocks
Microwave cavity attached to microwave oscillator, controlled by microprocessor	atomic clocks



(a) verge escapement



(b) anchor escapement



(c) Verge with balance wheel

(d) Verge with spring

Figure A.3: Escapement mechanisms

REFERENCES

Anchor escapement (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Anchor_escapement (accessed 11/11/2018).

Anderson, M. (n.d.). The Mathematics of the Calendar (Lecture course MA 110, Colorado College, USA). Retrieved from https://faculty1.coloradocollege.edu/~manderson/calendar/calendar2.htm (11/11/18).

Auerbach, D. (1995). Leap-times and the fall of the second. *European Journal of Physics*, *16*, 42-44.

Barbin, É. (1997). Histoire des Mathématiques: Pourquoi? Comment? *Bulletin de l'Association Mathématiques du Québec*, *37*(1), 20–25.

Beveridge, R. W. (n.d.) *Friday the 13th and the Mathematics of the Gregorian Calendar*. Retrieved from https://www.clatsopcc.edu/sites/default/files/file/Friday_13th_Gregorian_Calendar.pdf (11/11/2018)

Borst, A. (1993). *The Ordering of Time*. Cambridge: Polity Press.

Clark, K.M., Kjeldsen, T.H., Schorcht, S., Tzanakis, C., & Wang, W. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, T. Hausberger (Eds.), *Proc. of the 2016 ICME Satellite Meeting – HPM 2016* (pp.135-179). Montpellier: IREM de Montpellier.

Clark, K.M., Kjelden, T.H., Schorcht, S., & Tzanakis, C. (2018). Introduction: Integrating history and epistemology of mathematics in mathematics education. In K.M. Clark, T.H. Kjelden, S. Schorcht, & C. Tzanakis (Eds.), *Mathematics, Education and History: Towards a harmonious partnership* (ch.1, pp.1-23). Cham: Springer.

Clark, K.M., Kjelden, T.H., Schorcht, S., & Tzanakis, C. (2019). History of mathematics in mathematics education: An overview. *Mathematica Didactica*, *42*(1), 3-27.

Clock (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Clock (accessed 11/11/18).

Dauben, J.W. (2007). Chinese Mathematics. In V.J. Katz (ed), *The Mathematics of Egypt, Mesopotamia, China, India and Islam : A Sourcebook* (ch.3, pp. 187–384) Princeton: Princeton University Press.

Dence, J.B, & Dence, T.P. (1999). *Elements of the Theory of Numbers*. San Diego: Academic Press.

Dominical Letter (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Dominical_letter (accessed 11/11/18).

Doomsday (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Doomsday_rule (accessed 11/11/18).

Drake, S. (1995). *Galileo at work: His scientific biography*. New York: Dover

Duncan, D. E. (1998). *Calendar: Humanity's Epic Struggle to Determine a True and Accurate Year*. New York: Avon Books.

Dutka, J. (1988). On the Gregorian revision of the Julian calendar. *The Mathematical Intelligencer*, **10**(1), 56-64.

Eisenbrand, F. (2012). Pope Gregory, the Calendar, and continued fractions. *Documenta Mathematica*, Extra volume ISMP, 87-93.

Escapement (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Escapement (accessed 11/11/18).

Evolute (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Evolute (accessed 11/11/18).

Fong, C., & Walters, M. K. (2011). Methods for Accelerating Conway's Doomsday Algorithm (part 2). *7th International Congress of Industrial and Applied Mathematics*. Melville (N.Y.): American Institute of Physics. Retrieved from https://arxiv.org/ftp/arxiv/papers/1010/1010.0765.pdf (11/11/2018).

Fraser, J. T. (1987). *Time: The familiar stranger*. Redmond (WA): Tempus Books.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, **10**(4), 391–408.

Furinghetti, F. (2004). History and mathematics education: A look around the world with particular reference to Italy. *Mediterranean Journal for Research in Mathematics Education*, *3*(1–2), 1–20.

Furinghetti, F. (2019, in print). History and Epistemology in mathematics education. *International Journal of Mathematical Education in Science and Technology*. DOI:10.1080/0020739X.2019.1565454

Furinghetti, F., Jahnke, H. N., & van Maanen, J. (Eds.). (2006a). *Report No22/2006 on the Mini-workshop on studying original sources in mathematics education*. Oberwolfach: Mathematisches Forschungsinstitut Oberwolfach.

Gauss, C.F. (1801). *Disquisitiones Arithmeticae*. Leipzig: G. Fleischer (in Latin); Retrieved from https://gdz.sub.uni-goettingen.de/id/PPN235993352 (11/11/18).

Gerland, E. & Traumüller, F. (1899) *Geschichte der physikalischen Experimentierkunst*. Leipzig: Wilhelm Engelmann, p.182, fig.182 & p.179, fig. 178. Retrieved from *Pendulum clock* (n.d.) (11/11/2018).

Goldstein, C., Schappacher, N., & Schwermer, J. (eds). (2007). *The Shaping of Arithmetic: after C. F. Gauss's Disquisitiones Arithmeticae*. Berlin: Springer.

Grabovsky, Y. (n.d.) *Modern calendar and continued fractions*, Retrieved from https://math.temple.edu/~yury/calendar/calendar.pdf (11/11/18).

Grattan- Guinness, I. (2004a). The mathematics of the past: distinguishing its history from our heritage. *Historia Mathematica*, **31**, 163–185.

Grattan-Guinness, I. (2004b). History or heritage? An important distinction in mathematics for mathematics education. *The American Mathematical Monthly*, *111*(1), 1–12.

Hairer, E. & Wanner, G. (1996). *Analysis by its history*. New York: Springer

Hannah, R. (2005). *Greek and Roman calendars: Constructions of Time in the classical world*. London: Duckworth.

Heath, T.L. (1991). *Greek Astronomy*. New York: Dover

Holford-Strevens, L. (2005). *The history of time: A very short introduction*. Oxford: Oxford University Press.

Huygens, Ch. (1673/2013). *Horologium Oscillatorium: sive de motu pendulorum ad horologia aptato demonstrationes geometricae* English translation. I. Bruce (transl.) Retrieved from http://www.17centurymaths.com/contents/huygenscontents.html (11/11/2018)

Horologium Oscillatorium (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Horologium_Oscillatorium (retrieved 8/11/2018).International Astronomical Union (IAU) (2017). *Standards of fundamental*

Astronomy. http://www.iausofa.org/sofa_ts_c.pdf. (accessed 11/11/18).

International System of Units (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/International_System_of_Units (accessed 11/11/18).

Involute (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Involute (accessed 11/11/18).

Ireland, K., & Rosen, M. (1982). *A Classical Introduction to Modern Number Theory*. New York: Springer New York.

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., Silva da Silva, C. M., & Weeks, C. (2000). The use of original sources in the mathematics classroom. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study*, New ICMI Study Series, vol. 6 (pp. 291–328). Dordrecht: Kluwer.

Jankvist, U. T. (2009). A categorization of the “whys” and “hows” of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Jankvist, U. T. (2013). History, applications, and philosophy in mathematics education: HAPh—A use of primary sources. *Science & Education, 22*(3), 635–656.

Katz, V.J. (1998). *A History of Mathematics: An Introduction* (2nd ed.), Cambridge (MA): Addison Wesley Longman.

Khinchin, A. Ya. (1964). *Continued Fractions*. Chicago: University of Chicago Press.

Kraitchik, M. (1953). *Mathematical recreations*. New York: Dover

March equinox (n.d.). In *Wikipedia*, https://en.wikipedia.org/wiki/March_equinox (accessed 11/11/18).

Nelson, R.A., McCarthy, D.D., Malys, S., Levine, J., Guinot, B., Fliegel, H.F., Beard, R.L. & Bartholomew, T. R. (2001). The leap second: its history and possible future. *Metrologia 38*, 509-529.

Newton, R.G. (2004). *Galileo’s pendulum: From the rhythm of time to the making of matter*. Cambridge (MA): Harvard University Press.Ore, O. (1967). *Invitation to Number Theory*. Washington (DC): The Mathematical Association of America.

Ore, O. (1988). *Number theory and its history*. Washington (DC): The Mathematical Association of America.

Pendulum (n.d.). In *Wikipedia*, https://en.wikipedia.org/wiki/Pendulum (accessed 11/11/18).

Pendulum clock (n.d.). In *Wikipedia*, https://en.wikipedia.org/wiki/Pendulum_clock (accessed 11/11/18).

Richards, E.G. (1998). *Mapping Time: The calendar and its history*. New York: Oxford University Press.

Rickey, V. F. (1985). Mathematics of the Gregorian Calendar. *The Mathematical Intelligencer*, *7*(1), 53-56.

Sawer, C. (2002). Years, Calendars and Continued Fractions. Lecture delivered at Liverpool University, 26/1/2002. Retrieved from http://www.maths.liv.ac.uk/~mathsclub/talks/20020126/talk2/continued.pdf (accessed 11/11/2018).

Schwerdtfeger, B.E. (2010). *Gauss’ calendar formula for the day of the week*. Retrieved from http://berndt-schwerdtfeger.de/wp-content/uploads/pdf/cal.pdf (11/11/2018).

Sigler, L. (2002). *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*. New York: Springer.

Smart, W.M. (1971). *Textbook on Spherical Astronomy*. Cambridge: Cambridge University Press.

Smyntyna, O. V. (2009). Chronology. In H. James Birx (ed). *Encyclopedia of Time: Science, Philosophy, Theology, & Culture* (Vol.1, pp.183-186). Thousand Oaks (CA): Sage Publications.

Stillwell, J. (1989). *Mathematics and its history*. New York: Springer.

Swetz, F.J. (1987). *Capitalism & Arithmetic: The New Math of the 15th Century*. La Salle (IL): Open Court.

Tzanakis, C., & Thomaidis, Y. (2012). Classifying the arguments & methodological schemes for integrating history in mathematics education. In B. Sriraman (Ed.), *Crossroads in the history of mathematics and mathematics education* (ch.12). Charlotte, NC: Information Age.

Tzanakis, C. (2016). Mathematics & Physics - An innermost relationship: Didactical implications for their teaching and learning. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016*

ICME Satellite Meeting – HPM 2016 (pp.79-104). Montpellier: IREM de Montpellier.

van Helden, A. (1995). Pendulum Clock. Retrieved from http://galileo.rice.edu/sci/instruments/pendulum.html (11/11/2018).

Verge escapement (n.d.) In *Wikipedia*, https://en.wikipedia.org/wiki/Verge_escapement (accessed 11/11/2018).

Vinogradov, I. M. (1954). *Elements of Number Theory*. New York: Dover.

Weekday (n.d.). In *Wikipedia*, https://en.wikipedia.org/wiki/Determination_of_the_day_of_the_week (accessed 11/11/2018).

Whitrow, G.J. (1972a). *The Nature of Time*. Harmondsworth: Penguin Books.

Whitrow, G.J. (1972b). Reflections on the History of the Concept of Time. In J. T. Fraser, F. C. Haber, G. H. Müller (Eds.) *The study of Time*. Berlin: Springer.

Whitrow, G.J. (1988). *Time in history*. Oxford: Oxford University Press.

Zeller's congruence (n.d.). In *Wikipedia*,https://en.wikipedia.org/wiki/Zeller's_congruence (accessed 11/11/2018).

Topics in the history of mathematics education

THE FUSION OF PLANE AND SOLID GEOMETRY IN THE TEACHING OF GEOMETRY

Textbooks, Aims, Discussions

Marta MENGHINI
Dipartimento di Matematica
Sapienza, Università di Roma, Piazzale Aldo Moro 5, 00185 Italy
marta.menghini@uniroma1.it

ABSTRACT

The idea of the fusion of plane and solid geometry originated from projective and descriptive geometry, which worked with projections in space and sections. Different authors of textbooks (starting from Bretschneider in 1844 to Méray in 1874/1903; de Paolis in 1884; Lazzeri & Bassani in 1891, also translated into German by Treutlein in 1911) adopted this idea, mixing plane and solid considerations. For instance, the chapter on the properties of incidence also referred to the mutual position of a plane and a straight line, while homothety was defined in space and then on the plane. Pupils were supposed to have a better intuition of spatial relations when passing from space to plane, and to reason by analogy. Moreover, proofs could be presented of plane theorems using projections in space of simple known configurations. In the textbook of Lazzeri and Bassani we can see that one of the aims of the authors is to prove plane theorems with the help of considerations in space that allow to avoid part of the congruence axioms and the theory of proportions. This is not a novelty within history of mathematics, the development of conic sections is linked to this point, and Monge, too, used it in 1799. The question was also considered at the ICMI Congress of 1911—within the more general theme of the fusion of different branches of mathematics—by giving examples of successful textbooks (Fehr, 1911; Barbin & Menghini, 2013). This paper will discuss the methodological question of the fusion of plane and solid geometry bringing examples from different textbooks, and presenting some of the discussions on the subject, with particular reference to Italy, where there was even talk of a *fusionist school* (Borgato 2006 and 2016).

1 Introduction

This paper concerns the *fusion* of plane and solid geometry in the teaching of mathematics; that is the simultaneous use of plane and solid considerations when presenting and proving geometric properties at school. In Italy, at the turn of the 19th and 20th century this methodological question was deeply debated and there was even talk of a "fusionist school" (Borgato, 2006 and 2016).

We will consider the history of mathematics education starting from about two centuries ago, in the era of Gaspard Monge. However, the method of "fusion" does not belong only to the history of mathematics education but to history of mathematics in general. Indeed, it was used already by Apollonius to determine plane properties of the conic sections.

In 1911 the first plenary meeting of the International Commission on the Teaching of Mathematics (IMUK / ICMI / CIEM) was held in Milan (Italy). On this occasion a broader concept of fusion was discussed. The report of the discussion (Fehr 1911) was published in the Journal *l'Enseignement Mathématique*, which was at that time the official organ of the Commission. The report, based on an overview presented by Charles Bioche, refers to the teaching in various countries, and to the way in which they realize the different types of "fusion": geometry and arithmetic, plane geometry and trigonometry, plane and solid geometry (which is of interest for us), solid and descriptive geometry (which, as we will see, has also some interest for our question).

As to the *fusion of plane and solid geometry*, we read in the report (p. 469) that "the *fusionists* do not wait to have finished the treatment of all plane geometry before starting with spatial considerations"[1], and that "generally the two teachings are separated, excluding the entrance classes, because of the programmes. But "notable works appeared on geometric *fusionism"*. The mentioned textbooks are those written by the German Anton Bretschneider (1844), the French Charles Méray (1874/1903), and the Italians Giulio Lazzeri &Anselmo Bassani (1891).The report does not say very much about the educational pros or cons of the different of proposals.

Concerning the fusion of *solid* geometry and *descriptive geometry*, the report notes that these teachings are generally separated, and often not given by the same teacher (the report refers mainly to *Realschulen* in German speaking countries).

In the same number of the Journal we find a book review (Book review 1911) of Lazzeri & Bassani's *Elemente der Geometrie*, the German edition of the textbook, translated by Peter Treutlein and published by Teubner.

The review underlines that the idea of the fusion of plane and solid geometry is not new, since this methodology had already been presented in the Journal by "one of the main founders, Ch. Meray".

The first edition of his [Méray's] *book dates back to 1874, while the first Italian work – established on different bases –was published by De Paolis in 1884.The present work broadly follows the order traced by De Paolis* (Book review 1911, p. 429).

The review lists the contents of the book but does not describe the methodology used. Rather, it seems aimed at proving the priority of Méray (who died in 1911) with respect to "fusionism".

But the important sentence in the review is "established on different bases***"*** What does it mean?

We do not know what the author of the review was meaning, but surely we can distinguish between two kinds of approach:

- a methodological/educational approach based on a "new order" of the content allowing a "neighbourhood" of analogous properties in plane and space (mainly the one of Bretschneider, but also of Méray and de Paolis, with some exceptions). We describe this approach in section3.
- a mathematical/foundational approach based on the proof of plane theorems by means of space configurations (mainly the one of Lazzeri & Bassani).

This second approach is shown in the next chapter.

2 Proofs by means of space configurations

The link between plane and space dates back to the beginning of the history of conic sections, which are defined as plane sections of a solid. It is not only a question of definitions: their properties can be proved looking at their position with respect to the cone.

2.1 Apollonius

One of the major works about conic sections is *The Conics* by Apollonius of Perga (3rd-2nd cent. B.C.). In Book 1, Proposition XI **(**ver Eecke, 1963, p.22) we find a sort of "equation of the parabola".

[1] All translations are by the author.

The cone is defined by Apollonius as the set of straight lines that join a point *A* (the vertex) to the points of a circumference. It is therefore an *oblique circular double cone*. Fig. 2.1 (taken from Ver Eecke, 1963, p.22) represents the case in which the cone's section yields a parabola.

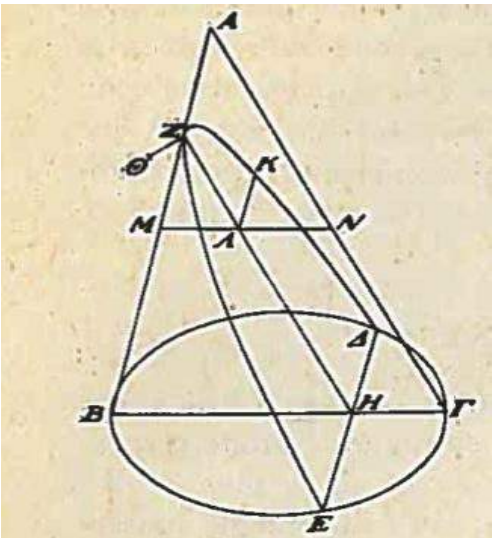


Figure 2.1

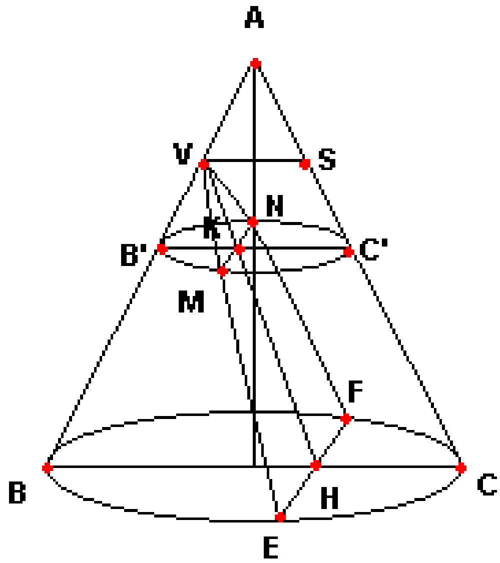


Figure 2.2

To slightly simplify the proof, let us consider a *right circular cone*, where *BC* is the diameter of the base circumference. In Fig. 2.2 we consider

$$EF \perp BC;\ H = EF \cap BC;\ VH \parallel AC.$$

The plane *EFV* is therefore parallel to a generatrix of the cone and cuts it in a *parabola*.

Now we chose any circumference whose diameter *B'C'* is parallel to *BC*, and take *MN* on the circumference and on *EFV* so that $MN \parallel EF$ and $K = MN \cap B'C'$

Euclid's "geometric mean theorem" holds for triangle *B'C'M*:

$$MK^2 = B'K \cdot KC'$$

What does it mean to find the equation of the parabola? We need a relation between two mutually perpendicular segments, in this case *MK* and *KV*, $MK \perp KV$,which correspond to our *x* and *y*.

To find this relation Apollonius considers the similarities between *VB'K* and *ABC*, and between *AVS* and *ABC*, obtaining

$$\frac{B'K}{B'V} = \frac{BC}{BA} \Rightarrow \overline{B'K} = \overline{B'V} \cdot \frac{BC}{BA} = \overline{KV} \cdot \frac{BC}{BA}$$
$$\frac{VS}{VA} = \frac{BC}{BA} \Rightarrow \overline{VS} = \overline{VA} \cdot \frac{BC}{BA} = KC'$$

In the previous formulas we replaced *B'V* by *KV* (two sides of an isosceles triangle, note that this is the only point in which we use the fact that the cone is a right cone) and *VS* by *KC'* (two opposite sides of a parallelogram). Substituting in the formula for the geometric mean theorem we obtain

$$MK^2 = \overline{KV} \cdot \overline{VA} \cdot \left(\frac{BC}{BA}\right)^2 \Rightarrow \boldsymbol{x^2 = p \cdot y}$$

In the previous equality we have considered that *MK* and *KV* are variables which depend on the changeable circumference whose diameter is *B'C'*. All the rest is constant and depends on the cone and the point in which the plane of the parabola cuts the cone.

The original proof is of course more difficult because Apollonius uses only proportions between geometric objects (for instance our p of the last equation corresponds to a segment – name dΘ in Fig. 2.1 – with certain properties). Apollonius finds analogous relations for the ellipse and the hyperbola, but what is interesting for us is that starting from a space definition and considering (always in space) elementary geometric properties we find a relation between two segments in the plane.

2.2 Gaspard Monge

In 1799 the *Géométrie Descriptive* by Gaspard Monge was published. The book is "pour l'usage des élèves de la première École Normale" and is devoted to future teachers.

Descriptive geometry deals with the representation of three-dimensional objects through drawings in two dimensions by projection and section (its *first aim*).In a certain sense it can be seen as a generalization of conic sections: the latter rise from a projection of a circle from a point and a successive section with a plane. Descriptive geometry deals with parallel or central projections of different geometric objects, and plane sections. The *second aim* of descriptive geometry is, according to Monge, "to research truth in geometry". The exactness of drawings and the research for truth render the content important for all the students of the French educational system (Barbin, to appear). So we have again to do with the history of mathematics education.

The first part of the book deals with the method of projections and shows how to determine the position of a point in space. The second part concerns tangent planes and normals to surfaces. It requires the capacity of seeing relations in the space. Here we find very interesting proofs of plane theorems made with the help of space considerations.

Let's for instance consider one of the properties proved by Monge (1847, p. II, n. 39, see Fig. 2.3). Our proof follows the notation of Fig. 2.4.

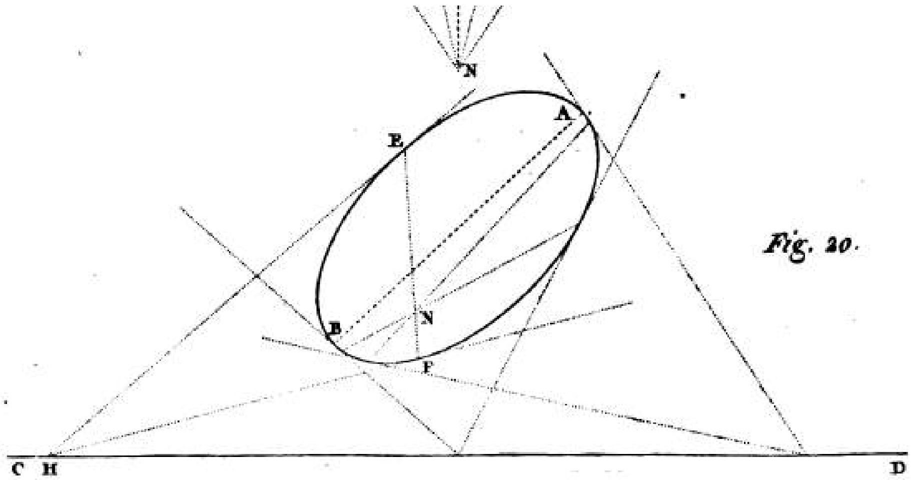
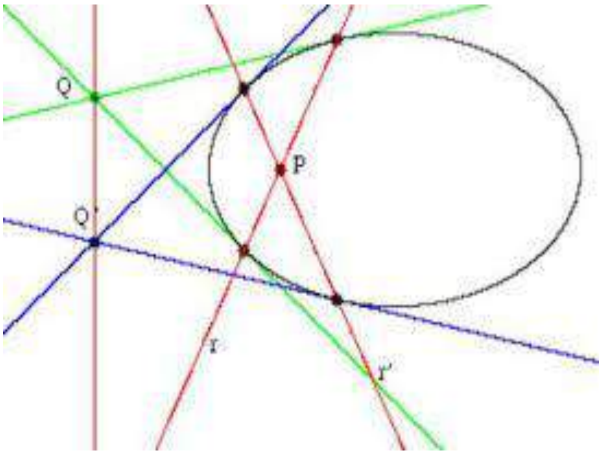


Figure 2.3

Let's take a line and points on it Q, Q', etc. From each point Q, Q', … we draw tangents to a given conic section (e.g. an *ellipse* E). For each pair of tangents, we draw line r cutting the ellipse in two points R and R'. All the lines constructed in the same way pass through a same point P.



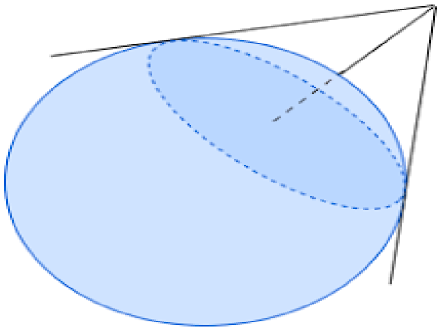


Figure 2.4 Figure 2.5

Proof: let the ellipse rotate about one of its axes, thus obtaining an ellipsoid. A cone with vertex Q touches the surface in an ellipse C (Fig. 2.5).

If the two tangent planes trough the line *QQ'* touch the ellipsoid in two pointsP_1, P_2 (see Fig. 2.6, where P and P_1 are exchanged), the ellipse C passes through P_1 and P_2.

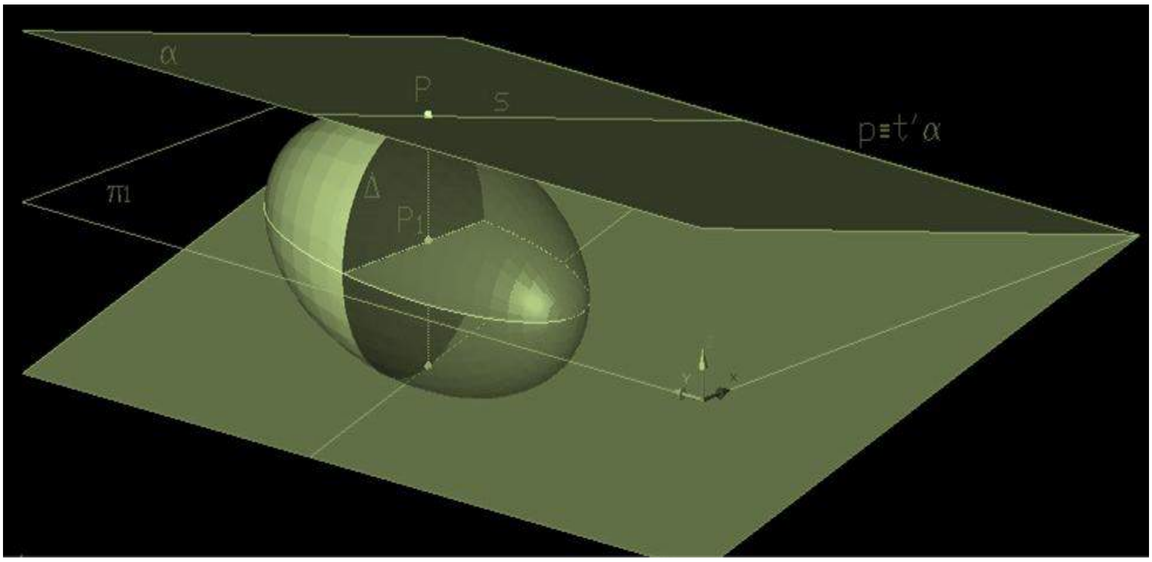


Figure 2.6[2]

The plane of C is ⊥ to π_1 (the plane containing the original ellipse E), and $C \cap E = R$, R' (note that C is any circumference rotating about the line P_1P_2, while the dark circumference indicated with Δ in figure 2.6 is a limit case of C when Q is the point at infinity of the line common to the two planes). The intersection $P_1P_2 \cap RR$' yields the point P. This happens for any point Q, so the theorem is proven.

2.3 Pierre Germinal Dandelin

In a paper of 1822 Pierre Germinal Dandelin, a former student of the *Ecole Polytechnique*,

[2] Retrieved from https://commons.wikimedia.org

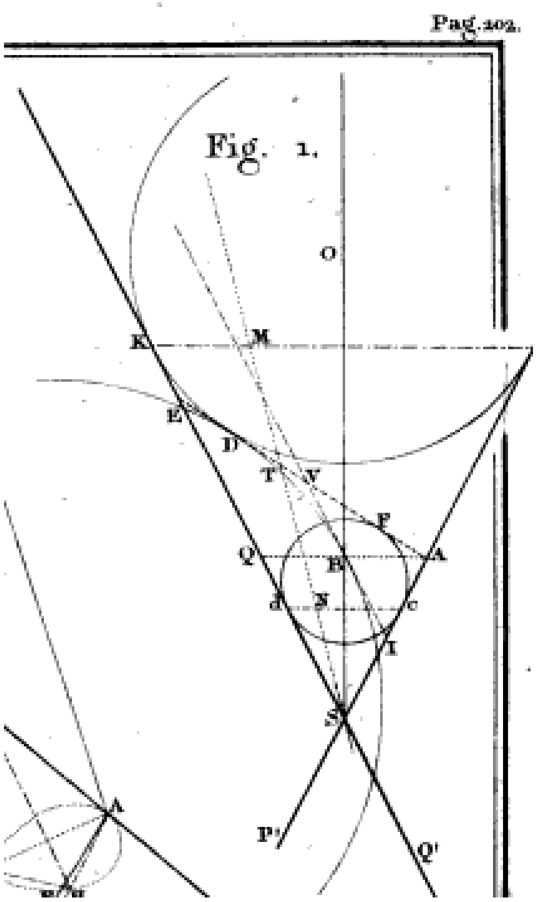


Figure 2.7[3]

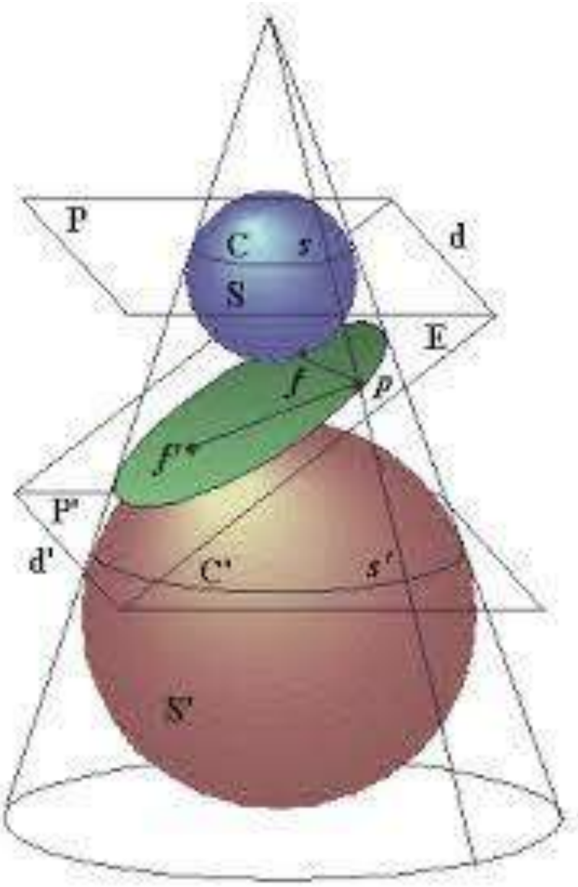


Figure 2.8[4]

presents a well-known proof that links the definition of a conic as a section of a cone to its definition as a locus of points. We are in a period in which the development of descriptive geometry brings with it also a revival of synthetic geometry.

For the proof, Dandelin considers two spheres tangent to a cone and to the plane that yields a conic section. We will consider the case of an *ellipse,* following the notation of Fig. 2.8:

A plane cuts a cone in an ellipse *E*. The sphere *S* touches the cone in a circumference *C*, and touches the plane containing *E* in *f.* The sphere *S*' touches the cone in a circumference *C*' and touches the plane containing *E* in *f*'. Take *p* on *E*. The generatrix through *p* touches *S* in *s*, and *S*' in *s*'. It holds

$pf = ps$; $pf' = ps'$ (equal tangent segments to the spheres)

$\Rightarrow pf+pf' = ps + ps' = ss'$

The distance *ss*' = constant. We thus obtain that for any point *p* on *E* the sum of the distances *pf* + *pf* ' is constant. This is the definition of an *ellipse as locus of points.* As for the proof by Apollonius, we used elementary geometric properties in space to find a relation in the plane.

This proof can be found in some textbooks, but strangely I did not find it in books which present a *fusionist* approach; instead I found it in the part concerning *solid geometry*

[3] From Dandelin, 1822, p. 169

[4] Retrieved from https://xavier.hubaut.info/coursmath/2de/belges.htm

of books as Henrici & Treutlein (1891/1901) and Cateni & Fortini (1958).

3 A new order in the textbooks

3.1 Carl Anton Bretschneider

One of the first textbooks to present a new order allowing a better integration of plane and space considerations is the one by Carl Anton Bretschneider in 1844. The aim is clearly stated in the introduction:

> *Basing the synthetic part of my book on the division into geometry of position, of form, measure, and organic geometry, which is offered by the nature of this science, the separation of the matter into the two main sections of plane and solid geometry could not be allowed anymore [...]The pedagogical value cannot be denied* (Bretschneider, 1844, p. VI).

Let's see in which way Bretschneider groups the various topics; in the following list of contents the chapters 1, 3, 4 of Book one concern plane geometry, the other concern geometry of space. In Book 2, the first five chapters are about plane geometry, the others about solid geometry.

Book 1. *Geometry of position*	Book 2. *Geometry of form*
Ch. 1 the straight line	Ch. 1 plane figures
Ch. 2 the plane	Ch. 2 plane triangles
Ch. 3 plane angles	Ch. 3 quadrilaterals
Ch. 4 parallelism in the plane	Ch. 4 circles
Ch. 5 wedges[dihedral angles in the space]	Ch. 5 circumscribed and inscribed circles
Ch. 6 Angles between lines and planes	Ch. 6 solid angles
Ch. 7 parallelism in space	Ch. 7 polyhedra in general
	Ch. 8 pyramids
	Ch. 9 prisms
	Ch. 10 the sphere

Book 3 concerns the *Geometry of measure* and also contains the theory of proportions and of similarity. The first 6 chapters are about plane measures, the 4 last chapters are about volumes and surfaces and lengths in the space.

The second part of the book is on *analytic geometry*, more precisely:

Book 4 and 5 are on plane goniometry and trigonometry,

Book 6 is on coordinates, and Ch. 5 considers *coordinates in space*.

Then we find five appendices, about geometric constructions in the plane; geometric loci in the plane, in particular conics; the method of *projections*; the area of a parabola and of an ellipse; the area of spherical triangles.

Appendix 3 deserves particular attention. It presents the method of *projections* (the "new geometry") in plane and space and contains interesting propositions, including the proofs of *Apollonius* for parabola, ellipse and hyperbola. Other plane propositions, proved by Monge using 3D geometry, are proved here using the theory of polars in the plane.

So, we can see that in the textbook of Bretschneider there is still a separation between plane and solid considerations, but similar topics are – when possible - the one near the other. The only really "fusionist" argumentations are the proofs by Apollonius, which are presented using the proportions among similar triangles, as shown in chapter 1.

3.2 Charles Méray

Another very interesting book is the one by Charles Méray, written in1874, which reached its major success in 1901, when it was revived thanks to the new programmes for the teaching of mathematics of France.

In the introduction Méray criticizes the “disorder” of Euclid’s Elements. In particular, he states that the division between plane geometry and solid geometry makes no sense, because nature only presents objects in space (Méray, 1874, p. XI). In his text, Méray substitutes most axioms with intuitive properties of motions in space (folding a piece of paper on itself, translating an object, rotating about a line).

The subdivision of the matter is not very different from Bretschneider, as in all other fusionist books, but in some chapters there is a *better integration of plane and space* thanks to the use of *geometric transformations*.

The chapters from 1to 4 deal with intersection, perpendicularity, and parallelism of lines and planes, and with plane and dihedral angles.

T*ranslation* is defined as a motion of figures in space. Two lines are *parallel* if a *translation* maps the one onto the other (independently from being in a plane or in space). The same for *parallel* planes and for lines parallel to a plane.

A plane is *perpendicular* to a line if it is mapped onto itself by the rotation about the line. Two lines are perpendicular if they meet and each of them is on a plane perpendicular to the other.

So, we can see that in these chapters plane and space are treated simultaneously and, for instance, perpendicularity between straight lines is defined using the perpendicularity between a plane and a line.

Not all chapters present such an integration, but we find it again in chapter 5, dealing with the comparison of segments. The *intercept theorem* (*Thales* theorem) about the proportion of segments is given both for plane and space. In chapter 10 areas are compared by means of a *projection* of an area on a plane (using trigonometry) (Méray, 1874, p. 96, see Fig. 3.1).

Not all chapters present such an integration, but we find it again in chapter 5, dealing with the comparison of segments. The *intercept theorem* (*Thales* theorem) about the proportion of segments is given both for plane and space. In chapter 10 areas are compared by means of a *projection* of an area on a plane (using trigonometry) (Méray 1874, p. 96, see Fig. 3.1).

In chapter 13 *homothetic figures and similarity* are treated both in plane and in space. (ibid. p. 122, see Fig. 3.2)

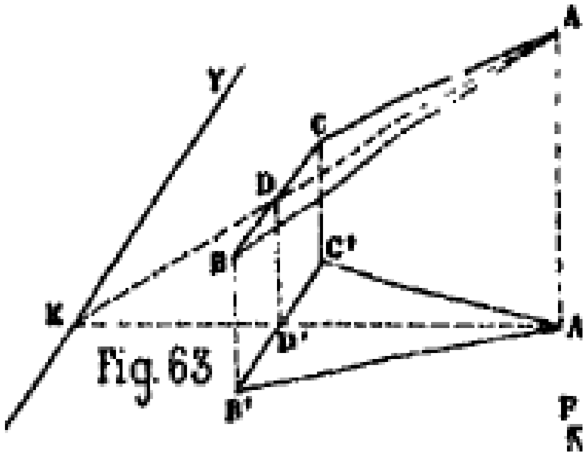


Figure 3.1

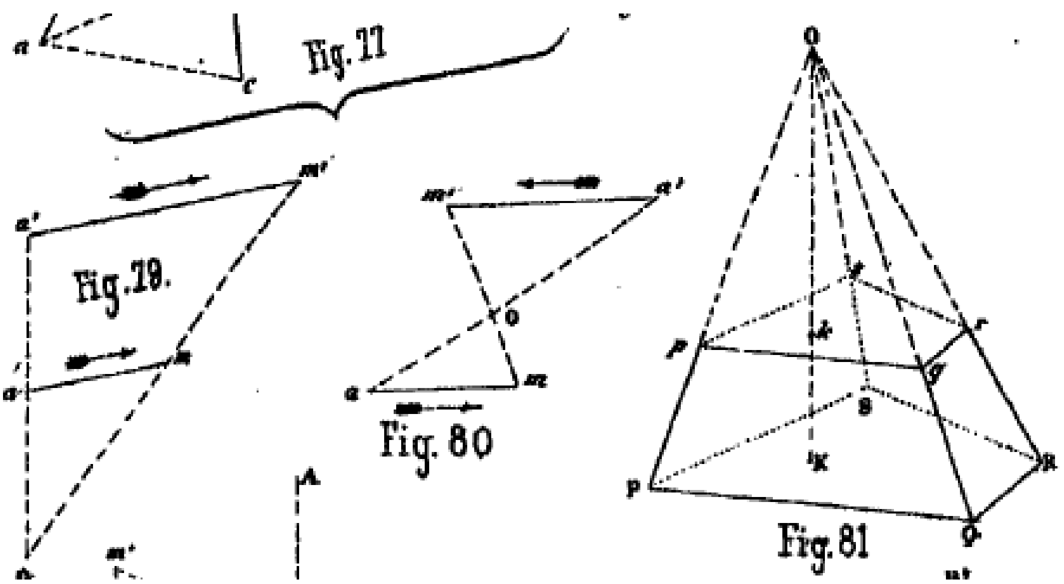


Figure 3.2

Let us note that Méray presents separately the parabola, hyperbola and ellipse referring to their eccentricity. He never uses the term "conic sections". Indeed, we can suppose that - being his a fusionist book – he would be then obliged to link the eccentricity to the definition of these figures as sections of the cone.

3.3 Riccardo De Paolis

The textbook by the Italian Riccardo De Paolis of 1884 is mentioned in the review of the textbook by Lazzeri & Bassani published in1911 in*l'Enseignement Mathématique*(see section 1).

In his introduction De Paolis writes:

There is a big analogy between certain figures in the plane and certain figures in the space; by studying them separately we renounce to know all that we can learn from this analogy, and we fall into useless repetitions. If we look for the properties of a line or a surface without being able to use the geometrical entities placed outside the line and the surface, we limit the forces we can dispose of and we renounce to geometric tools that would help to simplify constructions and proofs. In fact, how can you construct the midpoint of a given segment without leaving the segment itself? Instead, using the geometrical tools of a plane that contains it, the construction is known and very simple. How can one construct an isosceles triangle that has each of the two angles equal to twice the third? The triangle is easily constructed, and without applying the theory of equivalence or proportions, if we use the geometric objects placed outside its plane (page 92) (De Paolis, 1884, p. III-IV).

The proof mentioned by De Paolis is not as easy as he states, but what is important in this last sentence is the fact that proofs can be performed without applying the theory of equivalence or proportions. Indeed, it is necessary to observe that if – as was the case in Italy – the textbooks follow the books of Euclid, the theory of proportions takes much time and is quite difficult. Therefore the possibility to avoid it has a particular value.

Do not object that for beginners it is easier to conceive a plane angle than a dihedral angle; it is exactly because the mind of the students is forced to think and *draw only flat figures in the first years of their geometric studies, that they find difficulties afterwards* (ibid. p.IV).

In his book De Paolis gives much importance to geometric transformations, as Méray does. He also presents many interesting exercises and problems. But his proofs are often too long, being the author also much interested in rigour.

As in Méray, the first part of the book concerns properties of incidence and parallelism. Let's look, for instance, at the following theorem:

The angles formed by two intersecting lines are equal to those formed by two lines parallel to them, which meet (ibid., p. 33)

The formulation does not state if we are speaking of a plane theorem or of a theorem in solid geometry. This means that the theorem holds in both cases. The first part of the proof is performed in space and is based on the *sliding* of the pair of lines *APC* and *BPD* on the dihedral angle formed by the two couples of parallel lines *AC, A'C'* and *BD, B'D'* ...(Fig. 3.3)

Only afterwards De Paolis presents the proof for lines lying all in the same plane, which refers to previous theorems based on the properties of parallel lines.

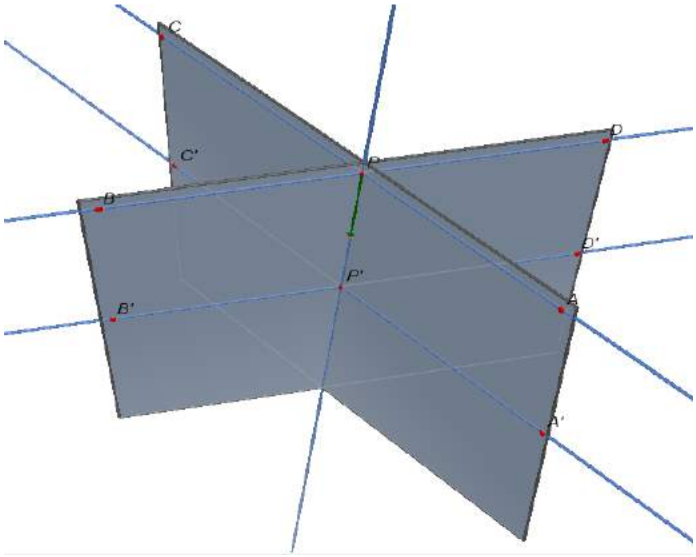
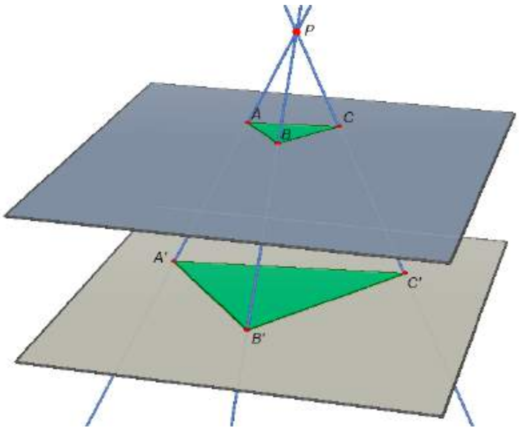


Figure 3.3

A further theorem presents what we could call a "fusionist" proof:

Given $ABC, A'B'C'$*. Suppose that* AA', BB', CC' *all meet in* P*, and that* $AB \parallel A'B'$*and* $AC \parallel A'C'$*; we want to prove that* $BC \parallel B'C'$(ibid., p. 92).



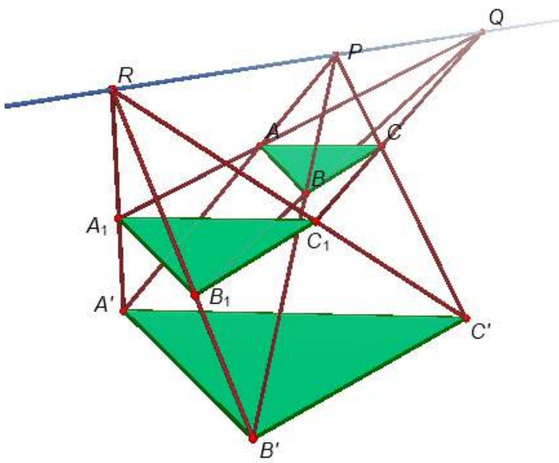


Figure 3.4

Figure 3.5

If the given triangles are *not* on the same plane, the proof is obvious: the two planes

$ABC, A'B'C'$ are parallel, because of the hypotheses, and hence also $BC \parallel B'C'$ (Fig. 3.4).

If ABC and $A'B'C'$ are on a same plane, take two points Q, R on any line (in *space*) through P, from which we project the triangles. So we have twice the solid case. For the transitivity of parallelism, the proof is completed (Fig. 3.5).

To follow the second part of the proof, we need to consider the same configuration of the first case, but looking at it differently: in the first case we “see” a solid configuration, in the second case a plane one.

Let us also note that this theorem is a particular case of the Desargues theorem, corresponding to the situation in which the intersection points of correspondent sides are on a line at infinity (i.e., correspondent sides are parallel). Hilbert shows that the Desargues theorem can be proven using only the incidence axioms for the space, and avoiding the congruence Axiom III, 5 (Hilbert, 1899).

3.4 Giulio Lazzeri and Anselmo Bassani

We arrive now to the last and most important book presenting a fusionist approach: the book by Lazzeri and Bassani, which was written in 1891 for the pupils of the *Accademia Navale* (naval academy) in Livorno – at that time a secondary technical school – and had a second edition in 1898 devoted also to the *Lycées*.

The introduction is very similar to the one of De Paolis. The authors mention - as their predecessors - *Bretschneider, De Paolis, Angelo Andriani*(Andriani 1887; another Italian fusionist book) and – above all –*Monge,*

> *who showed the utility of the fusion by proving, with the help of three-dimensional figures, many theorems concerning plane figures in a very simple way* (Lazzeri & Bassani, 1891, p. X).

Moreover, they add that this method of proving plane theorems with the help of solid geometry “is well accepted today in projective geometry […] and has now been realized also in elementary geometry” (ibid.).

Indeed, in the book by Lazzeri & Bassani the method is very often applied. Substantially there is no chapter in which plane and space are separated. Moreover, they state that they "succeeded in making many questions independent from the theory of proportions and of measures" (ibid., p. XI).A first example is given by the following theorem:

Two lines *r* and *r’* are given, with *r* parallel to *r*’, and *A*, *B*, *C*, *D* on *r*. Consider a point *O* and the lines *OA*, *OB*, etc. which cut *r’* in *A’, B’, C’*, *D*’. We want to prove that if *AB* = *CD* then *A’B’* = *C’D’*.

The theorem could be easily proven using proportions and similarity. But we can find a different proof that avoids the theory of proportions:

With reference to Fig. 3.6, move *OAB* to *O’CD,* so that *A’B’* = *A”B”*.

Take *V not on the plane* of the figure, and consider the tetrahedral *OCDV* and *O’CDV* and a plane containing *r’* parallel to the plane *VO'O"*: this plane intersects the triangle *VCD* (common to the two tetrahedra) in *HK*.

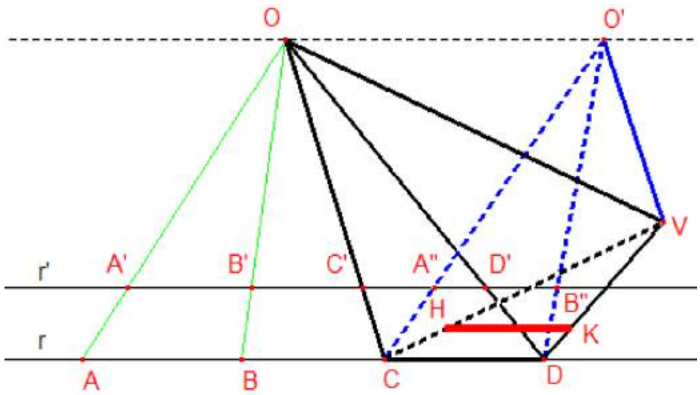


Figure 3.6

A previous theorem states that in a tetrahedron we can always consider a plane parallel to two opposite edges and at an intermediate distance from them. This plane cuts the tetrahedron in a parallelogram (fig. 3.7).

Thanks to this theorem we have that *HKC'D'* and *HKA''B''* are parallelograms; therefore *HK = A''B''= A'B'; HK = C'D'*, hence *A'B' = C'D'*.

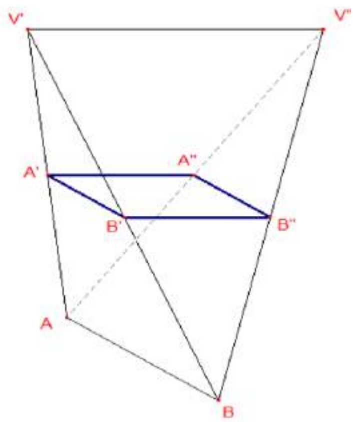


Figure 3.7

This proof only involves questions of parallelism and intersection. It is not a difficult proof, if we have the habit to "see" in space.

Let us look at second example, with the following theorem:

Given two circumferences c_1, c_2 ($c_1 \neq c_2$) on the plane, the locus of points such that the tangent segments led from each point to the two circumferences are equal is a straight line perpendicular to the line joining the two centres, and external to the circumferences if the circumferences are the one external to the other (Fig. 3.8)

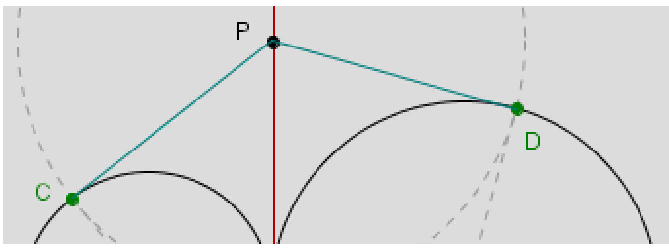


Figure 3.8

The proof refers to Fig. 3.9, which is taken from Lazzeri & Bassani (1891, p. 188). Consider two equal spheresS_1, S_2 passing through c_1, c_2, with centres O_1', O_2'. Take $\beta \perp \overline{O_1'O_2'}$ in its midpoint (plane of reflection of S_1, S_2).

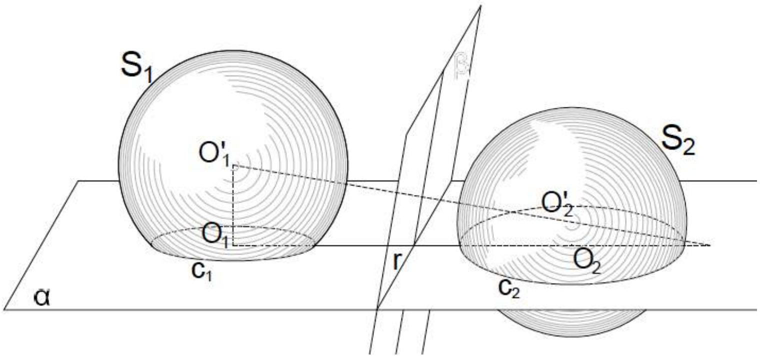


Figure 3.9

Call $r = \beta \cap \alpha$; r is the locus on α such that the tangent segments led from each point to the two spheres are equal and hence also to the two circles. We show that $r \perp \overline{O_1O_2}$:the perpendiculars to α through O_1', O_2' meet αin O_1, O_2 and forma plane $\gamma \perp \alpha$, containing $\overline{O_1O_2}$, $\overline{O_1'O_2'}$. Also β is perpendicular to γ being $\beta \perp \overline{O_1'O_2'}$, so planes αandβmeet in a line $r \perp \gamma$ and hence $r \perp \overline{O_1O_2}$.

The book by Lazzeri and Bassani presents very interesting and beautiful proofs. It is not easy to judge its difficulty without knowing in depth the teaching methods of such a topic in that period, and in particular of Lazzeri himself, who was a teacher in the naval academy. Indeed, at the time the book was well considered by teachers, but the question does not have a definite answer, as we can see in the next section.

4 Discussions about the question

In 1899 the Journal *l'Enseignement Mathematique* published a paper by Giacomo Candido, where the author describes the debate that takes place in Italy, presenting the arguments *against fusion* and the arguments *in favour of fusion* (Candido, 1899).

Against the fusion are the programmes of the *Lycées*, which present stereometry only in the third year, following the same order of Euclid; moreover the fusion seems too difficult, too much linked to systematization (with reference both to De Paolis and to the book of Andriani, who – according to Candido – found "non-existing connections" between plane and space).

In favour of fusion are the fact that it is time saving (it is not necessary to repeat certain parts of the school programs) and allows a simplification of some considerations on planimetry by explaining them through considerations in space. Furthermore, it allows a major harmony between the study of mathematics and that of other topics.

The author mentions the book by Lazzeri & Bassani among the arguments in favour of fusion. Indeed, it contains better proofs "through" space. The success in a technical institute (the *Accademia* Navale of Livorno, where Lazzeri was a teacher) brought the book to be used also in *Lycées* (Candido, 1899).

The Italian association of mathematics teachers *Mathesis* published different discussions and also asked to change the programs so as to allow fusion (Borgato 2006 and 2016).

Again in *l'Enseignement Mathematique*, Méray presents the 1903 edition of his *Eléments* (Méray, 1904) with a big critique to Euclid, as contained in the introduction of the book (see 3.2). He mentions the great Italian "*fusionist*" school, but notes that his book was written earlier.

In the second book of the series on *Elementary Mathematics from a higher standpoint*, first published in 1908 with a new English translation in 2016, Felix Klein presents the book by Scheffers & Kramer (1925):

> *The text is based on the view that for the development of the best possible space intuition, the fusion between planimetry and stereometry has to be dealt with more systematically and from an earlier time in school than has happened so far. If we start to realise this idea of fusion, we encounter soon the necessity to perform spatial constructions graphically and to imagine solids on the plane. The planimetry-stereometry-fusion urges therefore a broader notion of fusion, which comprises descriptive geometry* (Klein, 2016, p. 303).

In a book of 1928 discussing the teaching of geometry in German schools Kuno Fladt states:

> *Already in the 1840s the need was felt of merging stereometry closely with planimetry. Even if too much weight was given at that time to a scientific systematization, there was anyway an educational idea: that the pupil who has to do only with planimetry is almost educated to "space blindness". Both types of reasons lead in Italy to an extended "fusion" of planimetry and stereometry. But there was a setback: the too early and extended employment of stereometry turned out to be too difficult.*
>
> *This does not exclude that, on the one hand, in the first teaching of geometry plane objects are shown on solids, from which they are then abstracted, and that, on the other hand, when presenting new plane figures, we always consider and present the solid bodies in which they can be found. This is a "moderate fusion" as now required in the new programmes of Würtenberg of 1926/27* (Fladt, 1928, p. 126)

A compromise is indeed often a good solution, and this can happen also in the case of fusionism. A presentation of the incidence properties as presented in the first chapters by Méray is surely a wonderful help to space representation.

In the case of conic sections, proofs as the one by Apollonius or by Dandelin allow the link between different definitions of conic sections, which is not usual in schools.

But also other suggestions come from this historical overview, which could help the construction of a curriculum that avoids "space blindness".

REFERENCES

Andriani, A. (1887). *Elementi di geometria euclidea esposti con nuovo metodo*. Napoli: Pellerano.

Barbin, E. (to appear). Monge's Descriptive Geometry: His Lessons and the Teachings givenby Lacroix and Hachette. In E. Barbin, M. Menghini, &K. Volkert (Eds.), *Descriptive Geometry, the Spread of a Polytechnic Art*. Cham: Springer Nature.

Barbin, E., & Menghini, M. (2013). History of teaching geometry. Chapter XXVII in A. Karp & G. Schubring (Eds), *International Handbook on the History of Mathematics Education*. New York: Springer-Verlag.

Book review (1911). G. Lazzeri und A. Bassani. — Elemente der Geometrie (Unter Verschmelzung von ebener und räumlicher Geometrie). — Aus dem Italienischen übersetzt von P. Treutlein. — 1 vol. in-8°, XVI-491 p.; 14 M.; B. G. Teubner, Leipzig, 1911. *l'Enseignement Mathématique*, 13, 429 – 430.

Borgato, M. T. (2006). Il fusionismo e i fondamenti della geometria. In Giacardi L. (Ed.), *Da Casati a Gentile, momenti di storia dell'insegnamento secondario della matematica in Italia* (pp. 125-157). Lugano: Lumières Internationales.

Borgato, M. T. (2016). Il fusionismo: moda didattica o riflessione sui fondamenti della geometria? *Periodico di Matematiche*, s. XII 8(2), 45-65.

Bretschneider, C. A. (1844). *Lehrgebäude der niederen Geometrie: für den Unterricht an Gymnasien und höheren Realschulen*. Jena: F. Fromann.

Candido, G. (1899). Sur la fusion de la planimétrie et de la stéréométrie dans l'enseignement de la géométrie en Italie. *L'Enseignement Mathématique*, 1, 204 – 215.

Cateni, L., Fortini, R. (1958) *Il pensiero geometrico*. Firenze: Le Monnier.

Dandelin, P. G. (1822). 1° Mèmoire sur les propriétés remarquablesde la focale parabolique. *Nou. Mèm. De l'Acad. Des Sci. Bruxelles*, II, 169-202.

De Paolis, R. (1884). *Elementi di geometria*. Torino: Loescher.

Fehr H. (1911). Compte rendu du congrès de Milan. *L'Enseignement Mathématique*, 13, 437-511. A. II. — La question de la fusion des différentes branches mathématiques dans l'enseignement moyen. 468 – 470.

Henrici J., & Treutlein P. (1881-1883). *Lehrbuch der Elementar-geometrie*, 3 v. Leipzig: B.G. Teubner.

Hilbert, D. (1899). Grundlagen der Geometrie.Leipzig: Teubner.

Klein, F. (2016). *Elementary Mathematics from a Higher Standpoint*. Volume II: Geometry. Berlin/Heidelberg: Spinger.

Kuno Fladt (1928). Elementargeometrie: Der Stoff bis zur Untersekunda. Tübingen: Teubner.

Lazzeri, G., & Bassani, A. (1891). *Elementi di geometria*: libro di testo per la R. Accademia Navale. Livorno: Giusti (2d ed. 1898).

Méray, C. (1874/1903). *Nouveaux éléments de géométrie*. Paris: Savy.

Méray, C. (1904). Justification des procédés et de l'ordonnance des nouveaux Éléments de géométrie 1, *L'Enseignement Mathématique*, 6, 89 – 123.

Monge, G. (1799). *Géométrie descriptive*. Paris: Beaudoin.

Monge, G. (1847). *Géométrie descriptive*. Paris: Bachelier.

Scheffers, G., & Kramer,W. (1925). *Leitfaden der darstellenden und räumlichen Geometrie*. I Teil für Untertertia bis Untersekunda. Leipzig: Quelle & Meyer.

ver Eecke, P. (1963). *Les coniques d'Apollonius de Perge*. Paris: Librairie Albert Blanchard.

THE 'NEW MATH' IN MATHEMATICS TEACHERS TRAINING IN PORTUGAL (1957-1969)

Mária ALMEIDA
UIED Agrupamento de Escolas de Casquilhos
ajs.mcr.almeida@gmail.com

ABSTRACT

The introduction of the "new math" and teacher training are very interesting topics in the newer history of mathematics education. The "New Math" Reform was curricular change in secondary school mathematics that occurred in the early sixties. It spread through many countries attempting major changes in content and methods for teaching mathematics. In Portugal, the reform measures introduced in the 1960's were linked to economic considerations and the new initiatives in mathematical education were aimed at the more able senior student. Some of the new contents were modern algebra, sets, relations and vector calculus. Most of the mathematics teachers were not acquainted with these contents so they were not ready to teach them. This paper focuses on the training for secondary school (liceus) mathematics teachers, in Portugal, and aims to contribute to deepen our knowledge on this topic. We will briefly characterize the teacher education system, in the second half of the1950s and during the1960s. We attempt to clarify in what way the new ideas for the teaching of mathematics were discussed and being developed on teacher training courses held in Liceu Normal de D. Manuel II, between 1957 and 1963. Similarly, we will characterize the in-service training for teachers put in place during the Modern Mathematics' reform, in Portugal. The documents related to the mathematics teachers training in Liceu Normal de D. Manuel II, show that the trainees had become acquainted with the new math ideas, either on the application of new contents as on changes introduced in teaching practices. Regarding the in-service training courses, we have noticed that its main purpose was to improve the level of preparation of mathematics teachers on the new concepts and contents that they would be requested to give on a next future. The teacher training courses held at the Liceus Normais and the continuous in-service training for mathematics teachers did not made available an enough number of mathematically qualified teachers on "new mathematics". As the new material was presented to a much more diverse range of pupils, teaching the more abstract new ideas, particularly to weaker students, proved to be difficult for teachers.It is hardly surprising that the attempts to reform the mathematics curriculum did not meet with total success. This paper concerns the history of Portuguese mathematics education and it is based on printed and manuscript sources, as well as interviews.

Keywords: History of Mathematics Education, Mathematics teachers education, Modern Math.

DESCRIPTION OF OLD NEPALI MATHEMATICS BOOKS AND THEIR POTENTIAL IN IMPROVING CURRENT DAY TEACHING AND LEARNING

Deepak BASYAL
University of Wisconsin-Milwaukee, College of General Studies
Department of Mathematics and Natural Sciences, West Bend, Wisconsin, USA
basyal@uwm.edu

ABSTRACT

This paper is a result of a comparative literature study of four Nepali early mathematics books from the period 1880-1935. The books used in this study are the first mathematics book written in Nepali language *Vyaktachandrika* (Arithmetic) (1884) by Gopal Pandey; the first mathematics book written in Nepali poetry, *Ankendushekhar* (Pinnacle of numbers) (1900) by Pahalman Simh Swar; *Ganitchandrachandrika* (A treatise of mathematics) (1922) by Raghunath Pant and *Shishubodha Tarangini part II* (Series of lessons for children) (1933) by Tikaram Dhananjaya. A content analysis and comparison of these four books based on seven categories recognized a few historical features that can be made useful in the teaching and learning of mathematics today.

1 Introduction

The teaching and learning of mathematics in Nepal is not a very new phenomenon. However, the formal teaching of mathematics in schools in Nepal started quite recently at the end of the nineteenth century. The start of the first formal school in 1885 for the public (Sharma, 1990), and a slow-paced increase in the number of schools thereafter motivated the production of teaching and learning materials of mathematics. A handful of mathematics books came into existence during the next 70 years, some of these books found their way to antiquarian sections of local libraries, and some of them were lost. A historical account of mathematics teaching and learning situations in Nepal can be found in these articles (Lardner, 1967; Tuladhar & Jha, 2002; Jha, Adhikari & Pant, 2006; Maskey, 2013). Readers interested to learn history of Nepal are recommended to see (Whelpton, 2005). A historical narrative of education situation in Nepal can be found in (Vaidya, Manandhar & Joshi, 1993, pp. 269-329).

Several studies support the use of old mathematics in teaching and learning of current mathematics contents and concepts. Research suggests that using history to teach mathematics generates positive attitude among students for learning mathematics (Marshall, 2000; Tözlüyurt, 2008; Goktepe & Ozdemir, 2013), motivates abstract concepts (Laubenbacher, Pengelley & Siddoway, 1994), creates interest in learning of mathematics (Tözlüyurt, 2008), and engages student in metalevel analysis (Barnett, Lodder & Pengelley, 2014). The use of original sources in the mathematics classroom has been proven beneficial to students in many aspects (Van Mannen, 1997; Jahnke et al., 2002). In spring 2011, I had an opportunity to take a graduate number theory course. This class, taught by professor David Pengelley, at New Mexico State University was based on Sophie Germain’s translated manuscript. Over the last few years, I have used primary historical source projects (Otero, 2017; Klyve, 2017) produced by TRIUMPHS (TRI, 2015) in my classrooms. These experiences made me realize the value and importance of primary historical sources in teaching and learning of mathematics. In the beginning of the twentieth century, Barwell (1913) suggested to use history of mathematics in teaching and learning of mathematics in the west, whereas Nepali mathematicians were just starting to produce mathematics books in Nepali language at the moment. In the west, the use of history of mathematics in teaching and learning was happening in the 1960s and 1970s (Fried, 2001), while Nepal was still working on its first mathematics curriculum. Nowadays, many countries in the world are advocating the use of history of mathematics in teaching and learning (Fauvel & Maanen, 2000); on the other hand, the modern curriculums of Nepal (MoE, 2014; MoE, 2009; MoE, 2007) and school sector development plans (MoE, 2016) does not even foresee the history of mathematics in teaching and learning as a possibility.

In 2017, a newspaper article (THT, 2017) reported that traditional teaching and learning techniques currently employed in the majority of Nepali institutions is a catalyst for the rapid decline in students' interest in mathematics. Moreover, it is also observed that current Nepali mathematics curriculum is formal, abstract, decontextualized and disconnected from everyday life (Luitel & Taylor, 2007; Wagle et al., 2008). The historical sources considered in this study are rich in cultural and contextual problems. These culturally based mathematics curriculum has a potential to increase student's understanding of mathematical concepts (Lipka & Adams, 2004), and possibly remove barriers to motivation and engagement (Miller & Roehrig, 2016). The growing interest in history of mathematics education research in Nepal can be realized from these two recent PhD dissertations (Acharya, 2015; Subedi, 2017), and this paper seeks to initiate a more formal discussion on the use of old Nepali primary sources in teaching and learning of mathematics in Nepali institutions. A comparative content analysis of Nepali mathematics books from the period 1880-1935 is carried out in this study in order to identify useful features that can be made beneficial in teaching and learning of mathematics today.

2 Methods

The suggestions for analyzing the printed documents are followed from (Robson, 2002). Robson suggests a content analysis of printed documents, the textbooks in this case, with the following specific steps: start with a research question, decide on sampling strategy, define the recording unit, construct categories for analysis, test the coding on samples of text and assess reliability, and carry out the analysis. With a major research question on how to identify the useful features from these sources that can be beneficial in modern day teaching and learning of mathematics, a sampling strategy is initiated. Based on published newspapers and research articles, a list of books published in between 1880-1935 was formed. A total of 17 books were recorded. Robson further suggests reducing the task to a manageable dimension from population of interest. However, in this case, the options were limited. The major challenge was to locate a sample of these books from local libraries in Nepal. To locate a sample of books published in that period, the website of local libraries and their collections were searched. Google search was not helpful. The collection of Madan Puraskaar Pustakaalaya (MPP), Keisher Pustakaalaya, and Tribhuvan University Central Library were searched. These libraries are located in Kathmandu. MPP houses a collection of old Nepali mathematics books. Ten out of seventeen books from my list is found in MPP website. A total of six books were received. The library takes months to produce a digital copy. They charge cash amount to be paid in exact change. There is always a need for a person to go there on my behalf. A few friends came forward to help me. All six books were received as a digital copy in different times. There might be other books in other libraries outside Kathmandu. I did not have easy access to those materials. Taking suggestions from Frejd (2013), I confined the data collection to those sources, which I have an easy access to from my working location in the US. After receiving the sources, I worked on another crucial step of categories construction. Robson (2002) suggested creating mutually exclusive and exhaustive categories; however, I followed suggestions from Frejd (2013) in this regard too. As Frejd suggested, I considered the curriculum contents, which are purposed in a different level of current Nepali mathematics curricula (MoE, 2014; MoE, 2009; MoE, 2007), as well as pragmatic considerations, for example, the contents must be possible to compare (Frejd, 2013). Two books out of six were not considered for this study for the following reasons. *Bhasvati* (1931) by Tikaram Dhananjaya is dropped because it has more astrology content than mathematics, and *Ganitasagara* (1921) by Gangaprasad Shrestha is not considered because it only lists questions and answers throughout the book. It was not easy to compare this book with other books based on the seven categories discussed below and shown in Table 3.1. This study attempts to answer the following specific questions: What does the preface say? What does the table of contents look like? What is the structure of the textbook? Are there any definitions/procedures explained? Are there any story problems? What is the presentation style of the book? How do the authors treat zero? Are there any answers/solutions provided?

3 Results and Discussion

A brief discussion about the books and their authors is provided in this section. This section also attempts to answer the questions considered in each category.

3.1. Books and Authors

The first book *Vyaktachandrika* (Arithmetic) was originally published in year of Vikram known as Vikram Samvat 1940 which is equivalent to 1883/1884 AD. A digital copy of second edition of *Vyaktachandrika* was received. Record shows that this book was in demand for about 30 years, and a several editions of the book were produced. The second edition was published in 1895, third edition in Hindi was published in 1907, and the fourth edition was published in 1914 (Maskey, 2013). The second edition considered for this study was published by Nirnayasangara Chhapakhana located in Mumbai, India. The front page of the book has a rubber stamp mark ‘Kamalko Nepali Samgraha’ indicating that the book was collected in the Nepali collections of Kamal. Late Kamal Mani Dixit was the founder of the Madan Puraskaar Pustakaalaya. It is not clear how this book landed in Kamal’s collection.

Gopal Pandey (?1847-1921), the author of the book Vyaktachandrika, was a teacher at Sanskrit *pathasala* located in Kathmandu for more than four decades. He was taught by Bapudeva Shastri (1821-1900), a prominent mathematician and astronomer in India. Pandey has written several other books. Interested readers are suggested to see (Panta, 1980) for more information.

The second book *Ankendushekhara* (Pinnacle of numbers) was written in the Falgun month of Vikram Samvat 1956 on Tuesday, which is equivalent to the beginning of 1900 AD. The date was given in page six in the form of shloka. The copy from MPP has a handwritten name Prof. SR Pant, who is a former department head of the Central Department of Mathematics at Tribhuvan University. It appears that the book was in his possession at some point. The second edition of the book was published in 1953 AD by Sarba Hitaishi Company located in Banaras, India. This book was quite popular in the first decade of 20th century Nepal and India (Subba, Sinha, Nepal & Nepal, 2009).

Pahalman Simh Swar (1879-1933) is considered the first drama writer in the Nepali language. He worked for the government of Nepal for some time and moved to India. He had written several books on drama, poetry, story and spiritual writings. More on his life and work can be found in this biographical publication (Swar, 1982).

Pandey (1884) Swar (1900) Pant (1922) Dhananjaya (1933)

Figure 3.1 : The cover pages of the books

The third book *Ganitachandrachandrika* (A treatise of mathematics) was published in 1922 AD according to the MPP website. The digital copy received is the second edition of part I, which was published by Bhagawati Press, Kathmandu, Nepal in 1933 AD. The book was written by Raghunath Pant.

It was not possible for me to find any relevant sources to write biographical information about Raghunath Pant, the author of the book *Ganitachandrachandrika*.

The fourth book *Shishubodha Tarangini Part II* (Series of lessons for children) was believed to be published in 1991 Vikram Samvat which is equivalent to 1933/34 AD. There is a handwritten date 1991 in the cover page of the book. There is no other printed publication date provided for this book. The purported author of the book is Chandrakala Dhananjaya, who did not receive any formal education in her lifetime. Later, when she was interviewed by Modnath Prashrit she revealed that she did not write the book (Prashrit, 2002). She also confirmed that the book was written by her husband Tikaram Dhananjaya. I found this book in MPP, and it is probably the one out of three-surviving copies of the book. Part I seems to be lost. Although, the part III was in the plan for writing, it is possible that the part III was never written because of sudden death of Tikaram at the age of 26. Interested readers can find more on this in Basyal (2015).

Tikaram Dhananjaya (1909-1936) was a poet, mathematician, translator, astrologer, commentator, writer, grammarian and an inventor of a sign language. In his short life of about 26 years he produced a good number of books on different area of knowledge. Tikaram earned his high school degree from Banaras, India. After returning back from Banaras, he worked as a mathematics and astrology teacher in a local school.

3.2. Categories

In this section, the categories being considered are discussed and commented on what is observed in each category. Table 3.1 provides a summary of the results. Now, each category will be discussed in detail with examples and explanations.

Table 3.1: Summary of findings

Books / Categories	Arithmetic (1884) by Gopal Pandey	Pinnacle of Numbers (1900) by Pahalman Swar	A treatise of mathematics (1922) by Raghunath Pant	Series of lessons for children (1933) by Tikaram Dhananjaya
Preface	✓	✓	✗	✗
Table of contents	✓	✗	✓	✓
Definitions/Procedures	✓	✓	✓	✓
Story problems	✓	✓	✓	✓
Presentation style	Normal 𝄞 (Sanskrit)	𝄞 (Nepali)	Normal	𝄞 (Nepali)
Treatment of zero	Eight operations with zero	Placeholder	Placeholder	Eight operations with zero
Answer/Solutions	Few detailed worked out examples	Few worked out examples	A long list of problems with answers	Few worked out examples

What does the preface say?

Only two books provide the preface. In *Arithmetic*, it is stated that the first edition was well received by the readers and all 500 copies were sold soon after it was published. It does not state the number of copies published in the second edition.

My translation of the preface in Gopal Pandey's book is given here:

> The calculations we make with one, two, three and more such numbers, is called Arithmetic. The European learned Arithmetic from Arabian, and Arabian learned from Indians and they still call it 'Hindu-mathematics'. …
> […] People in various countries have written arithmetic in their own languages to teach their children. People might think that Gorkhali does not know mathematics because we still have no arithmetic written in Parvatiya language. Therefore, I am writing this book with my limited knowledge to prevent the situation. (Pandey, 1985, p. 1)

In the quote above *Gorkhali* and *Parvatiya* both refers to Nepali. In the preface, Pandey further goes on to indicate that the most contents for his books are taken from Bhaskaracharya's Lilavati. He also cites his teacher Bapudeva Shastri's work on page 43 for the method 'casting out by eleven'. As promised in the preface, Pandey tried to simplify the rules and provided thoughtful and useful examples. He also hoped that the book will be much easier for learners to follow as the book is written in local language.

The preface in Pahalman Swar's book roughly translates to:

> Dear gentleman, for a long time, I had a desire to write mathematical shlokas in Nepali vernacular. I have realized that it would make a special contribution to write

mathematical shlokas. Here, I present a few mathematics in shloka for gentleman. As my readers are aware of the fact, writing anything new is not an easy task, so writing mathematics is not easy either. With my limited knowledge, I am taking this daunting task. (Swar, 1953, pp. 2-3)

After reading these prefaces one can conclude that these books were written to fulfill the lack of mathematical resources in Nepali language. Note that Pandey managed to credit his sources; however Swar did not mention anything about his sources.

What does the table of contents look like?

Three books provide a table of contents: The content in *Arithmetic* is given in three parts, the third part being geometry. The *Pinnacle of Numbers* does not provide a table of contents; however, a table of contents is created by observing the content given in book. The order of contents is written as it was presented in the book. The *Series of Lessons for Children* provides 63 titles in mathematics and astrology in the table of contents. However, only mathematics titles are translated here.

A closer look at Table 3.2 suggests that these books were written to teach the basic rules and word problems related to arithmetic. The addition, subtraction, multiplication and division of whole numbers are presented in every book. The rule of three, five, seven etcetera, compound interest, mixtures, barter and series were also common topics of the time. These basic rules of arithmetic were used for illustrating how some real-life problems can be solved using these arithmetic tools. The work-rate problems, interest calculation, and conversion of local money and land units are also considered. These topics given in Table 3.2 can be identified in current Nepali curriculums (MoE, 2014; MoE, 2009; MoE, 2007), so there is an opportunity to create guided classroom worksheets and projects and use them in the classroom to test the usability of these contents.

Table 3.2: Table of Contents of the books

Arithmetic (1884)	**Part I:** Numeration and pronunciations of numbers, addition, subtraction, multiplication, division , exponentiation, taking roots, greatest common divisor, least common multiple, dividing by a same number, addition of fractions, [subtraction] of fractions, addition of fractions with its parts, multiplying fractions, compound fractions, addition and subtraction of a number with its parts, dividing by fraction, division of fractions, exponentiation of fractions, taking roots of fractions, eight rules with zero, rule of three, rule of five, seven and etcetera, barter, mixtures, working backwards, supposition, double false position, using sum and difference to find the two numbers, quadratic [Diophantine] problems **Part II** Method for checking if multiplication is correct, addition of decimals, subtraction [of decimals], multiplication of decimals, division [of decimals], exponentiation of decimals, repeating decimals, taking roots of decimals, taking root of fractions, taking cube root [of fractions], ratio, adding whole in ratio, **Part III** Definitions, area and volume for simple shapes, triangles, rectangles, ratio with areas, circle, length measure.
Pinnacle of Numbers (1900)	Invocation, definitions, numeration, local values, addition, subtraction, multiplication, division, dividing in five, four, and seven equal parts, money exchange, rule of three, inverse rule of three, rule of five, seven, nine and eleven, compound interest, mixtures, shadow measures, series, miscellaneous problems
A treatise of mathematics (1922)	Definitions, number writing, pronunciations of numbers, local values, addition, addition of whole numbers, subtraction, subtraction of whole numbers, multiplication, multiplication of whole numbers, division, division of whole numbers, role of grouping symbols, several topics, concerning addition and subtraction, concerning multiplication and division, working backwards, concerning addition and subtraction, money, rate and miscellaneous, examples workout and various question, examples, answers
Series of lessons for children (1933)	Invocation, definitions, eight fundamental rules of arithmetic concerning whole numbers, greatest common divisor, least common multiple, eight fundamental rules of arithmetic concerning fractional numbers, multiplying fractions, adding or subtracting a fraction and a whole number, Increasing and decreasing a fraction by a fractional amount, rule for addition and subtraction of fractions, eight fundamental rules of arithmetic concerning zero, eight fundamental rules of arithmetic concerning decimals, working backwards, method for checking if multiplication is correct, method of supposition, using the sum and the difference of two numbers to find the numbers, the rule of three, simple interest, Investigation of mixtures, series

Are there any definitions/ procedures explained?

All books show a procedure to write numbers. The mathematical rules and working procedures are explained either in prose or in verse in all books.

(२) दुइ भैंसीले एक खेतको घास पांच दिनमा र तीन गाइले सात दिनमा खाइ सकतछन् भने एक भैंसी र एक गाइले मिलेर खाया भने कति दिनमा सकलान् ? उत्तर ६$\frac{२४}{३१}$ दिन.

(३) एक तलाउमा पानी आउन्या चारौटा धारा रहेछन् तिनमा एक लाइ छाडदा दुइ घडिमा, दोसरो लाइ छाडदा तीन घडिमा, तेसरो लाइ छाडदा चार घडिमा, चौथोलाइ छाडदा पांच घडिमा, त्यो तलाउ भरीदो रहेछ भने सबै लाइ एकहि पल्ट छाडदा कति बेरमा भरियला ? उत्तर ४६$\frac{५८}{७७}$ पल्.

(४) एक तलाउमा एक धारो लाइ छाडदा एक घडिमा, दोसरो लाइ छाडदा दुइ घडिमा, भरिदो रहेछ तेसको निकास लाइ खोलि दिदा तीन घडिमा खालि हुदो रहेछ भने दुइ धारा र निकास लाइ एकहि पल्ट छाडदा कति बेरमा भरियला ? उत्तर ५१$\frac{३}{७}$ पल.

(७) एक पोखरीमा तीन ओटा पानी आउन्या कुला रहेछन् तिनमा पहिला र दोसरा लाइ छाडदा आठ घंटामा, अरु पहिला र तेसरा लाइ छाडदा बारह घंटामा, अरु दोसरा र तेसरा लाइ छाडदा पन्द्रह घंटामा, त्यो पोखरी भरीदो रहेछ भने तीन ओटा एकहि पल्ट छाडदा कति बेरमा भरियला ?

यसमा ६ कुलाले $\frac{४०}{११}$ तव तीन कुलाले $\frac{८०}{११}$ घंटामा

A few contextual examples in Pandey (1884)

(३) मैले एउटा मानिसलाई घर किन्न २५०१ रुपिञा र ४७५ रुपिञा दिएँ। अघि त्यस्ले घरखर्चलाई २५० रुपिञा पनी लगेको थियो। हाल त्यस्ले आफुले लिएका सब रुपिञा र त्यस्को व्याज ६२ रुपिञा समेत तिर्न ल्यायो भने, त्यस् सँग मैले कति रुपिञा बुझिलिनुपर्ला ?

(४) एउटा धनीका पहिलो बंकमा १२५००० रुपिञा, दोस्रो बंकमा पहिलोमा भन्दा ३५००० रुपिञा बढता, तेस्रो बंकमा दोस्रोमा भन्दा ४५५०० रुपिञा बढ्ता हालेका रहेछन्। त्यस्का साथमा ७२०७५ रुपिञा पनी रहेछ भने, त्यो धनी कति धनको मालिक रहेछ ?

(५) १६३८ सालमा जन्मेको मानिस कुन् सालमा ७२ वर्षको होला?

(६) एउटा थैलीमा ४६८ कम्पनी, १२५० मोहर, ५६१ पैसा र ६५५ कम्पनीपैसा मिसिएकाछ भने, त्यस थैलीमा थान गन्ती कति थान सिक्काहुनुपर्दछ?

(७) एउटा महाजनले ५०० गरीबलाई २ पैसाका दरले, १५५ गरीबलाई ३ पैसाका दरले र ८१६ गरीबलाई १ पैसाका दरले दियो भने जम्मा त्यस्का कति पैसा खर्च भएछन्? कति जना गरीबहरूले पाएछन् ?

A few contextual examples in Pant (1922)

Figure 3.2: A few contextual story problems

In Table 3.5, I provide a translation of mathematical procedure known as Rule of Three. The working rules with zero are provided in Table 3.4. In Table 3.7, the definition of Greatest Common Divisor and Least Common Multiple from both old book and modern book is provided. These procedures and definitions can be compared with the definition given in modern curriculums to gain some pedagogical insights.

Table 3.3: Translation of contextual problems in Figure 3.2

Pandey (1884)	Pant (1922)
(2) Two buffalos can finish eating grass of a field in five days. Two cows can finish in seven days. How many days does it take for a cow and a buffalo to finish eating the grass? (3) There were four taps connecting to a pond. First tap can fill the pond in two ghadi, second one in three ghadi, third one in four ghadi, fourth one in five ghadi. If all taps are open at a time, how long will it take to fill the pond?	(3) I lend 2501 rupiya and 475 rupiya to a person to buy a house. He also received 250 rupiya for household expenses. Now he is paying back all he received and 12 extra rupiya for an interest. How much should I get from him? (4) One rich person has 125000 rupiya in the first bank, 35000 more than that of in the first bank is in the second, 45500 more than that of second bank is in the third bank. If he have 72075 rupiya with him. How much money does this rich person have?

Are there any story problems?

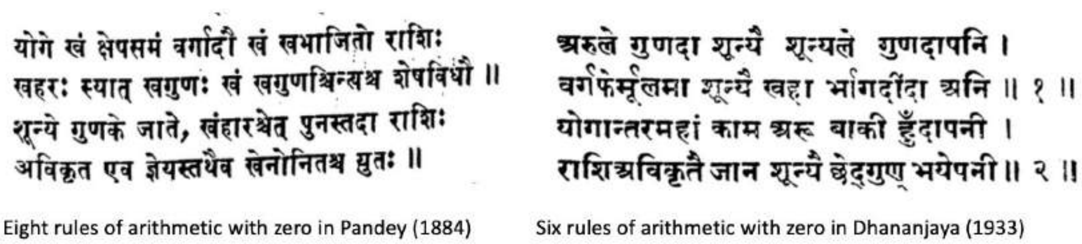
Yes. A few contextual story problems can be found in each book, the third one being the richest in the contextual problems. A few contextual story problems are given in Figure 3.2. A translation of the problems in Figure 3.2 is provided in Table 3.3. Studying how these problems are similar and different mathematically in the current context could be a lesson for today's students. These problems are comparable with the problems given in modern textbooks. A sample of problems in modern textbooks is given in Table 3.6.

Table 3.3 provides a rough translation of first two problems from each book given in Figure 3.2. Pandey's problems are taken from (Pandey, 1895, pp. 34-35), and Pant's problems are taken from (Pant, 1932, p.22).

Looking at these representative problems from these books, one can get a sense of market products, rates, measurement and monetary units of that time. These problems where one talks about buffalos, *rupiya*, *ghadi* and the person with less than 1 million Nepali rupees would be considered a rich person are all foreign and old-fashioned to today's student. Similar problems can be seen in modern textbooks. Table 3.66 shows a few representative problems from textbooks used today in Nepali schools in various grades. These books are mostly used in private schools where the language of instruction is English. These books are also written in English.

What is the presentation style of the book?

Arithmetic is written in a mixture of Sanskrit shloka (verse) and Nepali prose, which consists of topics on arithmetic, algebra and geometry. The *Pinnacle of Numbers* and the *Series of Lessons for Children* are rich in Nepali shloka, whereas the *A Treatise in Mathematics* is rich in word problems. The *Series of Lessons for Children* book would be a great motivational and mnemonic resource as it lists 137 shlokas to explain mathematical rules, procedures and problems.


योगे खं क्षेपसमं वर्गादौ खं खभाजितो राशिः
खहरः स्यात् खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ ॥
शून्ये गुणके जाते, खंहारश्चेत् पुनस्तदा राशिः
अविकृत एव ज्ञेयस्तथैव खेनोनितश्च युतः ॥

अरुले गुणदा शून्यै शून्यले गुणदापनि ।
वर्गफेर्मूलमा शून्यै खहा भागदींदा अनि ॥ १ ॥
योगान्तरमहां काम अरू बाकी हुँदापनी ।
राशिअविकृतै जान शून्यै छेदगुण भयेपनी ॥ २ ॥

Eight rules of arithmetic with zero in Pandey (1884)

Six rules of arithmetic with zero in Dhananjaya (1933)

Figure 3.3: Rules in Shlokas

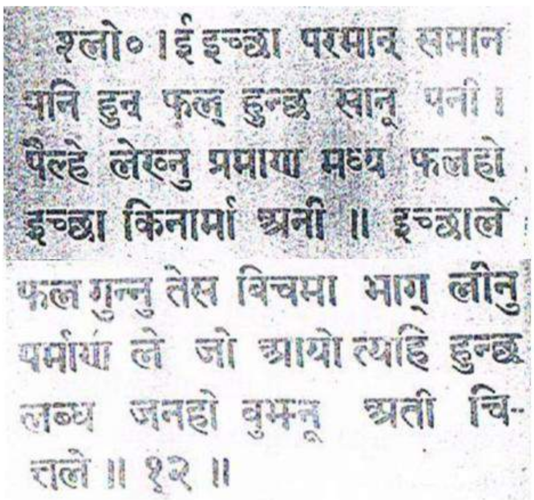
How do the authors treat zero?

Interesting treatment of zero can be found in (Pandey, 1884) and (Dhananjaya, 1933). Figure 3.3 provides a screenshot of rules with zero given in Pandey and Dhananjaya. Pandey copied the Sanskrit shloka from Lilavati as it is. Dhananjaya gave a completely new shloka in Nepali to explain these rules. One change I noticed in Nepali shloka is that Dhananjaya removed the part where it talks about addition and subtraction with zero. Therefore, the rule reduced down to six rules from eight rules given in Sanskrit shloka. Dhananjaya provides examples in which addition and subtraction to and from zero is necessary. Table 3.4 provides a translation of the Sanskrit shloka taken from (Patwardhan, Naimpally & Singh, 2001, p. 47). My translation of Nepali shloka in Figure 3.3 is provided in the right side of Table 3.4.

It is fair to say that Dhananjaya was translating Sanskrit shloka into Nepali shloka without changing much information given in the original Sanskrit shlokas. Even the coherence of the presentation is kept intact. Out of 137 shlokas given in *Series of Lessons for Children,* a good number of shlokas seem to be a direct translation of Sanskrit shlokas in Lilavati. Other two books considered in this study treat zero as a placeholder and do not mention the multiplication and division by zero at all.

Table 3.4: Translation of Shlokas in Figure 3.3

Pandey (1884)	Dhananjaya (1933)
If zero is added to a number, the result is the same number; the square etc. (i.e., square, square-root, cube, cube-root) of zero is zero; any (non-zero) number divided by zero is *khahara*, i.e., infinite; the product of a number and zero is zero. (If in some mathematical calculations, multiplication and division by zero are likely to occur frequently then, though a number multiplied zero is zero,) one should maintain the form of multiplicand and multiplier zero in rest of the operations (until the final operation is reached). This is because if a number is multiplied by zero and divided by zero then the result is the (former) number. (Patwardhan, Naimpally & Singh, 2001, p. 47)	Zero multiplied by any [number] is zero. [Any number] multiplied by zero is zero. The square and square root of zero is zero. [Any number] divided by zero is infinity. If there is more work to be done, the zero divided by zero remains unchanged.



श्लो० । ई इच्छा परमान् समान
भनि हुन्न फल् हुन्छ सानू पनी ।
पैल्हे लेख्नु प्रमाण मध्य फलहो
इच्छा किनार्मा अनी ॥ इच्छाले
फल गुन्नु तेस बिचमा भाग् लीनु
पर्माणा ले जो आयो त्यहि हुन्छ
लब्ध जनहो बुझ्नू अती चि-
त्तले ॥ १२ ॥

Rule of Three in Swar (1900)

श्लोकः । पान्सौ दारिमको
पचास रुपिया सौ दारिमैं को
कती ? हज्जार् आँप पचासिमा
बिकि गयो चार्सौको दाम् हो
कती ? पच्चिस् द्रव्यदिया पचासि
बढिया मिल्छन् अनारै पनी ।
दशद्रव्यै दिइयो यदी भनिभन्या
मिल्लन् कती ती पनी ॥ १३ ॥

Example related to Rule of Three in Swar (1900)

Figure 3.4: Rule of Three and Examples in Swar (1900)

Table 3.5 provides a translation of shlokas given in Figure 3.4. The rule of three on the left side of Figure 3.4 and the example related to rule of three given in right side of Figure 3.4 are both written in shloka (Swar, 1953, pp. 51-54).

The given words *Ichhya*, *praman*, *phala* and *labdha* translates to desire or the requisition, scale, fruit, and the desired result. The rule is essentially saying if we were to calculate a desired result given three things, we need to multiply desire by fruit and divide by the scale.

An equivalent rule from current grade 8 book reads:

In direct variation, the value of unit quantity is obtained by dividing the value of any quantity by that quantity and the value of any quantity is obtained by multiplying the value of unit quantity by that quantity. (Shrestha, Karki, & Bhandari, 2015d, p.60)

Table 3.5: Translation of shlokas in Figure 3.4

Rule of Three in Swar	Example related to Rule of Three in Swar
Ichhya, praman are of similar type. *Phala* is the smaller one. First we write the *praman* [on the left side], *phala* in the middle, and *ichhya* on the edge. Multiply *phala* by *ichhya* and divide by *praman*. The result is a *labdha*.	If five hundred pomegranates cost fifty *rupiya*, how much does it cost for one hundred pomegranates? If one thousand apples are sold in eighty-five [*rupiya*], how much does it cost for four hundred [apples]? Twenty-five *drabya* can buy eighty-five good pomegranates, then how many pomegranate does ten *drabya* buy?

Table 3.6: Problems in current Nepali mathematics textbooks

A few exercise problems in current mathematics books in various grades
Deepak had Rs 2645. Sony gave him Rs 4675. How much money does he have altogether? (Shrestha, Karki, & Bhandari, 2015, p.67) The monthly expenditure of Khadka family is Rs 12500 on food, Rs 2400 on fuel, Rs 6500 on education and Rs 9475 on miscellaneous. Find the total expenditure of the family in a month. (Shrestha, Karki, & Bhandari, 2015, p.67) Hari earned Rs 85,250 in a year and spend Rs 78,375. How much money did he save in the year? (Shrestha, Karki, & Bhandari, 2015a, p.59) If 6 men can finish a piece of work in 12 days, in how many days would 18 men finish the same work? (Shrestha, Karki, & Bhandari, 2015b, p.116) In a barrack, 250 soldiers have food enough for 40 days. How long would the food enough for only one soldier? (Shrestha, Karki, & Bhandari, 2015c, p.102) Tap A can fill a tank in 3 hours and Tap B can fill it in 6 hours. If they are open together, in how many hours the tank can be filled completely? (Shrestha, Karki, & Bhandari, 2015d, p.67)

Are there any answers/solutions provided?

Pandey and Pant provide a list of practice problems at the end of each chapter with answers, whereas, other two books provide mostly the worked-out problems.

Table 3.7: Definitions of GCD and LCM from old and modern books

A number which will divide two or more numbers evenly is called a common divisor [of the numbers]. The largest of such divisors is called the greatest common divisor. (Dhananjaya, 1933, p.9)
The greatest number that divides the given numbers without remainder is called H.C.F of those numbers. (Shrestha, Karki, & Bhandari, 2015b, p.42)
By reducing [dividing] all the numbers [by some numbers] until there is no remainder, keep the numbers separately. The product of these numbers with the numbers [which appear to be remainders] is called the least common multiple by mathematicians. (Dhananjaya, 1933, p.12)
Lowest Common Multiple (L.C.M) of two or more numbers is the smallest number that can be divided by those numbers without leaving any remainder. (Shrestha, Karki, & Bhandari, 2015b, p.45)

4 Concluding Remarks

These considered categories and identified historical features such as: the contextual problems, old fashioned units of measurements, presentation style of the book and problems and solutions may be a useful resource for current teaching and learning of mathematics. These may be directly used in the classroom, in teacher trainings, and in textbook writing. Avital's (1995) suggestions to replace current problems with old ones might improve instruction and learning of mathematics. Empirical research is needed to confirm the usefulness of these sources and features, which is clearly a future direction. However, as discussed in Jankvist (2009) these primary historical sources can be used both as 'history as a tool' and 'history as a goal'. As suggested in Frejd (2013) these historical contexts can be used as a 'history as a tool' to assist students and teachers in the learning and teaching of mathematics, and 'history as a goal' to show that mathematics is a part of evolution of society and that mathematics has been developed and changed over time. As noted by (Schubring, 2011), these features can be helpful in teacher education to provide the meta-knowledge of mathematics so that it will eventually help increase the meta-knowledge in teacher and student. Glaz and Liang (2009) observed an improvement in student learning by using poetry as a pedagogical tool in a mathematics classroom, I do see an ample opportunity to use these shlokas in classrooms to enhance the teaching and learning of mathematics for both teachers and students alike. Moreover, these primary sources can be a resource for multidisciplinary study such as literature, linguistics, poetry, sociology, history, and Sanskrit study. Many topics observed in Table 3.2 can be identified in modern day primary schools' curriculums and textbooks (MoE, 2007; MoE, 2009; MoE, 2014; Shrestha, Karki, & Bhandari, 2015, 2015a, 2015b, 2015c, 2015d), and the topics such as series, compound interest, false position, double false position and topics in geometry are present in current school curriculum, so there is an opportunity of using these contents in different levels of school today. The creation of guided task based projects similar to TRIUMPHS projects, and use it in an empirical study to test the usability and usefulness of these resources is my future direction.

Acknowledgement: I would like to express my sincere appreciation to two anonymous reviewers for their suggestions and comments for the improvement to this paper. I like to thank Allen Learst for proofreading the earlier draft of the paper. Also, I thank Ghanshyam Bhatt for thoroughly reading the paper and providing much needed suggestions for improvement in the latest draft.

REFERENCES

Acharya, E. R. (2015). *Naya Raj Pantaka Ganitiya Kritiharuko Adhyayan*. Doctoral dissertation, Nepal Sanskrit University.

Avital, S. (1995). History of mathematics can help improve instruction and learning. In F. Swetz, J. Fauvel, O. Bekken, B. Johansson, & V. Katz (Eds.) (1995). *Learn from the masters!* (pp. 3-12). Washington, DC:The Mathematical Association of America.

Barnett, J. H., Lodder, J., & Pengelley, D. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education*, *23*(1), 7–27.

Barwell, M. E. (1913). The advisability of including some instruction in the school course on the history of mathematics. *The mathematical gazette*, *7*(104), 72-79.

Basyal, D. (2015). *A 1933 Nepali mathematics and astrology book Śiśubodha Taraṅgiṇī II: Translation and commentary on mathematics chapters*. Doctoral dissertation, New Mexico State University.

Dhananjaya, T. (1933). *Shishubodha Tarangini (part II).* Benares: Gorkha Pustakalaya.

Fauvel, J., & van Maanen, J. (Eds.). (2000). *History in mathematics education: The ICMI Study*, New ICMI Study Series, vol. 6. Dordrecht: Kluwer.

Frejd, P. (2013). Old algebra textbooks: a resource for modern teaching. *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, *28*(1), 25-36.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391–408.

Glaz, S., & Liang, S. (2009). Modelling with poetry in an introductory college algebra course and beyond. *Journal of Mathematics and the Arts*, *3*(3), 123-133.

Goktepe, S., & Ozdemir, A. S. (2013). An Example of Using History of Mathematics in Classes. *European Journal of Science and Mathematics Education*, *1*(3), 125-136.

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., ... & Weeks, C. (2002). The use of original sources in the mathematics classroom. In J. Fauvel, & J. van Maanen, J. (Eds.).*History in mathematics education: The ICMI Study* (pp. 291-328). Dordrecht: Springer.

Jankvist, U. T. (2009). A categorization of the “whys” and “hows” of using history in mathematics education. *Educational studies in Mathematics*, *71*(3), 235-261.

Jha, K., Adhikary, P. R., & Pant, S. R. (2006). A history of Mathematical Sciences in Nepal. *Kathmandu University Journal of Science, Engineering and Technology*, *2*(1), 1-6. https://www.academia.edu/3753852/A_HISTORY_OF_MATHEMATICAL_SCIENCES_IN_NEPAL (accessed 18 August 2019)

Klyve, D. (2017). *The Derivatives of the Sine and Cosine Functions*. https://digitalcommons.ursinus.edu/triumphs_calculus/1/ (accessed 2 March 2019).

Lardner, T. J. (1967). Mathematics in Nepal. *The American Mathematical Monthly*, *74*(1), 67-72.

Laubenbacher, R., Pengelley, D., & Siddoway, M. (1994). Recovering motivation in mathematics: Teaching with original sources. *Undergraduate Mathematics Education Trends*, *6*, 1-7.

Lipka, J., & Adams, B. (2004). *Culturally Based Math Education as a Way to Improve Alaska Native Students' Math Performance*. Appalachian Collaborative Center for Learning.

Luitel, B. C., & Taylor, P. C. (2007). The shanai, the pseudosphere and other imaginings: Envisioning culturally contextualised mathematics education. *Cultural Studies of Science Education*, *2*(3), 621-655.

Marshall, G. L. (2000). *Using history of mathematics to improve secondary students' attitudes toward*

mathematics. Doctoral dissertation, Illinois State University.
Maskey, S. M. (2013). *Mathematics in Nepal*. https://issuu.com/danbdr.budha/docs/final_sau_paper_mathematics_in_nepa. (accessed 3 November 2018).
Miller, B. G., & Roehrig, G. (2016). Indigenous cultural contexts for STEM experiences: snow snakes' impact on students and the community. *Cultural Studies of Science Education*, *13*(1), 31-58.
MoE (2007). *National Curriculum Framework for School Education in Nepal*. Kathmandu: Ministry of Education and Sports.
MoE (2009). *Primary education curriculum*. Kathmandu: Government of Nepal, Ministry of Education.
MoE (2014). *Secondary education curriculum*. Kathmandu: Government of Nepal, Ministry of Education.
MoE (2016). *School Sector Development Plan Nepal, 2016/17–2022/23*. Kathmandu: Government of Nepal, Ministry of Education.
Otero, D. (2017). A Genetic Context for Understanding the Trigonometric Functions. https://digitalcommons.ursinus.edu/triumphs_precalc/1/ (accessed 2 March 2019).
Patwardhan, K. S., Naimpally, S. A., & Singh, S. L., (2001). *Lilavati of Bhaskaracharya: A Treatise of Mathematics of Vedic Tradition with rationale in terms of modern mathematics largely based on N. H. Phadke's Marathi translation of Lilavati*. Delhi: Motilal Banarsidass publishers.
Pandey, G. (1895). *Vyakta Chandrika.* Mumbai: Nirnayasagara Chapakhana.
Pant, R. (1932). *Ganitachandrachandrika (pahilo bhaga): Arithmetic in Gorkhali.* Kathmandu: Bhagawati Press.
Panta, N. (1980). *Pandita Gopala Pande ra unako ghanamula lyaune riti*. Kathmandu: Rajakiya Prajna-Pratishthana.
Prashrit, M. (2002). *Unnayana* [special edition] In Adhikari, Acyutaramana (Ed.). Kathmandu: Unnayana Prakasana.
Robson, C. (2002). *Real world research*. 2nd edition. Malden: Blackwell Publishing.
Schubring, G. (2011). Conceptions for relating the evolution of mathematical concepts to mathematics learning—epistemology, history, and semiotics interacting. *Educational Studies in Mathematics*, *77*(1), 79-104.
Sharma, G. (1990). The impact of education during the Rana period in Nepal. *Himalaya, the Journal of the Association for Nepal and Himalayan Studies,* *10*(2), 3-7. https://digitalcommons.macalester.edu/cgi/viewcontent.cgi?referer=https://scholar.google.com/&httpsredir=1&article=1255&context=himalaya (accessed 18 August 2019).
Shrestha, S., Karki, S., & Bhandari, R. (2015). *Prasiddha's Functional Mathematics Book-3.* Kathmandu: Prasiddha Prakashan.
Shrestha, S., Karki, S., & Bhandari, R. (2015a). *Prasiddha's Functional Mathematics Book-4.* Kathmandu: Prasiddha Prakashan.
Shrestha, S., Karki, S., & Bhandari, R. (2015b). *Prasiddha's Functional Mathematics Book-6.* Kathmandu: Prasiddha Prakashan.
Shrestha, S., Karki, S., & Bhandari, R. (2015c). *Prasiddha's Functional Mathematics Book-7.* Kathmandu: Prasiddha Prakashan.
Shrestha, S., Karki, S., & Bhandari, R. (2015d). *Prasiddha's Functional Mathematics Book-8.* Kathmandu: Prasiddha Prakashan.
Subba, T.B., Sinha, A. C., Nepal, G. S. & Nepal, D. R. (Eds.). (2009). *Indian Nepalis: issues and perspectives.* New Delhi: Concept Publishing Company.
Subedi, N. (2017). *The Development of Numeral System in Ancient Nepal*. Doctoral dissertation, Nepal Sanskrit University.
Swar, P. S. (1953). *Ankendusekhara.* Benares City: Malik Pustakalaya.

Swar, P. S. (1982). *Pahalamanasimha Svara smirti grantha.* Kasthamandap: Sajha Prakasana.
THT (2017). *Student losing interest in science, mathematics in Nepal.* https://thehimalayantimes.com/kathmandu/students-declining-interest-science-mathematics-lamented/ (accessed 4 November 2018).
Tözluyurt, E. (2008). *The perceptions of senior high students regarding the lessons, in which activities chosen from history of mathematics are used on the subject of numbers learning area*. Master's Thesis, Gazi University.
TRI (2015). *TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS)*. http://webpages.ursinus.edu/nscoville/TRIUMPHS.html (accessed 4 November 2018).
Tuladhar, B. M., & Jha, K. (2002). A brief report on mathematics education in Nepal. *Ganita Bharati*, *24*(1-4), 160-174.
Vaidya, T. R., Manandhar, T., & Joshi S. L. (1993). *Social history of Nepal*. New Delhi: Anmol Publications.
Van Maanen, J. (1997). New maths may profit from old methods. *For the Learning of Mathematics*, *17*(2), 39-46.
Wagley, M. P., Sharma, T. N., Koirala, B. N., Ramos, S. Y., Taylor, P. C., Luitel, B. C., ... & Poudel, K. P. (2008). *Developing Culturally Contextualised Mathematics Resource Materials*: *Capturing Local Practices of Tamang and Gopali Communities*. Kathmandu: Lusha Press.
Whelpton, J. (2005). *A history of Nepal*. Cambridge: Cambridge University Press.

DEVELOPMENT OF SCHOOL ALGEBRA – A COMPARISON BETWEEN THE 1980 AND 2011 SWEDISH MATHEMATICS CURRICULA

Kajsa BRÅTING
Department of Education, Uppsala University, Sweden
kajsa.brating@edu.uu.se

ABSTRACT

In search of the reasons for Swedish students' low achievement in algebra in international evaluations, this study analyses and compares the algebraic content in the 1980 and 2011 Swedish curricula for the grades 1-9. In order to characterize the algebraic content, Blanton et al.'s (2015) five big ideas of algebra have been applied as an analytical tool. The results show that algebra is introduced earlier in 2011 compared to 1980 and that the big idea generalized arithmetic is weakly represented in both curricula. There is a high emphasis on practical and everyday mathematics in both curricula and there seems to be an increased emphasis on verbal abilities as opposed to computational skills over time.

1 Background

During the past decades, research on algebraic thinking at primary school level has gained increasing interest within the research field of mathematics education (for an overview, see Kieran, 2018). The assumption that young pupils are not capable to think algebraically has been challenged by several mathematics educators (e.g. Blanton, et al., 2015; Carraher, Schliemann, Brizuela & Earnest, 2006). Also, the idea that students' development of learning algebraic concepts would be reflected in the historical development of algebra has been questioned (Bråting & Pejlare, 2015; Schubring, 2011). In fact, recent studies reveal that it is not only possible, but also beneficial, to start working with algebraic ideas and generalizations in parallel with arithmetic from the very beginning (Mason, 2018).

This development is also visible in steering documents where several countries have revised their mathematics curricula in order to introduce students to algebraic thinking already in the early grades (NCTM, 2006). This is also the case in Sweden, which is the focus of the study reported in this paper. In Sweden, algebra has always been a part of mathematics that has caused school students major difficulties. In the international evaluation TIMSS (Trends in International Mathematics and Science Study), Swedish students' results in algebra have been below average ever since the 1960s (Hemmi et al., 2018). Even in the TIMSS test from 1995, where the overall result was the best ever for Sweden, the result in algebra was still below the international average. In the two TIMSS tests from 2007 and 2011, when the Swedish overall result decreased significantly, algebra was the topic that deteriorated the most (Yang Hansen et al., 2014). In the recent TIMSS evaluation from 2015, the Swedish overall result slightly increased, but the result in algebra remained poor (National Agency for Education, 2016).

Besides the revisions in the curriculum documents, there have been various attempts to improve school algebra teaching in Sweden through in-service training projects for teachers, and the "Algebra for All" movement ensuring that all school students are studying algebra before graduating high school. In connection with the latter, a specific textbook (see Bergsten, Häggström, & Lindberg, 1997) was compiled and used in teacher education at several universities in Sweden. However, it is not possible to discern a general positive effect of these efforts on Swedish students' learning in algebra, at least not if we consider the results in the TIMSS evaluations.

The study reported in this paper is part of an ongoing research project aiming at characterizing Swedish school algebra (Hemmi, et al., 2018). Both diachronic and synchronic studies are being conducted focusing on both formulation and realization arenas (Lindensjö & Lundgren, 2000) in order to identify the specific teaching tradition developed in Swedish school algebra (Bråting, 2015). The formulation arenas refer to steering documents and curriculum materials, and the realization arenas to schools and teachers who develop and maintain their own more or less tacit traditions. The overall purpose of the project is to find reasons for the failure to raise the quality of algebra teaching in Sweden, but also to find possible ways to improve the situation.

The study in this paper focuses on the formulation arena as it investigates Swedish curriculum documents in a historical perspective. The aim is to analyze and compare the algebraic content in the 1980 and 2011 Swedish mathematics curricula by using Blanton et al.'s (2015) so called big ideas of algebra as an analytical tool. In a longer perspective, the intention is to analyze the algebraic content in all Swedish mathematics curricula from the implementation of primary school ("grundskolan") in 1962 until today. In total, there have been five curriculum reforms in Sweden during this time period; in 1962, 1969, 1980, 1994 and 2011. An investigation of all these curricula provides valuable knowledge of how school algebra has traditionally been treated in Sweden with respect to the formulation arena. The study reported in this paper is a first step towards that goal.

2 Research on curriculum documents and school algebra in Sweden

During recent years, there has been an increased interest of research on mathematics curriculum and policy documents in Sweden with respect to the formulation arena (c.f. Boesen et al., 2014; Prytz, 2015; Bergqvist & Bergqvist, 2017). For instance, Prytz (2015) has given an overview of the Swedish mathematics curricula between the years 1850 to 2014 regarding structural aspects such as length (number of pages and words), the amount of time allocated to mathematics, and variation of mathematical topics through the years. The results show that the three curricula from 1980, 1994 and 2011 contain 50% more words compared to the curricula from 1962 and 1969. One reason is that the number of topics included in Swedish school mathematics has increased over the years and the descriptions of what it means to know that something has become more versatile. Moreover, Bråting and Österman (2017) illuminate a development from numerical and computational skills toward an increased emphasis on verbalizations and practical uses of mathematics in the Swedish school mathematics through the years.

Another study on the formulation arena is Boesen et al.'s (2014) investigation regarding mathematics teachers' response to the implementation of mathematical competency goals (see NCTM, 2000; Niss & Jensen, 2002) in the Swedish mathematics curriculum documents. The results reveal that the teachers are positive to the competences but it is difficult for the teachers to identify the meaning of the competence message by using national curriculum documents and national tests. Even though the competences were introduced fifteen years ago, Boesen et al. (2014) argue that the implementation should still be viewed as an ongoing reform in curriculum documents as well as in textbooks, assessment and teaching. Drawing on Boesen et al's (2014) study, Bergqvist and Bergqvist (2017) have investigated to what extent and how clearly the national policy documents convey the competence message. The results show that the message is present to a large extent in the policy documents, but that it is vague and formulated with complex wording.

Although there is research on the formulation arena in connection to Swedish mathematics curricula, this research has not been conducted specifically on school algebra, the focus of the study in this paper. An exception is Bråting, Madej and Hemmi's (2019) investigation of the algebraic content in current Swedish textbooks in mathematics for grades 1-6, which is included in the same research project as the present study. In conformity with the study in this paper, Bråting et al.'s (2019) study applies Blanton et al.'s (2015) big ideas as a base for an analytical tool. However, in Bråting et al.'s study the focus is limited to the grades 1-6 and the five big ideas are merged into three. In this study, we use all five big ideas and we take all grades from 1 to 9 into consideration. The results of Bråting et al.'s (2019) study show that the big ideas "functional thinking", and "inequalities, expressions, and equations" are well represented both in the Swedish textbooks for grades 1-6. Meanwhile, the big idea "generalized arithmetic" is poorly developed in the textbooks. Apparently, this result is significant and helpful for the study in this paper.

Furthermore, Jakobsson-Åhl (2008) has conducted a historical study regarding the development of algebraic content in Swedish textbooks for upper secondary school between the years 1960–2000. Jakobsson-Åhl (2008) states that over the years the algebraic content has become more integrated with other school subjects, the level of complexity of algebraic expressions in textbook exercises has decreased, and algebra has more often been considered as a tool for solving practical and everyday problems.

There are some additional Swedish research studies focusing on school algebra within the realisation arena. For instance, Häggström (2008) compares how algebra is taught in Sweden and China, focusing on the treatment of systems of linear equation in grade 8 mathematics classrooms from the perspective of variation theory. The result reveals that the tasks used in the Chinese mathematics textbooks showed extensive variation in many relevant aspects, while the Swedish textbooks contained very similar tasks that did not open many dimensions of variation. Furthermore, a case study by Kilhamn (2014) shows that two Swedish grade 6 teachers using the same textbooks introduced variables in very different ways. The differences mainly depended on the two teachers' different views of the meaning of the variable concept, but also the meaning of algebra. A longitudinal study on students' algebraic understanding at upper secondary school is conducted by Persson (2010), who followed the same class of students for three years. Persson (2010) identified five main factors for success in algebra learning: pre-knowledge, concept development, instruction, time for learning, and interest, attitudes and feelings.

3 Methodology

We have analysed and compared the algebraic content in the 1980 and 2011 Swedish national curricula in mathematics for grades 1-9. In this section, a brief characterization of the two curricula will be given[1], followed by a description of the analytical tool and the procedure of the data analysis.

3.1 The two curricula – Lgr80 and Lgr11

The 1980 Swedish mathematics curriculum for compulsory school, included in "*Läroplan för grundskolan" (Lgr80),* consists of the two sections: 1) Goals, and 2) The main content. In the first section, the following two main goals are prescribed:

1. The teaching in mathematics should be based on the students' experiences and needs, and prepare them for the role of adult citizens. Students should therefore, in the first place, acquire an ability to solve such mathematical problems that usually occur in everyday life. This means that the students, by means of the teaching, should acquire
    - numerical abilities with and without technical resources,
    - skills in mental arithmetic and estimate calculations,
    - knowledge primarily in percentage calculations, practical geometry, units and unit transformations, and descriptive statistics.
2. By means of the school activities the pupils will also acquire mathematical knowledge and skills usable for studying other subjects, to further studies after primary school, at leisure and in working life. This requires, in addition to the above, that the students acquire knowledge about
    - the real numbers,
    - geometric relationships,
    - algebra and basic knowledge about functions,
    - statistics and probability, and
    - the usage of computers and computer knowledge (Lgr80, 1980, p. 98).

Section 2, the main content, is divided into nine topics where each topic is described for the grade levels 1-3, 4-6 and 7-9. These nine topics consists of problem solving, arithmetic, real numbers, percentages, measurements and units, geometry, algebra and basic functions, descriptive statistics and probability, and computer knowledge. The topic Algebra and basic functions is, besides computer knowledge, the topic with the least space in this section. In the study reported in this paper, all three sections of the curriculum are considered but only section 2, the main content, is included in the investigation.

The 2011 Swedish mathematics curriculum for compulsory school, included in "*Läroplan för grundskolan, förskoleklassen och fritidshemmet*" (*Lgr11)* consists of the three sections: 1) Introduction to the subject, 2) Central content, and 3) Knowledge criteria. Section 1 is the same for all grades 1-9 and includes a historical background to the subject and a description of the aim of school mathematics, which is summarized in terms of five competencies in the following way:

The teaching in mathematics will give students the opportunity to develop their ability to

- formulate and solve problems by using mathematics as well as evaluate selected strategies and methods,
- apply and analyse mathematical concepts as well as relations between concepts,
- select and use appropriate mathematical methods to make calculations and solve routine tasks,
- conduct and follow mathematical reasoning, and

[1] For a more detailed summary regarding the overall structure of the Swedish curricula between 1850-2014, see (Prytz, 2015).

- use mathematical expressions to discuss, argue and account for issues, calculations and conclusions (Lgr11, 2011, p. 2).

Furthermore, sections 2 and 3 are split between the grade levels 1-3, 4-6 and 7-9. Section 2, the central content, is divided into six mathematical topics; Number sense and the usage of numbers, Algebra, Geometry, Probability and statistics, Relationships and change, and Problem solving. Section 3, the knowledge demands, is based on the five mathematical competencies in section 1, mentioned above.

In the present study, all three sections are considered but only section 2, the central content, is included in the investigation.

3.2 The analytical tool and data analysis

In order to characterize the algebraic content in the material, Blanton et al.'s (2015) so called *big ideas* has been used as a base for an analytical tool. These are the areas that should be developed through the grades as students develop their algebraic thinking. Next, we give a brief description of each big idea and how these have been interpreted in this study.

1. *Equivalence, expressions, equations, and inequalities (EEEI)* include relational understanding of the equal sign, representing and reasoning with expressions and equations, and relationships between and among generalized quantities (Blanton et al., 2015, p. 43). An example of a task within this category is the solving of the open number sentence: 8+5=__+4 and being able to reason based on the structural relationship in the equation. Number sentences such as 8+5=__ have not been included in this category since this kind of tasks consider the ability to calculate.
2. *Generalized arithmetic (GA)* involves reasoning about structures of arithmetic expressions (rather than their computational value) as well as generalizations of arithmetical relationships, which includes fundamental properties of numbers and operations (e.g., the commutative property of addition) (Blanton et al., 2015; Kaput, 2008). In this study, this category also includes relations between operations, such as multiplication defined as repeated addition. Sometimes the term generalized arithmetic is referred to as the bridge between arithmetic and algebra (Fujii, 2003). A more detailed description of generalized arithmetic can be found in Bråting, Hemmi and Madej (2018).
3. *Proportional reasoning (PR*) refers to opportunities for reasoning algebraically about two generalized quantities that are related in such a way that the ratio of one quantity to the other is invariant (Blanton et al., 2015, p. 43). In this study, some specific applications of proportional reasoning, such as scaling and similarity are also included.
4. *Functional thinking (FT)* involves generalizations of relationships between covarying quantities, and representations and reasoning with relationships through natural language, algebraic (symbolic) notation, tables, and graphs (Blanton et al., 2015, p. 43). For instance, this can mean generating linear data and organizing it in a table, identifying recursive patterns and function rules and describing them in words and using variables, and using a function rule to predict far function values.
5. *Variable (VAR)* refers to "symbolic notation as a linguistic tool for representing mathematical ideas in succinct ways and includes the different roles variable plays in different mathematical contexts" (Blanton et al., 2015, p. 43). One typical

example within this category is the ability to use variables in order to represent arithmetic generalizations.

The data analysis was conducted in the following way. The two curricula were first analysed separately in the original language. The unit of analysis was a statement or part of a statement that addressed an issue connected to one of the big ideas. The results of this process were written down in five tables (one for each big idea) for each curriculum. Each table was divided into the three grade levels 1-3, 4-6 and 7-9 and consisted of all the statements connected to a specific big idea. For instance, the table representing the big idea FT for the 1980 curriculum was structured as in Table 3.1.

Table 3.1: FT-categorized content in the 1980 curriculum

FT in Lgr80		
Grades 1-3	**Grades 4-6**	**Grades 7-9**
Statement 1 Statement 2 …	Statement 1 Statement 2 …	Statement 1. Statement 2. …

In a few cases, it was not all clear whether a statement represented a certain big idea or not. For instance, some statements connected to the decimal system were excluded because they were considered to belong to the development of number sense rather than algebra. In these cases, the interpretation had to be reconsidered which led to minor corrections.

After this procedure, the statements from the two curricula were compared for each big idea. In order to do that, the five tables from the 1980 curriculum were merged with the five tables from the 2011 curriculum, as in Table 3.2 below.

Table 3.2: FT-categorized content in the 1980 and 2011 curricula

FT in Lgr80 and Lgr11			
	Grades 1-3	**Grades 4-6**	**Grades 7-9**
Lgr80	Statement 1 Statement 2 …	Statement 1 Statement 2 …	Statement 1 Statement 2 …
Lgr11	Statement 1 Statement 2 …	Statement 1 Statement 2 …	Statement 1 Statement 2 …

Within this process, specific features, gaps, similarities and differences between each curriculum were identified. The merged tables for each big idea are presented in the result section here below.

4 Results

The results of the analysis are presented separately for each big idea. Every section commences with a table displaying the categorization of the authentic expressions identified in the curriculum documents. This is followed by a comparison of the two curricula with respect to how they address the big idea in question at the different grade levels.

4.1 EEEI – Equivalence, Expressions, Equations and Inequalities

Table 4.1 shows the distribution of algebraic content categorized as EEEI with respect to the grade levels 1-3, 4-6 and 7-9.

Table 4.1: The distribution of EEEI-categorized content in 1980 and 2011 curricula

	Grades 1-3	**Grades 4-6**	**Grades 7-9**
Lgr80	Solving simple equalities by trial and error.	Solving simple equations mainly by trial and error and on the basis of problems.	Setting up, simplifying and calculating algebraic expressions. Expressions with parentheses, factorization, and identities of the binomial squares are treated, with particular consideration to the students' maturity, interest and needs. First order equations, including unknowns on both sides of the equality sign and with parenthesis and fractional numbers. Problem solving with simple equations. Systems of linear equations and simple second order functions mainly in connection with problem solving and preferentially with graphical solution.
Lgr11	Mathematical equalities and the importance of the equal sign.	Methods for solving simple equations. Simple algebraic expressions and equations in situations that are relevant for the students.	Algebraic expressions, formulas and equations in situations that are relevant for the students. Methods for solving equations.

In grades 1-3, the topic equalities are included in both curricula. In Lgr80, it is emphasized to solve simple equalities by trial and error, while in Lgr11 the term equality is mentioned together with the importance of the equal sign. In grades 4-6 equalities are still considered in both curricula, but the focus is directed to equations. While the Lgr80 document explicitly prescribe trial and error based on problems as a solution method, the Lgr11 document only mentions that methods of solving equations should be presented but does not pinpoint the character of the methods.

The topic algebraic expressions are considered already in grades 4-6 in Lgr11, while in

Lgr80 algebraic expressions first appear in grades 7-9. In Lgr11 it is pointed out that equations and algebraic expressions should be considered in situations that are relevant for the students. It is noticeable that the Lgr11 document mentions simple algebraic expression for grades 4-6, but leaves the decision of what to do with the expressions to the teachers.

In grades 7-9, the big idea EEEI consists of equations and algebraic expressions in both curricula. However, the Lgr80 document includes more detailed descriptions than Lgr11 and emphasizes abilities such as being able to *set up*, *simplify* and *calculate* algebraic expressions. It is also specified that expressions with parentheses, factorization, and identities of the binomial squares should be treated. Meanwhile, the Lgr11 document only refers to "algebraic expressions and formulas" (Table 4.1). Also in connection with equations, the content is more specified in Lgr80 compared to Lgr11. The latter refers to "methods of solving equations" and that the equations should be connected to situations that are relevant for the students. Meanwhile, the Lgr80 document prescribes that first and second order equations as well as systems of equations should be treated, followed by a specification for each kind of equation.

Both curricula highlight that the mathematical content should be relevant for the students and connected to the students' interests. In Lgr80, it is also stated that students' maturity and needs should be taken into account (Table 4.1).

4.2 GA – Generalized arithmetic

Table 4.2 shows the distribution of algebraic content categorized as GA with respect to the grade levels 1-3, 4-6 and 7-9.

Table 4.2: The distribution of GA-categorized content in the 1980 and 2011 curricula

	Grades 1-3	**Grades 4-6**	**Grades 7-9**
Lgr80			
Lgr11	Properties and relations of the four arithmetical operations and their usage in different situations.		

GA is the least represented big idea in both curricula. In fact, in Lgr80 we could not find any content connected to generalized arithmetic. One item could be found in Lgr11: in grades 1-3 the properties and relations of the four arithmetical operations are included. As within EEEI, the usage in different situations is pointed out.

4.3 PR – Proportional reasoning

Table 4.3 shows the distribution of algebraic content categorized as PR with respect to the grade levels 1-3, 4-6 and 7-9.

In grades 1-3, simple examples of enlargements and reductions are PR-categorized content in both curricula. The term scale is introduced already in grades 1-3 in Lgr11, while in Lgr80 scale first appears in grades 4-6. In Lgr11, different proportional relationships, including doubling and halving, are also included in grades 1-3. It is noticeable that in Lgr80 the term proportion first appears in grade 7-9 in connection with linear functions (see Table 4.4 below).

In grades 4-6, the main topics within PR in both curricula are percentage and scale. In both curricula, the relations between fractions, decimal numbers and percentages are emphasized. In Lgr11, the relation between proportionality and percentage is also considered. As already mentioned, in Lgr80 the term proportion first appears in grades 7-9.

In grades 7-9, the PR-categorized topics percentage and scale are still considered in both curricula. It is noticeable that the Lgr80 curriculum emphasizes calculations with percentages while in Lgr11 percentage is used to express change and change factor as well as calculations in everyday situations. Furthermore, in both curricula uniformity is introduced in grades 7-9. In Lgr80 the term congruence is also used. Both curricula point out the connection between the mathematical content and students' interests across all grades 1-9. In Lgr11 the expression "in situations relevant for the students" is used while in Lgr80 uses the terms "practical usage" and "everyday contexts". In Lgr80 the connection to other school subjects is also mentioned.

Table 4.3: The distribution of PR-categorized content in the 1980 and 2011 curricula

	Grades 1-3	**Grades 4-6**	**Grades 7-9**
Lgr80	Simple and practical examples of enlargements and reductions, for instance in connection with maps and handicraft objects.	Treatment of the percentage concept in connection with practical problems and other school subjects. Calculations with percentages. Relations between fractions, decimal numbers and percentages. Treatment of scale in everyday life.	Calculations with percentages, parts and the whole. Usage of the concept of scale mainly in practical contexts. Treatment of congruence and uniformity.
Lgr11	Different proportional relationships, including doubling and halving. Students can use and give examples of simple proportional relations in situations relevant for the students. Scale with simple enlargements and reductions.	Proportionality and percentage and their relationship. Percentages and the connection to fractions and decimal numbers. Graphs for expressing different types of proportional relations in simple investigations. Scale and the usage of scale in situations relevant for the students.	Percentage to express change and change factor as well as calculations with percentage in everyday situations and within other disciplines. Scale with enlargement and reduction of two- and three dimensional objects. Uniformity in the plane.

4.4 FT – Functional thinking

Table 4.4 shows the distribution of algebraic content categorized as FT with respect to the grade levels 1-3, 4-6 and 7-9.

Table 4.4: The distribution of FT-categorized content in the 1980 and 2011 curricula

	Grades 1-3	**Grades 4-6**	**Grades 7-9**
Lgr80		The function concept is introduced through practical experiments. Interpretations of simple functions in the first quadrant in a coordinate system. Calculations of functions values by inserting them into formulas, connected to everyday life or other school subjects.	Interpretations and constructions of graphs in the whole coordinate system. Linear functions, especially those that indicate proportionality.
Lgr11	How simple patterns in number sequences and simple geometrical forms can be constructed, described and expressed.	How patterns in number sequences and geometrical patterns can be constructed, described and expressed. The coordinate system and strategies for scaling coordinate axes. Tables and graphs.	Functions and linear equations. How functions can be used to investigate change, rate of change and other relationships.

In grades 1-3, the big idea FT is not represented at all in Lgr80. In the same grade level in Lgr11, FT-categorized content consists of simple patterns in number sequences and simple geometrical forms as well as how these can be constructed, described and expressed. This topic is also considered in grades 4-6 in Lgr11. The only difference is that in grades 4-6 the word “simple” (in connection with simple patterns) is removed.

Furthermore, in grades 4-6 the coordinate system and graphs are included in both curricula. In Lgr80, the term “function” appears in grades 4-6, while in Lgr11 the term “function” first appears in grades 7-9. In Lgr80, interpretations of simple functions in the first quadrant as well as calculations of function values by inserting them into formulas are emphasized in grades 4-6. In grades 7-9, not only interpretations of functions are emphasized but also constructions of functions in the whole coordinate system. Linear functions, especially those that indicate proportionality, are also mentioned in Lgr80. In Lgr11, the description of the content of functions are not as detailed as in Lgr80. However, the Lgr11 document prescribes the usage of functions in order to investigate change (Table 4.4).

4.5 VAR – Variables

Table 4.5 shows the distribution of algebraic content categorized as VAR with respect to the grade levels 1-3, 4-6 and 7-9.

Table 4.5 7. The distribution of VAR-categorized content in the 1980 and 2011 curricula

	Grades 1-3	Grades 4-6	Grades 7-9
Lgr80			
Lgr11		Unknown numbers and their properties and also situations where there is a need to represent an unknown number by a symbol.	The meaning of the variable concept and its use in algebraic expressions, formulas and equations.

In Lgr80, we could not find any content connected to the big idea VAR. In Lgr11, VAR-categorized content first appears in grades 4-6 and 7-9. Unknown numbers and their properties are included in the content in grades 4-6. It is also pointed out that situations where there is a need to represent an unknown number by a symbol should be treated. In grades 7-9, the focus is on the meaning of the variable concept as well as its use in algebraic expressions, formulas and equations.

5 Discussion

The result reveals both similarities and differences between the two curricula. Next, this will be discussed on the basis of the following issues: The earlier introduction of algebra over time, the representation of different big ideas, the movement from a focus on computational skills to verbal abilities between the years 1980 and 2011, and finally the high emphasis on practical and everyday mathematics in both curricula.

The result of the analysis reflects the recent international trend to integrate algebra in school mathematics already from primary school (Blanton et al., 2015; Carraher et al., 2006; NCTM, 2006). Several parts of the algebraic content are introduced earlier in the Lgr11 document compared to the Lgr80 document. For instance, in Lgr80 algebraic expressions and proportionality first appear in grades 7-9, while in 2011 algebraic expressions appear in grade 4-6 and proportionality already in grades 1-3 (Table 4.1 & 4.3). An exception in this study is the function concept which is introduced in grades 4-6 in Lgr80 and not until grades 7-9 in Lgr11. However, the Lgr11 document prescribes "patterns" already in grades 1-3 which might be viewed as a first step to understand the function concept (Blanton et al., 2015). That is, even though the function concept appears earlier in the Lgr80 document, the big idea 'functional thinking' (Blanton et al., 2015) is represented earlier in Lgr11 compared to Lgr80 (see Table 4.4).

Furthermore, the function concept is applied somewhat different in the two curricula. For instance, the Lgr80 document prescribes calculations of function values which is not mentioned in Lgr11. Instead, the Lgr11 document emphasizes the usage of functions in order to investigate rate of change and other relationships which is not included in Lgr80 (Table 4.4). One reason to this might be that 'Relationships and change' constitutes a new, separate category of mathematical content in the Lgr11 curriculum (see the Methodology section above, p. 5). In previous Swedish curricula, the content in 'Relationships and change' was distributed among the different topics, especially algebra. The emphasis on 'Relationships and change' is probably an effect of a recent international trend where 'Relationships and change' has been identified as one of the four broad mathematical content categories in the PISA framework for school mathematics (OECD, 2010). This is also reflected in the current Swedish textbooks in mathematics, where the big idea 'functional thinking' is a dominating content with a clear progression throughout the grades 1-6 (Bråting et al., 2019). Meanwhile, as mentioned in the Methodology section above, in Lgr80 'Algebra and basic functions' is one of the smallest topics in the curriculum and it is also pointed out that individualization based on students' ability is necessary within this topic (Lgr80, p. 105).

'Generalized arithmetic' is the least represented big idea in both curricula. Apparently, it is not represented at all in the content description of the Lgr80 document and very little in Lgr11 (Table 4.2). As already mentioned, generalized arithmetic is seen as one of the most important parts of school algebra by several researchers (Blanton et al., 2015; Kaput, 2008). Sometimes generalized arithmetic is considered as a bridge between arithmetic and algebraic thinking (Fujii, 2003), that is, as a development of "algebra as generalized arithmetic" throughout compulsory school. However, we cannot find any notion of building a bridge between arithmetic and algebra in the two curricula. In fact, the terms 'generalize' and 'generalization' do not appear in neither of the two curricula. It can also be noted that the term variable (or unknown) is not mentioned at all in the Lgr80 document which probably is a reaction to the great focus on abstract mathematics in connection with "New math" during the 70s (c.f Prytz, 2015).

The results reveal that the Lgr80 document emphasizes computational skills to a greater extent than Lgr11. Within the big idea EEEI, a typical example is the emphasis on *setting up*, *simplify* and *calculate* algebraic expressions in Lgr80 (Table 4.1). Furthermore, the Lgr80 document prescribes *solving* equations and specifies which kind of equations that should be treated. Instead, the Lgr11 document emphasizes *methods* for solving equations and *the meaning* of the equal sign (which is pointed out already in grades 1-3). Moreover, the Lgr11 document stresses abilities such as *expressing* and *describing*. For instance, within the big idea PR, percentage is connected to the ability to *express* change (Table 4.3) and within the big idea FT patterns should be *constructed*, *described*, and *expressed* (Table 4.4).

The emphasis on terms such as methods, expressing and describing in the Lgr11 reflects the implementation of the mathematical competency goals (NCTM, 2000; Niss & Jensen, 2002) in the Swedish mathematical curriculum. As already mentioned in the methodology section above (p. 5), the competency goals consist of abilities such as analysing mathematical concepts, evaluating selected strategies and methods as well as use mathematical expressions to discuss conclusions. These abilities cannot be found in the Lgr80 document where computational or operational aspects are in focus. Based on the results in this study, one could grasp a movement of focus from computational and operational abilities in Lgr80 to more verbal abilities in Lgr11 (c.f. Bråting & Österman, 2015). This is in accordance with the results of Jakobsson-Åhl's (2008) study which revealed that the level of complexity of algebraic expressions in Swedish textbook exercises had decreased over the years.

The implementation of the competency goals in the Swedish curriculum is an ongoing reform in Swedish school mathematics (Boesen et al., 2014) and according to Bergqvist and Bergqvist (2017) it is problematic for teachers to convey the message which is vague and formulated with complex wording. The focus on the implementation of the competencies in Swedish school mathematics might be one reason why the traditional emphasis on computational skills has been disregarded during the past decades. However, we believe it is important to have a balance between the emphasis on verbal abilities on the one hand and computational abilities on the other.

Apparently, both curricula frequently point out the importance of practical and everyday mathematics within the algebraic content. However, the view of how these aspects should be acquired differs between the two curricula. In the Lgr80 document, practical skills and the acquirement of everyday mathematics are considered as a part of an overall ability, which should be acquired by means of learning computational, numerical and geometrical skills. Let us consider the same citation of one of the two main goals with mathematics from Lgr80 which was cited in the methodology section above:

> Students should therefore, in the first place, acquire an ability to solve such mathematical problems that usually occur in everyday life. This means that the students, by means of the teaching, should acquire
>
> - numerical abilities with and without technical resources,
> - skills in mental arithmetic and estimate calculations,
> - knowledge primarily in percentage calculations, practical geometry, units and unit transformations, and descriptive statistics (Lgr80, p. 98).

Here, it is clearly stated that the teaching must be based on the students’ own experiences. However, the practical skills are closely linked to the specific mathematical content. This differs from the Lgr11 document where the specific mathematical content is more separated from the practical and verbal abilities within the descriptions of the goals (as in the citation of the competency goals on p. 5 in the methodology section above). Moreover, it is prescribed in Lgr11 that mathematics should be used as a tool to solve everyday and practical problems such as private economy, social life, and electronics. It seems that in Lgr11 the mathematical content is used to solve the practical and everyday problems, while in Lgr80 the practical and everyday problems are used as a platform to learn the specific mathematical content (see also Bråting & Österman, 2015). This is in accordance with Jakobsson-Åhl’s (2008) study where the results revealed that algebra has more often been considered as a tool for solving practical and everyday problems through the years and that algebraic content has become more integrated with other school subjects.

Acknowledgement:
This work was supported by the Swedish Research Council [Grant no. 2015–02043].

REFERENCES

Bergqvist & Bergqvist (2017). The role of the formal written curriculum in standards-based reform. *Journal of Curriculum Studies, 49*(2), 149–168.

Bergsten, C., Häggström, J., & Lindberg, L. (1997). *Nämnaren tema: Algebra för alla*. Göteborg: Nationellt Centrum för Matematikdidaktik (NCM).

Blanton, M., Stephens, A., Knuth, E., Murphy Gardiner, A., Isler, I., & Kim, J. (2015). The development of children’s algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education, 46*(1), 39–87.

Boesen, J., Helenius, O., Bergqvist, E., Bergqvist, T., Lithner, J., Palm, T., & Palmgren, B. (2014). Developing mathematical competence: From the intended to the enacted curriculum. *The Journal of Mathematical Behavior, 33*, 72–87.

Bråting, K. (2015). Teaching traditions in Swedish school algebra - a project description. In I. Bjarnadottir, F. Furunghetti, J., Prytz, & G. Schubring, (Eds.). *"Dig where you stand" 3. Proceedings of the Third International Conference on the History of Mathematics Education, ICHME3*, Sept 30-Oct 3, 2013, in Uppsala (pp. 81-92). Uppsala University.

Bråting, K., Hemmi, K., & Madej, L. (2018). Teoretiska och praktiska perspektiv på generaliserad aritmetik. In J. Häggström, Y. Liljekvist, J. Bergman Ärlebäck, M. Fahlgren & O. Olande (Eds.), *Perspectives on professional development of mathematics teachers. Proceedings of MADIF 11* (pp. 27–36). Gothenburg: NCM & SMDF.

Bråting, K., Madej, L., & Hemmi, K. (2019). Development of algebraic thinking: opportunities offered by the Swedish curriculum and elementary mathematics textbooks. *Nordic Studies in Mathematics Education, 24*(1), 27-49.

Bråting, K., & Pejlare, J. (2015). On the relations between historical epistemology and students' conceptual developments in mathematics. *Educational Studies in Mathematics, 89*(2), 251-265.

Bråting, K., & Österman, T. (2017). John Dewey and mathematics education in Sweden. In I. Bjarnadottir, F. Furunghetti, M. Menghini, J. Prytz, & G. Schubring (Eds.). *"Dig where you stand" 4. Proceedings of the Fourth International Conference on the History of Mathematics Education, ICHME4*, Sept 23–26, 2015, Turin (pp. 61-72). Roma: Nuova Cultura.

Carraher, D.W., Schliemann, A., Brizuela, B., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37*(2), 87-115.

Fujii, T. (2003). Probing students' understanding of variables through cognitive conflict problems: Is the concept of a variable so difficult for students to understand? In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education, PME 27* (Vol. 1, pp. 49-65). Honolulu: PME.

Hemmi, K., Bråting, K., Liljekvist, Y., Prytz, J., Madej, L., Pejlare, J., & Palm Kaplan, K. (2018). Characterizing Swedish school algebra – initial findings from analyses of steering documents, textbooks and teachers' discourses. In E. Norén, H. Palmér & A. Cooke (Eds.), *Nordic research in mathematics education. Papers of NORMA 17* (pp. 299–308). Gothenburg: SMDF.

Häggström, J. (2008). *Teaching systems of linear equations in Sweden and China: What is made possible to learn?* Doctoral thesis. Department of Education, University of Gothenburg.

Jakobsson-Åhl, T. (2008). Word problems in upper secondary algebra in Sweden over the years 1960-2000. *Nordic Studies in Mathematics Education, 13*(1), 7–27.

Kieran, C. (Ed.). (2018). *Teaching and learning algebraic thinking with 5- to 12-year-olds*. Cham: Springer.

Kilhamn, C. (2014). When does a variable vary? Identifying mathematical content knowledge for teaching variables. *Nordic Studies in Mathematics Education, 19*(3-4), 83-100.

Lindensjö, B., & Lundgren, U.P. (2000). *Utbildningsreformer och politisk styrning*. Stockholm: HLS förlag.

Mason, J. (2018). How early is too early for thinking algebraically? In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds* (pp. 329-350). Cham: Springer.

National Council of Teachers of Mathematics (NCTM). (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. Reston, VA: National Council of Teachers of Mathematics.

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklaering (competencies and mathematical learning).* Uddannelsestyrelsens temahaefteserie nr. 18-2002, Undervisningsministeriet.

Persson, P-E. (2010). Räkna med bokstäver: en longitudinell studie av vägar till en förbättrad algebraundervisning på gymnasienivå. Doctoral thesis, Luleå University of Technology.

Prytz, J. (2015). Swedish mathematics curricula, 1850-2014. An overview. In Bjarnadóttir, K., Furinghetti, F., Prytz, J. & Schubring, G. (Eds.), *Dig where you stand 3: Proceedings of the Third International Conference on the History of Mathematics Education* (pp. 309-325). Uppsala universitet.

Schubring, G. (2011). Conceptions for relating the evolution of mathematical concepts to mathematics learning – epistemology, history, and semiotics interacting. *Educational Studies in Mathematics, 77*(1), 79–104.

Swedish National Agency for Education. (1980). *National curriculum in Sweden (Läroplan för grundskolan, Lgr80).* Stockholm: Liber.

Swedish National Agency for Education. (2011). *National curriculum in Sweden (Läroplan för grundskolan, förskoleklassen och fritidshemmet, Lgr11).* Stockholm: Fritzes.

Swedish National Agency for Education (2016). TIMSS 2015 – huvudrapport. Svenska grundskoleelevers kunskaper i matematik och naturvetenskap i ett internationellt perspektiv. Stockholm: Kluwers.

Yang Hansen, K., Gustafsson, J-E., Rosén, M., Sulkunen S., Nissinen, K., Kupari, P., … Hole, A. (2014). *Northern lights on TIMSS and PIRLS 2011: differences and similarities in the Nordic countries.* Copenhagen: Nordic Council of Ministers.

MATHEMATICS TEACHING IN GYMNASIA AND REAL SCHOOLS IN POLAND IN THE YEARS 1795–1918

Schools with Polish and German as the language of instruction – comparison

Karolina KARPIŃSKA

L. & A. Birkenmajer Institute for the History of Science, Polish Academy of Sciences,
Nowy Świat 72, room A09, 00-330 Warsaw, Poland
karolinakarpinska001@gmail.com

ABSTRACT

This article is dedicated to the comparison of ways mathematics was taught in secondary schools with Polish and German as languages of instruction on the territory of Poland from 1795 to 1918. Particular attention will be paid to gymnasia (Pol. *gimnazjum*, Ger. *Gymnasium*) and real schools (Pol. *szkoła realna*, Ger. *Realschule*), which constituted the best types of secondary schools at that time. Mathematics teachers education, mathematics curricula and school leaving examinations, called also Matura examinations, will be compared. But firstly, there will be explained why there were schools with different languages of instruction on the territory of Poland from the end of the 18th to the beginning of the 20th century.

Keywords: mathematics, mathematics teaching in Poland, secondary schools, gymnasia, real schools, partitions of Poland.

1 Introduction

In the early 18th century, Poland was a prosperous country with its own mathematical culture and mathematical traditions. In 1773, the Polish government established the Commission of National Education. This Commission reorganized the system of Polish education, prepared new school curricula, new school textbooks, organized a system of teachers' education and school supervision system. Regulations prepared by the Polish Commission of National Education (*Ustawy Kommissyi Edukacyi Narodowej dla Stanu Akademickiego i na szkoły w kraiach Rzeczypospolitey przepisane*, 1783) were based partly on the rules of Jesuits and Piarist schools but they also took into consideration the pedagogical concepts the age of Enlightenment. New school curricula paid attention to mathematical and natural sciences, especially on: arithmetic, geometry, trigonometry and algebra. Those regulations also described the best method of teaching: professors should abandon the lecture method in favour of experiential teaching in which students will understand what they are learning (*Ustawy Kommissyi Edukacyi Narodowej dla Stanu Akademickiego i na szkoły w kraiach Rzeczypospolitey przepisane*, 1783, 36).

The Polish Commission of National Education was the first Ministry of Education in Europe, it was a secular organisation, independent of the Church. It was the first European attempt to transform chaotic teaching into an organized system of education – from elementary schools to universities. 10 years later everything was ready and implemented on the territory of Poland.

One year before establishing the Commission of National Education, so in 1772, the First Partition of Poland took place. Three empires, Prussia, Austria and Russia, occupied some territories of Poland as a result of the unstable political situation in Poland. In 1793 there was the Second Partition of Poland, in 1795 – the Third Partition of Poland. From the Third Partition of Poland to 1918 all territories of Poland were occupied[1] (Fig. 1.1). For 123 years Poland disappeared from the map of Europe.

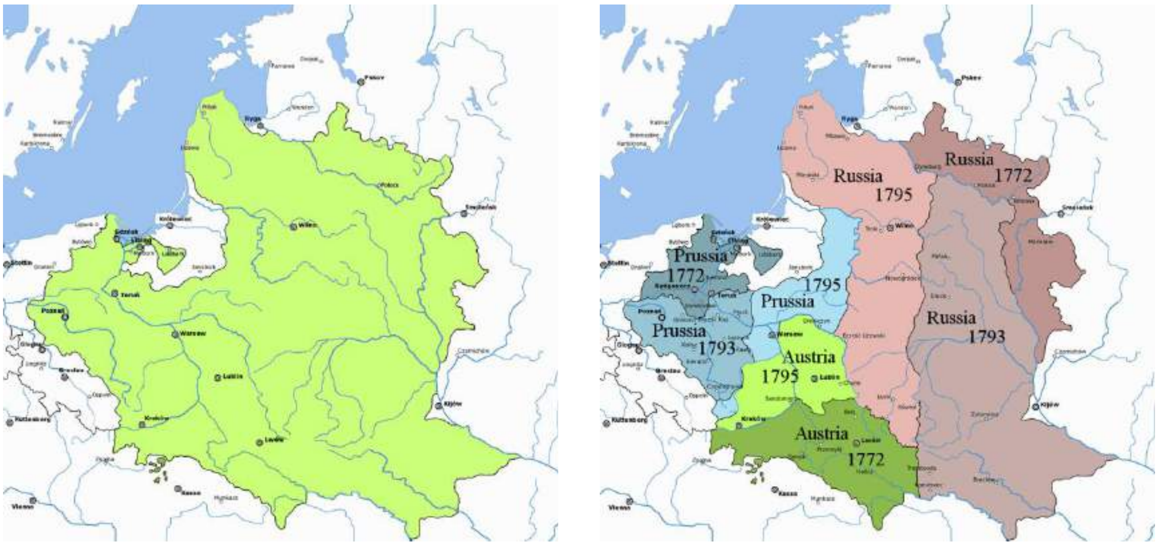


Figure 1.1: Partitions of Poland[2]

In 1795 Prussia, Austria and Russia began the process of germanization and russification of the territories of Poland.

Prussia and Austria almost immediately introduced German as a language of instruction to Polish schools, Polish teachers were replaced by those from Prussia and Austria, Prussian and Austrian curricula and textbooks were also introduced. In 1868 the territories of the Austrian Partition gained autonomy[3] in education and there was established the Polish Ministry of Education: National School Council (Pol. *Rada Szkolna Krajowa*), Polish was the language of instruction, there were Polish teachers, Polish textbooks and Polish curricula. In 1875 the freedom of Poles was limited and all decisions of the Polish Ministry of Education had to be approved by the Ministry of Education and Religious Affairs in Vienna (Świeboda, 1984, 111–112). In the Prussian Partition from 1795 to 1918 there were schools with German as a language of instruction and Prussian rules of education only.

In the Russian Partition from 1795 to 1831[4] there were Polish schools with Polish rules of education and Polish teachers. In 1831 freedom of Poles was limited and all decisions related to Polish schools had to be approved by the Department of Religious Affairs and Public Education of the Polish Provisional Government, which was under the supervision of Russian Emperor (Manteuffel, 1929, 26). The year 1839 is considered the beginning of the process of combining Polish education with the school organization of the Russian

[1] The Polish lands were occupied by Prussia, Austria and Russia almost for the entire time from 1795 to 1918. The exception are the years 1807–1815 when part of the Polish lands under the name Duchy of Warsaw were under the authority of the French emperor, Napoleon Bonaparte. After the fall of Napoleon, the Duchy of Warsaw was enlarged at the Congress of Vienna in 1815 and transformed into a Kingdom under the rule of the Tsar of Russia. Education in the Duchy of Warsaw remained in the hands of the Poles.

[2] Source of maps: https://pl.wikipedia.org/wiki/Rozbiory_Polski#/media/File:Partitions_of_Poland.png.

[3] Obtaining autonomy by the territories of the Austrian Partition was a process lasting several years: 1860–1873 (Majorek, 1980, XIII–XVI). It should be emphasized here that from the 1850s, there were schools on the territories of the Austrian Partition with Polish language of instruction, however there were only a few of them (*Sprawozdanie…*, 1885, 26–27).

[4] In 1831 November Uprising took place.

Empire (Królikowski, 2008, 285). Changes were introduced gradually – Russian teachers were successively brought to Polish lands and regulations concerning the functioning of Polish schools were issued, e.g. in 1851 another school reorganization was approved, under which Polish philology gymnasia were made similar to this type of schools in Russia (Królikowski, 2008, 287). After the January Uprising, taking place in 1863, Polish schools were revived for a short time thanks to Aleksander Wielkopolski, but already in 1867 they were thoroughly Russified (along with the introduction of the Russian language of instruction to all schools) (Manteuffel, 1929, 37-49). Until 1867, many Polish names appeared in the composition of the central educational authorities for the Polish territories of the Russian Partition (Manteuffel, 1929, 51-69). The managerial positions (such as the director, curator, minister, etc.) were usually filled by the Russians, however, also the Poles had a great influence on the form of teaching in the Polish lands.

This article will compare the teaching of mathematics in schools with Polish and German language of instruction functioning in Polish territories in all partitions. Particular attention will be paid to gymnasia and real schools, in which, in the 19th century, Matura exams were already carried out. Gymnasia were focused on teaching humanities subjects, and real schools – mathematics and natural science subjects. These were schools for the chosen, not the majority. These schools educated the intellectual elite of the country.

Young people starting gymnasia and real schools were specially selected. During the entrance exams to these schools, on the one hand, students had to demonstrate knowledge of the material obligatory in elementary schools, while on the other hand, they had to demonstrate high general abilities, whose determinants were intelligence, creative thinking, good memory and imagination. Only students with these skills could successfully continue their education in gymnasia and real schools and pass the 19th century Matura exam. Among students, especially in real schools, there were also people with special abilities in mathematics (see also: Karp, 2009). It allowed to implement the widest possible curricula, also in the field of mathematics.

Comparison of the method of mathematics education in gymnasia and real schools in the Polish territories during the partitions is therefore a comparison of the mathematical education of the intellectual elite in Poland with the way of its education in Prussia and Austria. It is a comparison of the mathematics education in the occupied country, seeking independence, and the way it is taught in the occupying countries.

Stefan Banach said that only countries that cherish mathematics can be strong and powerful. Mathematical education has often been an instrument of political reform (see, e.g., Karp & Funinghetti, 2016; Karp & Funinghetti, 2018). A lot has been written about this in the context of Prussia or France. Here we will focus on Poland. We will discuss the teaching of mathematics in a country that wants to free itself from slavery. We will try to answer the following questions: What kind of mathematics was taught to the Polish intellectual elites, which as a result, in 1918 raised Poland from the fall? Did Poles model themselves on the reaching systems implemented in Prussia and Austria or did they develop their own teaching system? Was the national element visible in teaching – was teaching adapted to the needs of the Polish society? Was mathematics aimed at practical application taught, or maybe the 19th century tendency was followed: "learning mathematics was intended to contribute to the development of one's mental capacities. Practical value or applicability of mathematics was typically much less important" (Smid, 580)? How were mathematics teachers educated?

The analysis carried out in this article is a contribution to the study of the history of mathematics teaching. It presents the means by which high effects of mathematical education were achieved in Poland in 1795-1918. It also allows to draw conclusions for modern educational practice.

Let us compare mathematics teaching in schools with Polish and German as languages of instruction in all partitions of Poland.

2 Mathematics teachers in schools with Polish and German as languages of instruction

2.1 Regulations related to teachers' education

2.1.1 Polish regulations

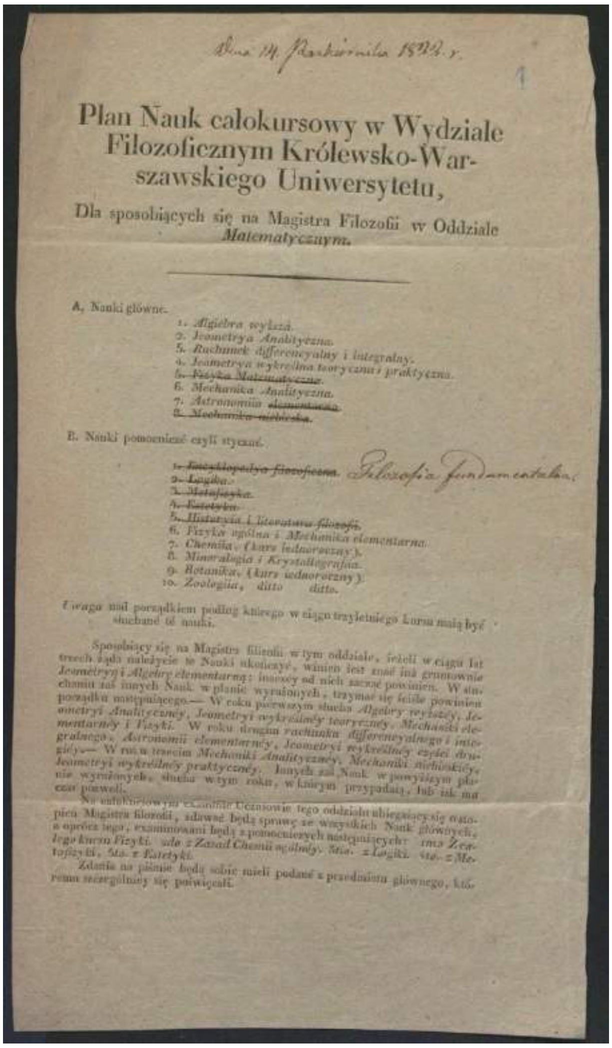
The first Polish regulations related to teachers' education were prepared by the Commission of National Education. It took place in the 1780s (see: Popławski, 1780; *Ustawy Kommissyi Edukacyi Narodowej dla Stanu Akademickiego i na szkoły w kraiach Rzeczypospolitey przepisane*, 1783, 15–18). The Commission operated until the Third Partition of Poland. However, its general regulations were in force for a dozen years after 1795. Moreover, its regulations related to teachers' education were kept for almost the entire first half of the nineteenth century in Polish schools[5].

The most important part of education, in the opinion of the Polish Commission of National Education, was teachers. According to the phrase, *the result of education will resemble the level of the teachers* (Popławski, 1780), the Commission of National Education has prepared a system for educating Polish teachers.

From 1780 those people who wanted to work in Polish secondary schools had to finish teachers seminars. The first Polish teachers seminar was opened in 1780 at the university in Cracow. Later, another seminars were opened – at universities in Vilnius and Warsaw[6]. Each teachers seminar lasted for 4 years and consisted of a one-year general pedagogical course and 3 years' university studies related to the main subject of the candidate's teachers course (Popławski, 1780).

Let's look at a schedule of 3 years' university studies in mathematics at Warsaw University in 1822 (Table 2.1). This is the framework of the 3-year part of the teachers seminar in mathematics.

[5] Certain regulations were also applied at the end of the 19^{th} century and the beginning of the 20^{th} century, an example here is delegating teachers to foreign scholarships in order to improve their skills.
[6] See: (Więsław, 2007, 110–114 and 256–276).



Dnia 14 Października 1822 r.

1

Plan Nauk całokursowy w Wydziale Filozoficznym Królewsko-Warszawskiego Uniwersytetu,

Dla sposobiących się na Magistra Filozofii w Oddziale *Matematycznym.*

A, Nauki główne.

1. *Algiebra wyższa.*
2. *Jeometrya Analityczna.*
3. *Rachunek differencyalny i integralny.*
4. *Jeometrya wykreślna teoryczna i praktyczna.*
5. ~~*Fizyka Matematyczna.*~~
6. *Mechanika Analityczna.*
7. *Astronomiia* ~~*elementarna.*~~
8. ~~*Mechanika niebieska.*~~

B. Nauki pomocnicze czyli styczne.

1. ~~*Encyklopedyia filozoficzna.*~~ Filozofia fundamentalna,
2. ~~*Logika.*~~
3. ~~*Metafizyka.*~~
4. ~~*Estetyka.*~~
5. ~~*Historyia i literatura filozofii.*~~
6. *Fizyka ogólna i Mechanika elementarna.*
7. *Chemiia, (kurs jednoroczny).*
8. *Mineralogia i Krystallografiia.*
9. *Botanika, (kurs jednoroczny).*
10. *Zoologiia, ditto ditto.*

Uwaga nad porządkiem podług którego w ciągu trzyletniego kursu mają być słuchane té nauki.

Main subjects[7]:

1. Higher algebra.
2. Analytic geometry.
3. Differential and integral calculus.
4. Descriptive geometry (theory and practice).
5. Mathematical Physics. (cancelled!)
6. Analytical Mechanics.
7. Astronomy.
8. Celestial mechanics. (cancelled!)

Associate subjects:

1. Philosophical encyclopedia. (cancelled!) Fundamental philosophy.
2. Logic. (cancelled!)
3. Metaphysics. (cancelled!)
4. Aesthetics. (cancelled!)
5. History and literature of philosophy. (cancelled!)
6. General physics and elementary mechanics.
7. Chemistry (one-year course).
8. Mineralogy and crystallography.
9. Botany (one-year course).
10. Zoology, *ditto ditto*[8].

Table 2.1: Teaching Plan at the Philosophy Department of the Royal Warsaw University for those who pursue a Master's degree in Philosophy at the Mathematical Department (1822)[9]

For example, let us discuss the program implemented during the lectures on differential and integral calculus. The then professor Adrian Krzyżanowski (1788–1852) during these lectures discussed the Taylor formula, power series, he determined the extrema of the functions of one variable and he introduced the basic theorems related to determination of integrals and their application. During the lectures he used Gottfried Leibniz's works (Więsław, 2007, 245). As historian says, mathematical studies at Warsaw University were at similar level of education to these at other European universities at that time (Duda, 2019, 115–117).

[7] On the whole-source Teaching Plan some subjects are cancelled, there is also one subject that was added in handwriting. This is probably a consequence of the fact that at that time, the teaching plans were prepared individually for each student (*Ustawy Kommissyi Edukacyi Narodowej dla Stanu Akademickiego i na szkoły w kraiach Rzeczypospolitey przepisane*, 1783, 16–17).

[8] *Ditto* is an Italian word that means "as above". *Ditto ditto* in this context should be understood as: (one-year course).

[9] State Archives in Łódź, sign. 39/ 592/0/2.1/734.

2.1.2 Prussian regulations

In Prussia teachers' education was finally settled at the beginning of the 19th century. In 1810 teachers examinations were introduced and they had to be passed after finishing university studies (Wiese, 1864, 545–547). Over the years, these ordinances have been improved, e.g. in 1831 the scope of material in mathematics which was applicable for the exam for mathematics and natural sciences teachers was determined: school arithmetic, geometry, plane trigonometry and algebra, higher mathematics, especially the application of mathematics to astronomy and physics (Kröger, 1837, 50–84).

2.1.3 Austrian regulations

In the Austrian empire in the first half of the 19th century the teachers of the highest grades of secondary schools, according to the regulations of 1786, were required to graduate from university studies, but in practice they were not often observed (Świeboda, 1984, 71–72). Teachers were employed on the basis of the so-called competition, which consisted of a written exam of subjects that the candidate was intended to teach. There were opinions that these exams often did not check the candidate's knowledge in a sufficient way, and did not check pedagogical knowledge and skills at all (Świeboda, 1984, 72). The first detailed regulations related to the teachers examinations in Austria were issued in the 1849. From then on, secondary school teachers had to graduate from university studies and pass a teacher's exam. These ordinances have been respected. In 1856, they were clarified (see: Seidl, Bonitz, Mozart, 1856, 673–686; Rakoczy-Pindor, 2012).

Since 1856 examinations could be passed in two subjects which the candidate intended to teach. Candidates had to prepare also a pedagogical, philosophical or didactical work at home on the topic given by the Examination Board (see: Rakoczy-Pindor, 2012, 164–167).

Requirements for mathematics teachers were as follows: higher algebra and number theory, elementary geometry, analytic geometry on the plane and in space, descriptive geometry, differential and integral calculus with applications, variational calculus, theory of functions (Rakoczy-Pindor, 2012, 165).

Austrian regulations were quite similar to Prussian. These regulations were also valid in Polish schools with the Polish language of instruction on the territory of the Austrian Partition.

2.2 Teachers characterization

In what way teachers on the territory of Poland under partitions can be characterized? Teachers in Polish schools from 1780, in schools under Prussian Partition from 1810 and in schools under Austrian Partition from 1849 can be characterized in a similar way. They published articles related to mathematics education, for example discussing arithmetic sequences of higher degrees[10], binomial theorem and properties of binomial coefficients,

[10] Definition of arithmetic sequences of the higher degrees (Koppe, 1869):
1. Sequence $(a_n^1)_{n\in N_+}$ is an arithmetic sequence of the first degree if it is a sequence of natural numbers.
2. Sequence $(a_n^2)_{n\in N_+}$ is an arithmetic sequence of the second degree if : $a_1^2 = 1$, $a_i^2 = a_i^1 + a_{i-1}^2$, $i \in N_{\geq 2}$.
3. Sequence $(a_n^3)_{n\in N_+}$ is an arithmetic sequence of the third degree if: $a_1^3 = 1$, $a_i^3 = a_i^2 + a_{i-1}^3$, $i \in N_{\geq 2}$.
4. Sequence $(a_n^k)_{n\in N_+}$ is an arithmetic sequence of the k^{th} ($k \in N_{\geq 2}$) degree if: $a_1^k = 1$, $a_i^k = a_i^k + a_{i-1}^k$, $i \in N_{\geq 2}$.

continued fractions and their applications (examples of teachers works related to Mathematics education: J. Słonimski, *O ułomkach ciągłych*, Kalisz, 1829; J. Piegsa, *Ein Beitrag zur Theorie der höheren arithmetischen Reihen*, Ostrowo, 1855). They published also school textbooks (examples of mathematical textbooks written by teachers: W. Karczewski, *Początki arytmetyki*, Kielce, 1822; E. Fassbender, *Anfangsgründe der beschreibenden Geometrie, der analytischen Geometrie, der Kegelschnitte und der einfachen Reihen*, Essen, 1860). Teachers very often conducted their own scientific work (examples of teachers scientific works: Z. Krygowski, *O pewnym zastosowaniu funkcji theta*, Przemyśl, 1890; K. Weierstrass, *Beitrag zur Theorie der Abel'schen Integrale*, Braunsberg, 1849). Sometimes they had Ph.D. degrees. In schools with Polish language of instruction in the second half of the 19th and at the beginning of the 20th century[11] it was about 7–8% of all teachers (Rakoczy-Pindor, 2012, 163). Teachers with Ph.D. degree in schools with German as a language of instruction can be roughly estimated at 30%[12]. Moreover, there were schools in which in some years the percentage of teachers with a doctorate was equal to 50%. For example this situation took place in Toruń Gymnasium in 1860/1861 school year – there were 20 teachers and a half of them had a Ph.D. degree, among them there were three teachers of mathematics: Eduard Fassbender (1816–1892), Rudolf Brohm (1807–after 1864), Hermann Rietze (1831–1862) (School reports, Thorn, 1861, 41–42).

There was one very important difference between Polish teachers and those who worked at schools with German as a language of instruction. The Polish ministries of education sent teachers abroad on scientific trips, for example to France, England and Prussia. During these trips teachers had to visit local schools and learn mathematical sciences at local universities[13].

It was a way to improve Polish teachers' education. After returning they were secondary school teachers during their whole educational career, for example[14]:

Ignacy Przybylski (1770–1838) – trip around Prussia; mathematics teacher in Płock and Kalisz.

Wincenty Wrześniowski (1800–1862) – trip around Europe; mathematics teacher in Radom and Warsaw.

August Bernhard (1804–1861) – trip around Europe; mathematics teacher in Piotrkowo and Warsaw.

Sometimes, after a few years in secondary schools, they were promoted to university teachers, for example[15]:

Franciszek Armiński (1789–1848) – visit to Paris; mathematics teacher in Warsaw; since 1816 astronomy professor at Warsaw University.

Adrian Krzyżanowski (1788–1852) – visit to Paris, mathematics teacher in Kielce and

---

[11] At that time, the most schools with Polish language of instruction were located in the Austrian Partition.

[12] In the Gymnasium in Trzemeszno in 1856–1862 people with doctoral degree accounted for approx. 27% of all teachers (School reports, Trzemeszno). In the Gymnasium in Braniewo in 187–1880, 1882–1912 and 1915, it was 27% (School reports, Braunsberg). In the Real Gymnasium in Bydgoszcz in 1888–1912 and 1915 the percentage of teachers with a doctorate was 34% (School reports, Realgymnasium zu Bromberg). In the Gymnasium in Bydgoszcz in 1856–1865 and 1869, 1871, 1873–1876, this mean amounted to 23% (School reports, Gymnasium zu Bromberg). In the Gymnasium and Real School (Real Gymnasium) in Toruń in 1860–1874, 1881, 1885, 1889–1892 and 1894–1900 it was about 41% (School reports, Thorn).

[13] See: (*Sprawozdanie…*, 1896, 6–7) and State Archives in Łódź, sign. 39/592/0/3.2/815.

[14] State Archives in Łódź, sign. 39/592/0/2.2/762, 327–328, 356–357, 148–149.

[15] Ibidem, 125–126, 275–278.

Warsaw; since 1821 mathematics professor at Warsaw University.

Kajetan Garbiński (1795–1847) – visit to Paris, since 1820 mathematics teacher at secondary schools in Warsaw and mathematics professor at Warsaw University.

At schools with German as a language of instruction mathematics teachers were also promoted to universities. I would like to mention three widely known names: Karl Weierstrass (1815–1897), Wilhelm Killing (1847–1923), who were teachers in Gymnasium in Braniewo and Martin Ohm (1792–1872) – teacher in Gymnasium in Toruń.

2.3 Conclusions

Teachers working in secondary schools preparing for Matura exams in the 19th and early 20th century had a very good professional education – university studies in the subject they taught. By publishing articles on didactics and preparing textbooks, the school tried to improve the level of teaching in secondary schools. On the other hand, they also took care of their scientific development. Often these were people with a PhD. In the Polish-language schools in the second half of the 19th century and at the beginning of the 20th century, people with a PhD constituted about 7-8% of all teachers, in German-language schools this percentage was greater – approx. 30%. Polish educational authorities, wanting to improve the education of teachers, decided on a solution that was not used in German-language schools – sending teachers to foreign internships. Teachers' scientific activity was often so advanced that they were appointed as academic teachers, where they also achieved significant successes. Mathematics teachers in schools on Polish lands during the partitions can be described as people who have a passion for mathematics. According to the statement *you have to burn in order to shine*, they had all predispositions to also light the passion for this subject in students. They were people with a predisposition to educate intellectual elites.

3 Mathematics teaching in real schools with Polish and German as languages of instruction

Now we are going to compare mathematics teaching in real schools on the territory of Poland under partitions. Real schools were those oriented on mathematics and natural sciences.

3.1 First real schools in Prussian, Russian and Austrian Partitions. First Polish real school

The first real schools were founded in Prussia in the 18th century. The so-called real school established by Christoph Semler in Halle in 1709 was considered to be the first school with a mathematical and natural profile (Heinen, 1863, 13).

The first Prussian ordinances regarding the organization of real schools, which were also valid for real schools opened in the Polish territories under the Prussian Partition, were issued on March 8, 1832 (Wiese, 1864, 27), but they have not yet been rigorously adhered to. After 1832, there are real schools in which the curricula were slightly different. This was the case, e.g., in the Real School in Krotoszyn (School reports, Krotoschin) and St. Peter School in Gdańsk (School reports, Danzig). Much influence on the teaching of mathematics was exerted by the Ordinance of October 6, 1859 (Der

Minister…, 1859) introducing the Matura exams in the selected types of real schools.

In 1864, there were 65 real schools operating in Prussia (Wiese, 1864, 46–48), including 10 in the Polish lands of the Prussian Partition (Wiese, 1864, 46–48), among them there were: St. John School in Gdańsk (Ger. *Johannisschule zu Danzig*), St. Peter School in Gdańsk (Ger. *Petrischule zu Danzig*) and real school in Toruń, Elbląg, Poznań, Międzyrzecz, Wschowa, Rawicz, Bydgoszcz and Grudziądz. Of these, only the real school in Grudziądz did not have the right to conduct Matura exams.

The first real school on the territory of Poland with Polish language of instruction was founded in 1840 under Russian Partition (ВЬДОМСТВО ПРОСВЬЩЕНІЯ…, 1868, 287). It was the Real Gymnasium in Warsaw. Polish students attended this school and the Polish language of instruction was used there. However, it cannot be called a Polish school. Both Poles and Russians influenced the curriculum in this school.

A special curriculum was prepared for the Real Gymnasium in Warsaw. It was not imitative in relation to curricula in other European countries. It was adapted to the local, Warsaw, industrial, commercial and agricultural needs (*Sprawozdanie urzędowe Dyrektora Gimnazjum Realnego Karola Frankowskiego z upłynnionych trzech lat szkolnych, czytane na akcie publicznym dnia 26 czerwca 1844 roku*, 391). In 1840, arithmetics (1st grade – 6 hours a week, 2nd grade – 6 hours, 3rd grade – 6 hours), elementary geometry applied to practice (3rd grade – 5 hours, 4th grade – 5 hours), algebra (4th grade – 4 hours, 5th grade – 3 hours), trigonometry (5th grade – 3 hours), descriptive geometry (5th grade – 5 hours, 6th grade – 4 hours) and conics (6th grade – 2 hours) was lectured in the Real Gymnasium in Warsaw. In the seventh grade, there was no mathematical theory and only and applications of mathematics in chemistry, construction, painting and machine construction (ВЬДОМСТВО ПРОСВЬЩЕНІЯ…, 1868, 301).

In 1845, the Russian Ministry of Public Enlightenment in Petersburg issued regulations concerning curricula in two types of schools in the Polish lands under Russian rule was: real schools (four grades) and higher real schools (six grades). (See: ВЬДОМСТВО ПРОСВЬЩЕНІЯ…, 1868, 343–457).

In Austria, including the Polish territories of the Austrian Partition, real schools were founded since the beginning of the 19th century, e.g. in 1817 there were founded Real School in Lviv (with German language of instruction) (*Bericht der Handels-…*, 1859, 136). But the first general regulations about organisation of the real schools were issued there only in 1849, under the name *Entwurf der Organisation der Gymnasien und Realschulen in Oesterreich* (*Sprawozdanie…*, 1885, 57; Ministerium…, 1849).

The first completely Polish real school was founded on February 19, 1856 in Lviv (*Sprawozdanie…*, 1885, 59). Initially, only the Polish rules of education were in force there. Since 1875 the freedom of Poles under Austrian Partition was limited and the curricula of this school and other Polish real schools had to be approved by occupant (*Sprawozdanie…*, 1885, 61). In 1869, two real schools functioned in the Polish territories under the Austrian annexation with a full seven-grade cycle of teaching: in Lviv and Krakow. There were also three real schools with a three-grade teaching cycle: in Tarnopol, Jarosław and Śniatyń (*Galizisches Provinzial-Handbuch für das jahr 1869*, 1869).

We will divide the analysis of mathematics teaching in real schools in Polish territories during the partitions of Poland into two periods: before 1868 and after 1868, that is before and after obtaining autonomy in teaching by the Polish territories under the Austrian Partition.

3.2 Mathematics teaching before 1868

Before 1868, most schools with Polish language of instruction were in the Russian Partition. The best real school in the Polish lands under the Russian Partition was then the Real Gymnasium in Warsaw. It was a school elite with an extremely wide curriculum. Higher real schools were schools of a more common nature. Thus, the analysis of the curricula in the real schools will allow to present a better, more popular image of mathematical education in the Polish territories under the Russian rule.

This part of the article will compare: ordinances regarding teaching mathematics in higher real schools in the Russian Partition (1845) (ВЬДОМСТВО ПРОСВЬЩЕНІЯ…, 1868, 349–457), ordinances regarding teaching mathematics in the Austrian Partition (1849) (Ministerium…, 1849) and mathematics curricula in two schools under Prussian rule – the Real School in Krotoszyn (School reports, Krotoschin) and St. Peter School in Gdańsk (School reports, Danzig). The analysis will cover the years 1849-1854. The starting data is conditioned by the issuance of Austrian ordinances regarding the functioning of real schools. The year 1854 was chosen as the ending date. It can be assumed that after 1854, the process of including Polish schools in the Russian education system was so advanced that there were no signs of Polishness in them anymore.

In 1849-1854, higher real schools in the Russian Partition, real schools in the Austrian Partition, the Real School in Krotoszyn and St. Peter School in Gdańsk were seven-grade schools. In Krotoszyn and Gdańsk, learning in some classes lasted for two years. The table below illustrates a weekly hour timetable for mathematics.

<table>
<tr><th colspan="2" rowspan="2">school</th><th colspan="9">grade</th><th>number of hours in a week in the education cycle</th></tr>
<tr><th>1st</th><th>2nd</th><th>3rd</th><th colspan="2">4th</th><th colspan="2">5th</th><th colspan="2">6th</th><th></th></tr>
<tr><td colspan="2">Russian Partition</td><td>4</td><td>6</td><td>8</td><td colspan="2">8</td><td colspan="2">7</td><td colspan="2">8</td><td>41</td></tr>
<tr><td colspan="2">Austrian Partition</td><td>5</td><td>4</td><td>4</td><td colspan="2">5</td><td colspan="2">4</td><td colspan="2">4</td><td>26</td></tr>
<tr><td rowspan="2">Prussian Partition</td><td>Krotoszyn</td><td>4</td><td>3</td><td>3</td><td colspan="2">5</td><td>4</td><td>4</td><td>5</td><td>5</td><td>33</td></tr>
<tr><td>Gdańsk</td><td>5</td><td>5</td><td>5</td><td>6</td><td>6</td><td colspan="2">6</td><td colspan="2">5</td><td>38</td></tr>
</table>

Table 3.1: Weekly hour timetable for mathematics in schools on the territory of Poland under partitions

Over half of the time reserved for mathematics classes in all these schools involved discussing arithmetic-algebraic issues. Most time to discuss these issues was devoted in the school in Krotoszyn – 82% of time, in Gdańsk – 74%, in higher real schools in Russia – 56%, and in real schools in Austria – 54%. In the three lowest classes, mainly four arithmetic operations were discussed on concrete and non-concrete numbers, on ordinary and decimal fractions, and the rule of three and its application. In three higher grades, the following was discussed: power, elements, logarithms, 1^{st}, 2^{nd}, 3^{rd} and 4^{th} degree equations, arithmetic and geometric progressions and their applications. In schools in the Russian and Austrian Partitions, as well as in St. Peter School in Gdańsk, additional exponential equations, combinations, Newton's binominal and probability theory and its application to calculating the expected life expectancy were discussed. In all schools, attention was paid to the use of arithmetic in everyday life – for calculating interest, discounts, pensions, retirement and calculations related to deposits and loans.

The remaining part of the curricula was filled with geometrical-trigonometric problems. The Austrian regulations from 1849 introduced geometry already in the lowest 1st grade in 2 hours a week. Discussing geometry started with the introduction of basic planimetric concepts. Stereometry appeared only in the highest 6th grade.

Regulations regarding higher real schools in the Polish territories under the Russian Partition introduced geometry a year later than in Austria – in the 2nd grade. The students first learned the basic planimetric concepts, stereometry was introduced in the 4th grade.

Geometry appeared at the latest in the curricula of schools under the Prussian rule. In Gdańsk, planimetry was usually introduced in the 3rd grade, the exception was year 1850/1851, in which education commenced in the 4th grade. Basic stereometric concepts appeared only in the 6th grade. In Krotoszyn, planimetry was always introduced in the 4th grade, the basic stereometric concepts were discussed a year later.

Geometry in the period under consideration was an important part of the curricula of schools under Austrian and Russian Partition. Almost half of all time devoted to teaching mathematics was devoted to discussing geometric issues. In schools under the Russian Partition, 18 hours week were dedicated to geometry in the education cycle, while at the school in Krotoszyn it was only 1/3 of that time.

When analysing the curricula included in school reports and relevant ministerial ordinances, it can be seen that all schools discussed similar planimetry issues, including, among others: equivalent conversion of figures, division of figures into a given number of parts with equal areas, similarity and congruence of figures, regular polygons, circles with theorems concerning chords and tangents, areas of figures, solving triangles. From the stereometric issues, the basic three dimensional figures were discussed with their areas and volumes were calculated. Conics and their properties were discussed in all schools.

Differences in the curricula were as follows:

1. In Krotoszyn and Gdańsk in 1850/1851 and 1852/1853 and in the Austrian Partition schools in 1849–1854 the spherical trigonometry was discussed. This issue was not present in the curricula of higher real schools under the Russian Partition. In addition, the applications of spherical trigonometry into the mathematical geography and astronomy were discussed in Krotoszyn and Gdańsk.

   In Krotoszyn in 1849–1851, gnomonics was additionally discussed – the construction of sundials and their setting. This issue was not found in other schools.
2. In Austria and Russia, the geometric constructions and their applications in geodesy was an important part of curricula – measuring instruments such as cords and measuring chains, a coal-casing and a measuring table were used and tasks, such as: find the distance between two places on the ground between which the river flows, were solved.
3. In the Russian Partition schools, 4 hours a week in the education cycle was exclusively devoted to the teaching of descriptive geometry. Among others, the tangent and normal to curved lines and curved surfaces were discussed. In 1853/1854, descriptive geometry appeared in the curriculum of the real school in Krotoszyn (5th grade). In the school report from Krotoszyn from this year, there was even an article of the teacher from this school about the need to introduce descriptive geometry to real schools and gymnasia. This is a preview of the upcoming changes in the curricula of Prussian schools – in 1859 descriptive geometry became an obligatory element of the teaching programs of real schools

conducting Matura exams (Der Minister der geistlichen, Unterrichts- und Medicinal-Angelegenheiten, 1859). In real schools in the Austrian Partition, descriptive geometry was not discussed.

After 1854, the russification of schools on the Polish territory under the Russian Partition was already very advanced. Schools were included in the Russian teaching system and the Polish language of instruction was removed. What was it like in the Prussian and Austrian Partitions? In the Prussian Partition, there were school with parallel grades with German and Polish as languages of instruction. Polish language of instruction was applicable only in the lower grades. Two highest grades were German-speaking. However, schools with two languages of instruction were rare. One of such schools was the Real School in Poznań. Comparing the 1857 curricula there from both types of grades, it can be observed that they were almost identical the Prussian teaching system was in force (School reports, Posen, 1857). Polish youth, even if it was taught in Polish, it was done according to the Prussian rules.

In 1856, the first completely Polish real school was created (*Sprawozdanie…*, 1885, 59). It was established in Lviv – in the Austrian Partition. After acquiring autonomy in teaching by the lands of the Austrian Partition (1868), other Polish schools joined it. In 1875, the freedom of Poles was limited and all decisions had to be approved by the invader.

3.3 Mathematics teaching after 1868

3.3.1 Austrian autonomy (1868–1875)

Only a few school reports have survived from the period of "Polishness" to the present day. It is hard to find materials from the 1868–1875 period regarding the teaching of mathematics in the three types of schools that we are interested in: Austrian Partition schools, Austrian schools under the Austrian Partition and Prussian Partition schools which would survive to this day. Nevertheless, it was possible to do so. Higher Real School in Lviv (Austrian Partition) from 1874 (School reports, Lwów, 1874), Higher Real School in Bielsko (Austrian Silesia) from 1874–1877 (School programs, Bielitz, 1875, 1877), Real School in Toruń from 1873–1874 (School reports, Thorn, 1873-1874) and Real School in Poznań (Prussian Partition) from 1874/1875 (School reports, Posen, 1875) were selected for further analysis. The table below presents the curricula implemented in these schools.

school grade	Higher Real School in Lviv		Higher Real School in Bielsko		Real School in Toruń		Real school in Poznań	
	curriculum	hours/week	curriculum	hours/week	curriculum	hours/week	curriculum	hours/week
1st	Arithmetic operations on integer numbers and decimal fractions. Divisibility and prime factorization. Greatest common divisor and least common multiple. Common fractions. Introduction to geometry: two- and three-dimensional figures, perspective, curves constructions.	8	Decimal system. Arithmetic operations on denominate and abstract numbers. Divisibility. Greatest common divisor and least common multiple. Common fractions and decimal fractions. Introduction to geometry: basic planimetric figures with their drawings – triangle, square, quadrangle, regular hexagon, circle and their combinations.	9	Arithmetic operations on denominate and abstract integers.	6	Arithmetic operations on denominate numbers and common fractions.	4
2nd	Ratios and proportions. The Rule of Three and its application. Percentages. Congruence and similarity of triangles. Curves: circle, ellipse, hiperbola, parabola, cykloid, and helical curves. Solids.	7	Ratios and proportions. Percentages and their applications. Calculating the area of plane figures. Drawings using the drafting tool, protractor and set square.	6	Arithmetic operations on denominate integers. Fractions. Counting in memory.	4	Arithmetic operations on common fractions. The Rule of Three and its application. Introduction to geometry: two- and three-dimensional figures, circle, sphere, cylinder, cone.	4
3rd	Percentages and their applications. Second powers and square roots. Third powers and cube roots. Congruence and similarity of	7	Percentages and their applications. Second powers and square roots. Third powers and cube roots. Arithmetic operations on algebraic expressions.	6	Arithmetic operations on common and decimal fractions. The Rule of Three.	3	Arithmetic operations on decimal fractions. The Rule of Three. Basic planimetric figures, triangles and	3

	polygons. Plane and three-dimensional geometrical constructions.		Basics of stereometry. Technical drawings.		Counting in memory.		parallelograms, construction tasks.	
4th	Arithmetic operations on algebraic expressions. Greatest common divisor and least common multiple. Common fractions. Equations of the first degree with one and two unknowns. Calculating the area of figures and volume of solids (with practical application). Converting a given polygon into another one in an equivalent way. Curves constructions. Geometry application to geodesy, measuring the area of the ground using measuring instruments. Orthogonal projection.	6	Arithmetic operations on algebraic expressions. Greatest common divisor and least common multiple. Common fractions. Equations of the first degree with one and two unknowns. Application of arithmetic operations on algebraic expressions to solving planimetric and stereometric tasks. Curves. Introduction to descriptive geometry.	7	Introduction to planimetry up to parallelogram and trapezium. Compound Rule of Three. Percentages. Decimal fractions. Tradesmen calculations.	6	Percentages and their applications. Powers. Arithmetic operations on algebraic expressions. Introduction of formulas: $(a+b)^2$, $(a-b)^2$, $(a+b)(a-b)$. Systems of equations. Parallelogram. Circle, chords and tangents. Tangent circles. Regular polygons. Equality of figures. Pythagorean theorem.	6
5th	Arithmetic operations. Divisibility of numbers. Fractions. Proportions and their application to tradesmen calculations. Powers. Roots. Logarithms.	8	Equations of the first degree with more than two unknowns. Diophantine equations. Number systems, especially the decimal system. Decimal fractions. Powers and roots. Complex	9	Geometry, especially theory of similarity and construction tasks. Arithmetic operations on	6	Powers. First degree equations. Everyday calculations. Similarity of figures. Calculating the area of plane figures. Area of the	6

	First and second degrees equations. Discussion of further planimetric problems. Descriptive geometry[16]: projections of a point, straight line and plane and their relations, projections of three-dimensional solids; sections of solids.		numbers and four arithmetic operations on complex numbers. Ratios and proportions. Quadratic equations with one and two unknowns. Construction tasks. Descriptive geometry: projections of a point, straight line and plane; projections of three-dimensional solids; sections of solids.		expressions using letters. First degree equations with one and two unknowns. Square and cube roots.		circe.	
6th	Equations of higher degrees, which can be reduced to the second degree equations. Continued fractions. Arithmetic and geometric progressions and their applications to pensions calculatings. Combinations, Newton's Theorem. Descriptive geometry: Curve lines, i.e. an ellipse, a hyperbola and a parabola; tangent planes to surfaces; surfaces and planes intersections.	8	Logarithms. Equations of higher degrees, which can be reduced to the second degree equations. Exponential equations. Arithmetic and geometric progressions and their applications to pensions calculatings. Series. Combinations, Newton's Theorem. Plane trygonometry. Calculating the area and volume of solids. Spherical trygonometry. Descriptive geometry: Curve lines; tangent planes to surfaces; surfaces and planes intersections;	8	Discussion of further planimetric problems. Trigonometry. Quadratic equations. Powers, roots and logarithms. Arithmetic and geometric progressions, calculation of interest and pensions.	6	Second degree equations. Roots. Medians and heights of triangles and their properties. Figures inscribed in a circle. Introduction to stereometry.	5

[16] Descriptive geometry in 5th and 6th grades was taught as a separate subject within 3 hours a week.

			learning about shadows.					
7th	Polynomial equations. Second and third degrees equations. Probability theory. Series. Spherical trigonometry and its applications to astronomy and stereometry. Analytic geometry. Geometric drawings: intersections of curve surfaces; tangent planes to curve surfaces.	8	Probability theory with application to calculate the expected life time. Continued fractions. Arithmetic sequences of the higher degrees. Curve lines, i.e. ellipse, hyperbola and parabola. Tangent planes to surfaces. Surfaces and planes intersections. Conics. Descriptive geometry: Perspective. Repetition of the material.	8	Descriptive geometry. Analytic geometry. Arithmetic sequences of higher degrees. Equations of higher degrees. Permutations, combinations, variations and binomial theorem. Exponential and logarithmic series.	6	Roots and powers with fractional exponents. Logarithms. First and second degree equations with one or more unknowns. Arithmetic and geometric progressions. Interest and pensions calculations. Calculating the area of plane figures. Area of the circe. Spherical trygonometry.	5
8th							Equations of the second degree with one or more unknowns. Calculation of interest. Permutations, combinations, variations. Figural numbers. Arithmetic sequences of higher degrees. Third degree equations. Plane and spherical trigonometry. Tasks in mathematical geography and astronomy.	5

Table 3.2: Mathematics curricula in Higher Real School in Lviv, Higher Real School in Bielsko, Real School in Toruń and Real School in Poznań

The comparative analysis of the above curricula allows to see the following differences:

1. Certain issues in different schools were introduced at various stages of education. For example, the logarithms in Lviv were discussed in the 5th grade, in Bielsko and Toruń in the 6th grade, in Poznań – in the 7th grade; Combinations and Newton's binominal in Bielsko and in Lviv were discussed in the 6th grade, in Toruń in the 7th grade, and in Poznań in the 8th grade.
2. One of the most important differences between the above curricula was the moment of introducing basic geometrical objects to school teaching. Basics of geometry in the Higher Real School in Lviv were already introduced in the lowest 1st grade and they included the simultaneous introducing planimetric and stereometric concepts. In Bielsko, geometric concepts were also introduced in the 1st class, with the fact that they only included planimetry. Stereometry in this school appeared only in the 3rd grade. In Poznań, introduction to geometry was planned for the 2nd grade and, just like in Lviv, the planimetric and stereometric concepts were introduced at the same time. In the Real School in Toruń, geometry was introduced in the 4th grade and only the basic planimetric concepts were discussed then. Stereometry has not been included in the above curriculum of the Real School in Toruń, but undoubtedly it was discussed there. Most probably, stereometry was discussed only in the highest grade of the Real School in Toruń (School reports, Thorn, 1873). In schools located in Prussia, also on the territories of Poland under Prussian control, the introduction of geometry was moved to the lowest grade of secondary schools at the beginning of the 20th century – as a result of the reform called Merano Programme (1905). Merano Programme managed the simultaneous introduction of planimetric and stereometric concepts.
3. There are also issues that are found in the curricula of some schools, and not in others. Usually, the lack of mentioning certain issues in curricula means that they were not taught at school. However, this is not always the case. Let us note here the applications of geometry in geodesy, spherical trigonometry and construction tasks.
   a) In the 5th grade in Lviv there are applications of geometry for solving measurement tasks. This issue is not in Bielsko, Toruń and Poznań.
   The application of geometry to geodesy in Higher Real School in Lviv was taught in such a way: teachers took students to forests or meadows and using measuring cords, goniometers and measuring tables, they solved tasks such as finding a distance between two places on the ground between which a mountain was. During solving tasks, they had to use theorems related to congruent and similar triangles. This issue wasn't mentioned in Bielsko, Toruń and Poznań curricula, but it can be supposed that geometry application to geodesy was taught there. This application was discussed in the majority of Prussian and Austrian schools, for example, it was taught by Karl Weierstrass in the Lower Gymnasium (Ger. *Progymnasium*) in Wałcz in the 1840s (School reports, Deutsch-Crone, 1843–1848; Grunert, 1851).
   b) Spherical trigonometry in Bielsko was introduced in the 6th grade, in Lviv and Poznań in the 7th grade. While spherical trigonometry is not found in the curriculum of the Real School in Toruń.

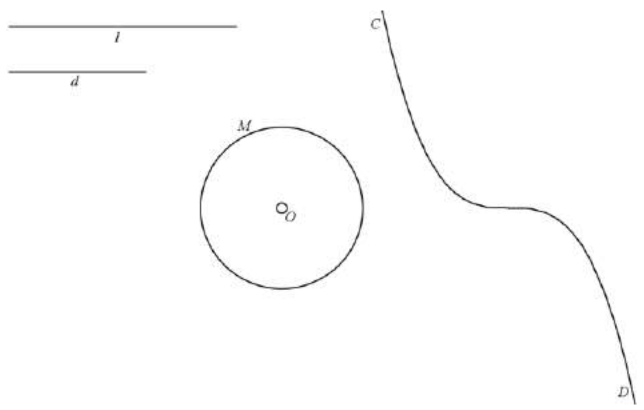
In 1874, Eduard Fassbender was the mathematics teacher of higher grades in the Real School in Toruń. Fassbender published the article *Die Kopernikanischen Sehnen- und Dreieckberechnungen* in 1872 (Fassbender, 1872). This article was a faithful representation of the three chapters of the first book of the *Nicolai Copernici Torinensis De Revolutionibus orbium coelestium, Libri VI,* which the astronomer devoted to the plane and spherical trigonometry. Fassbender, in his article, informed that students of the Toruń school were acquainted with the "world system" of Copernicus, and extensive fragments of the astronomer's work, after adapting to the knowledge and skills of students, were placed in school textbooks. Due to the fact that Copernican plane and spherical trigonometry constituted the basis for all the considerations regarding the structure of the universe, the Toruń school paid attention to them. Fassbender's article, constituting a compendium of Copernicus' geometry, was only a form of systematizing knowledge for students. Thus, although the 1874 curriculum did not mention spherical trigonometry, it was taught there. In the Real School in Toruń, spherical trigonometry was taught according to Copernicus, meaning solving the spherical triangles.

c) In Lviv, Bielsko and Poznań, great attention was paid to geometric constructions. The curriculum implemented in Toruń does not have a clear emphasis on this type of tasks. However, from an article *Beiträge für den Unterricht in der Geometrie* written in 1866 by one of the teachers from Toruń, Otto Reichel, it is known that great importance was attached to the construction there (Reichel, 1866). Reichel in his article placed a list of 27 construction tasks found in the school register. These tasks were not in school textbooks. They were independently prepared by teachers to improve the level of geometry teaching and to prepare young people better for university studies. Among them was the following task:

Exercise (Reichel, 1866, 14)[17]. A circle M and a curve CD are given. Construct a secant ABP of a given length l such that the length of the outer part of this secant is given and equal to d and its endpoint P is on the curve CD.

Solution

Data:



[17] This exercise was not solved by Otto Reichel. Solution of this exercise was prepared by the author of this article.

Sketch of the construction:

1. Construct an arbitrary chord AB of the circle M with a length equal to $(l-d)$.
2. Construct a circle Q with the center O tangent to AB.
3. Extend the chord AB by the segment of the length equal to d. As a result we receive the segment AF.
4. Construct a circle Q' with the center O and the radius OF.
5. The point of intersection of Q' and CD is the end of secant that we are looking for.
6. Construct the secant. Secant should be tangent to the circle Q.

Conditions under which the construction is possible to made:

1. $r \geq \frac{l-d}{2}$,
2. $Q'(O, \sqrt{r^2+ld}\,) \cap CD \neq \emptyset$,

where r is the radius of the given circle M, $Q'(O, \sqrt{r^2+ld}\,)$ is the circle with the center O and the radius $\sqrt{r^2+ld}$ and CD is a given curve.

4. Descriptive geometry was one of the mathematical problems, on which very high emphasis was placed in some schools, and was omitted in others altogether. Descriptive geometry was a very important issue in Higher Real School in Bielsko and Higher Real School in Lviv. In Bielsko it was taught as a separate subject in three highest grades, within 3 hours a week. In Lviv – in two grades: 5th and 6th, within 3 hours a week too. The 1874 report of the school in Toruń included descriptive geometry only as a slogan in the curriculum for the 7th grade. There are no written descriptions of the descriptive geometry discussed during classes. As it was mentioned before, Eduard Fassbender was the teacher in Toruń at that time. Fassbender in 1857 wrote another article *Abriss einer Einleitung in die beschreibende Geometrie* (Fassbender, 1857), in which he mentioned that descriptive geometry was taught by him in Real School in Toruń since 1855 and the lecture in this subject contained all issues discussed in this article. This issues were: the slope of the line to projection planes, calculating the length of a straight line contained between two given points (in other words: calculating the length of the segment), the mutual position of two straight lines and angles between them, the mutual position of a straight line and a plane and the mutual position of two planes. We can see that the descriptive geometry lecture in Real School in Toruń at that time was quite basic. Probably the same issues were discussed in 1874. In Lviv there were taught additionally for example: projections of solids and tangents to curve surfaces.

   Descriptive geometry was an important part of teaching mathematics in Bielsko and Lviv, it was also taught in Toruń, while it was not included in the school curriculum in Poznań.
5. Other issues that are included in the curricula of some schools and were omitted in others include:
   a) In Bielsko, in the 5th grade, complex numbers and arithmetic operations on these numbers are discussed. This issue is not found in Lviv, Toruń and Poznań, but certainly complex numbers were taught there – in school teaching

in the 2nd half of the 19th century, equations were always solved in complex numbers.

b) The exponential equation was discussed in Bielsko in the 6th grade. This was not in Lviv, Toruń and Poznań.
c) The series were discussed in Bielsko in the 6th grade. In Lviv and Toruń, they were introduced in the 7th grade. They are absent in the curriculum implemented in Poznań.
d) Analytical and conical geometry are discussed in Lviv, Bielsko and Toruń. These issues are not in the curricula of the school in Poznań.

The above analysis allows to note discrepancies in the teaching of mathematics carried out in the Polish Higher Real School in Lviv, the Austrian Higher Real School in Bielsko, and schools from the Prussian Partition: Real School in Toruń and Real School in Poznań. The method of teaching mathematics in a Polish school was not imitative in relation to the teaching of this subject in Austrian schools[18] and Prussian schools. The curricula differed, but so did the methods and time of introducing of certain concepts, e.g. in Lviv greater attention was paid to teaching geometry, it was already introduced in the lowest grade, in a visual way – the analysis of stereometric objects enabled the introduction of planimetric objects. Such a solution was not observed in other schools. Although geometry was also introduced in Poznań in an illustrative manner, but it was done a year later – in the 2nd grade.

3.3.2 Mathematics teaching after 1875

After 1875, mathematics curricula in schools with the Polish language of instruction in the Austrian Partition did not change much. The hourly timetables and mathematics curricula in individual classes were almost identical. Comparing curricula from the Higher Real School in Lviv from 1874 (School reports, Lwów) to the curricula for teaching mathematics at the Higher Real School in Krakow in 1876 (School reports, Kraków, 1876), it can be observed that the only significant difference concerned the introduction of geometry. The Lviv illustrative introduction to geometry in the 1st grade was changed in Krakow to introduction of planimetry in the 1st grade, and stereometry only in the 2nd grade.

After 1875, differences between teaching mathematics in Polish-speaking schools under Austrian rule and Austrian German-speaking schools were already small and usually concerned slight shifting of certain issues to the curricula of earlier or higher grades. When analysing teaching mathematics in the Higher Real School in Krakow and Higher Real School in Bielsko to the beginning of the 20th century (School reports, Kraków; School reports, Bielitz), it can be observed that certain issues were added to curricula in the school in Krakow, e.g. in 1899 rational and irrational numbers and exponential equations were added to the curriculum of the 5th grade, and in 1904 the logarithmic equations. The school curriculum in Bielsko actually remained unchanged –

[18] Bielsko, in the period under consideration, was part of the Austrian Silesia. The curriculum in the Higher Real School in Bielitz listed in the table was based on the Austrian regulations of July 19, 1870, August 9, 1873 and Silesian ordinances of November 5, 1874 (these ordinances did not apply to the teaching of mathematics) (School reports, Bielitz, 1875). The curricula in the Real School in Lviv differed from the curricula implemented in Bielsko. Curricula in the Real School in Lviv were also not in accordance with the Austrian regulations of 1849.

no new issues were added to it. In 1899, in both schools the method of introduction to geometry was unified – geometry was introduced in the 1st grade in the demonstrative manner: from stereometry to planimetry.

In schools under Prussian rule, program changes were also small. In 1884/1885, analytical geometry, cones and series were added to the curricula of the Real School in Poznań (School reports, Posen). In Toruń, in 1899/1900 minima and maxima were added to the curriculum of the highest grade (School reports, Thorn). The remaining changes were cosmetic.

The way of introducing geometry to school education and the scope of its discussion was still the basic difference in curricula. In schools under Prussian rule, descriptive geometry was discussed symbolically or was not discussed at all, whereas in schools under Austrian rule, descriptive geometry was taught as a separate subject 3 hours a week.

3.4 Conclusions

In 1849-1854, higher real schools, in which Polish youth educated, were mainly in the Russian Partition. 41 hours of mathematics a week was planned in them in the whole cycle of education. It was the largest number of math hours out of all real schools at that time in the Polish lands. While in the Prussian Partition schools the vast majority of time was devoted to discussing arithmetic-algebraic issues, in schools with the Polish language of instruction under Russian Partition and schools in the Austrian Partition, arithmetic-algebraic and geometric-trigonometric issues were treated equally – almost the same amount of time was devoted to their discussion. In curricula of schools located in all partitions, a lot of attention was paid to the use of mathematics, especially to perform the so-called civic settlements, i.e. calculation of interest, profit, loss, pension, etc. One of the basic applications of geometry was to solve measurement tasks. In schools under the Prussian and Austrian rule, spherical trigonometry and its applications to mathematical geography and astronomy were discussed; in Krotoszyn, the construction of sundials was even performed – these issues were not in Polish-language schools under the Russian Partition. On the other hand, the Polish-speaking schools paid a lot of attention to the descriptive geometry needed by future architects and constructors. Let us recall that descriptive geometry was not in schools under Austrian rule at that time, it was also rate in schools under Prussian rule.

The second important period from the point of view of the Polish education in Polish territories during the partitions was in the years 1868-1875. At that time, Polish schools were under the Austrian rule. Here, the difference in the teaching method is clear. In Polish schools, the geometric concepts in the lowest 1st grade were introduced in an illustrative manner – the abstract planimetric concepts were introduced based on stereometric concepts known to students from everyday life[19]. A similar solution was used in Poznań, but it was not a systemic solution, but a unitary solution. In Austrian schools, demonstrative teaching of geometry was introduced in the 1890s. In Prussian schools – only at the beginning of the 20th century. Polish schools paid a lot of attention to mathematical theory. They discussed most of these issues provided for teaching in

[19] At the end of the 18th century, school textbooks on illustrative geometry were published. A very popular textbook in the Austrian Partition, used, e.g. in the Nowodworski School in Krakow, was the textbook: F. Močnik, *Geometria poglądowa dla klas niższych szkół średnich*, Lwów, 1896.

secondary schools in the 19th century, together with analytical geometry and cones, which in Poznań were introduced in 1884/1885. In 1868-1875, in schools of all partitions, the concern to show the applications of mathematics to civic settlements continued, as well as to solving measurement tasks, along with the discussion of spherical trigonometry with its applications. A lot of attention was paid to structural tasks. In Polish and Austrian schools, descriptive geometry was a very important part of the curriculum.

When analysing the teaching of mathematics in Polish territories during the partitions, one can assume that the success of Polish-language schools under Russian Partition, and then Polish schools under the Austrian Partition, paying a lot of attention to teaching geometry was a success (these schools were the leaders in teaching descriptive geometry). Looking in general terms on the 19th century, schools for Polish students paid more attention to teaching geometry than in the German-speaking schools of the countries occupying Poland, that is, in Austria and Prussia.

It is worth mentioning there that geometry was a field of mathematics, which has been particularly valued by didactics for centuries. Samuel Dickstein spoke about geometry as an indispensable part of the school and home educational system. In geometry he saw the science that develops the mind of a young man, forces him to look for connections between different geometric forms, and *abstract, seemingly dry and non-absorbing claims, with a skilful lecture, become alive, arouse curiosity and prepare the mind for self-use of own strength* (Dickstein, 1889, 67). According to this statement, the construction tasks and descriptive geometry taught young people independent and logical thinking. Learning of mathematics by action – the self-made constructions – also arouses interest in the subject and helps to increase motivation for learning.

4 Mathematics teaching in gymnasia with Polish and German as languages of instruction

Now we are going to analyse mathematics teaching in gymnasia with Polish and German as languages of instruction.

The best way of checking both: the level of mathematics teaching and the most important topics in mathematics curricula, is to analyse the then secondary school-leaving examinations (Matura examinations) in mathematics. For almost the entire period which is took into consideration in this article Matura examinations tasks were prepared by the teachers of the schools in which these examinations were conducted[20]. It means that each school had an individually prepared set of examination tasks.

Regulations introducing the world's first examinations at the end of secondary schools enabling university studies were introduced in 1788 in Prussia. According to these regulations, mathematics wasn't an obligatory subject of Matura examinations (Domoradzki & Karpińska, 2017).

First Polish regulations introducing Matura examinations were published on February 17, 1812 (Potocki, 1812). A few months later, on June 25, 1812, the second Prussian Matura regulations were introduced. Much later, Matura exams were introduced in other European countries, e.g. in Austria these examinations were introduced in 1849, and in France in 1852 (Majchrowicz, 1869, 17).

[20] The sets of tasks had to be approved by the Chairman of the Examining Board, who was a member of a national body supervising schools.

4.1.1 Polish and Prussian Matura regulations from 1812

First Polish Matura regulations were introduced by the Directorate of National Education of the Duchy of Warsaw on February 17, 1812 (Potocki, 1812). From now on, only people with the Matura exam certificate could undertake studies at Polish universities. Polish regulations established mathematics as an obligatory subject of oral Matura examinations for all students. Moreover, mathematics was an obligatory subject of written Matura examinations in case of those students who wanted to study architecture. The introduction of Matura examinations required the unification of curricula in those schools in which Matura exams were carried. In the Polish Duchy of Warsaw, the framework curricula was set up in the 1812 regulations (Potocki, 1812).

Prussian Matura regulations from 25^{th} of June 1812 established mathematics as an obligatory subject of every written and oral Matura examination for all students (Wiese, 1864, 484–485). These regulations set the requirements for secondary school graduates, but the curricula in individual classes were done a bit later as a consequence of the Humboldt reform.

It shows us that mathematics was as important a part of the Polish education as it was in Prussia.

Poland (Potocki, 1812)	Prussia (Wiese, 1864, 485)
• first, second and higher degrees equations, • logarithms, powers, roots, • planimetry and stereometry, • plane and spherical trigonometry, • application of mathematics to everyday life calculations, application of mathematics to physics and geography.	• first and second degree equations, • logarithms, powers, roots, • planimetry and stereometry, • plane trigonometry, math tables, • application of mathematics to everyday life calculations.

Table 4.1: 1812 requirements to the mathematics Matura examination

The scope of the material obligatory for the Matura exam in mathematics in Poland was broader than the scope obligatory in Prussia with the following issues: higher degrees equations, spherical trigonometry and application of mathematics to physics and geography. This allows to assume that Polish schools preparing for Matura exams have implemented wider curricula than in Prussian schools, including schools located in Polish territories under Prussian rule.

The quoted curriculum of the two highest grades of the Polish Warsaw Secondary School (Pol. *Liceum Warszawskie*) from 1812 (this year, the first Matura exam took place there) (School reports, Warszawa):

> 5^{th} grade: The largest common divisor and the smallest common multiple, roots of the second and third degree, equations higher than the 2^{nd} degree, stereometry, spherical trigonometry with the application to astronomy.
>
> 6^{th} grade: Combinations, Newton's binominal, 3^{rd} degree equations using to the division of the circle into 3, 5, 7 etc. equal parts, solving the 4^{th} level equations with the discussion of the Bombelli and Euler rules, solving equations of any degrees by approximation, conics and their applications in optics, astronomy, civil and military architecture, etc., basics of descriptive geometry according to G. Monge.

The mathematics curriculum at the Warsaw Secondary School was very extensive, as for the school with a humanistic profile. The applications of 3rd degree equations to dividing the circle into 3, 5, 7 etc. equal parts or conics applications in optics, civil and military architecture, were not even discussed at the real schools established at a later time. Also descriptive geometry was rarely taught in humanistic schools. The establishment of real schools meant that in the humanistic schools, mathematics curricula and the scope of issues applicable at the Matura exams have been limited over the years.

After 1812, both Polish and Prussian, new Matura regulations were issued, which did not change the general organization of Matura exams, but refined, among others, examination subjects, the scope of the Matura material and the duration of exams. The regulations issued in Poland in 1819 were of particular importance, under which mathematics became a compulsory examination subject for all students (Potocki, 1867, 473–475). In Prussia in 1834, ordinances were issued according to which the written examination in mathematics was to last 4 hours and consisted of two arithmetic and two geometrical tasks (Wiese, 1864, 492–504). Further regulations were introduced, among others, in 1859, 1866, 1874.

4.1.2 Matura examinations under Austrian Partition

In Austria, the Matura exams were introduced in 1849. In the same year, they were introduced in schools located in the Polish lands under the Austrian Partition. According to the ordinances, mathematics was a compulsory subject of written and oral Matura examinations. These regulations were very generally related to the Matura requirements. In the case of mathematics, it was assumed that by taking part in the Matura exam, the student was to know all the mathematical theorems discussed at school along with their proofs and all the tasks that were solved during the classes. During the Matura exam, students were required to apply knowledge acquired at school to solve new problems or to prove theorems that were not discussed at school. During the written exam in mathematics, it was recommended to solve tasks in the field of planimetry and trigonometry, and in particular tasks that at the same time checked the geometric, arithmetic and trigonometric knowledge were recommended (Ministerium…, 1849, 195–196). In the 1850s and 1860s, a series of successive Austrian Matura ordinances was issued, which, among others, increased the number of subjects of Matura examinations and clarified the scope of the material used during the exams (Matauschek, 1864, 125–130). After obtaining the autonomy by the territories under the Austrian annexation, the Polish Ministry of Education modified the general regulations of the Matura examinations. At the beginning of the 80s, the number of subjects passed during the oral Matura examinations was limited in Polish schools to five: Latin, Greek, Polish, German and mathematics (*Sprawozdanie…*, 1885, 172).

4.2 Matura tasks in mathematics

In schools under the Austrian Partition, Matura exams were introduced in 1849, which is why further comparative analysis will cover the second half of the 19th century. When preparing this article, it was impossible to find mathematics Matura exams conducted in schools with the Polish language of instruction under the Russian rule. Thus, the comparative analysis will cover the Matura exams from the Austrian and Prussian Partitions.

We will begin the analysis with a discussion of written mathematics Matura exams conducted on the Polish territory under the Austrian Partition before it was granted autonomy.

In 1864, in the Gymnasium in Przemyśl, which was then a German-language school included in the Austrian education system, the written Matura exam was as follows[21]:

Exercise 1. Solve the equation: $3\sqrt{12-\sqrt{x-3}}=\sqrt{5x+21}$.

Exercise 2. Two forces P=37˙5 N and Q=89˙23 N are directed towards each other at an angle of $72°43'21˙7''$. What is their resultant force?

Exercise 3. A certain capital bears the 4% interest. How will it increase after 20 years of interest capitalization?

Examples of oral exercises were as follows[22]:

Exercise. We know the height and the bases of a truncated pyramid, calculate the volume of this solid.

Exercise. Calculate the surface of the circle.

It can be seen that apart from the mathematical theory, much attention was paid to the applications of mathematics in Przemyśl. Two Matura exam tasks are related to the applications: one task concerns the use of trigonometry in physics, and the second the applications of arithmetics to the accounts related to deposits. The purely theoretical task of the written exam is to solve the equation. It can be said that during the written exam, one of the basic ministerial orders has not been implemented here – paying attention to geometry. This postulate was implemented only at the oral exam.

In 1875, after 8 years of autonomy, the freedom of Poles under Austrian occupation was again limited, and curricula in schools with Polish language of instruction were similar to those implemented in Austrian schools (this was previously shown on the example of real schools). And what about mathematics Matura exams? It has been described above that in the 1880s, new Matura regulations concerning the Matura exams in the Polish schools under Austrian Partition were implemented. Did they introduce any change to the mathematics Matura exams?

In 1889, the Gymnasium in Przemyśl was already a Polish-language school. The mathematics Matura exam looked as follows (School reports, Przemyśl):

Exercise 1. Solve the equations:

$$\frac{\sqrt{3x}}{x+y}+\frac{\sqrt{x+y}}{3x}=2$$
$$xy-(x+y)=54$$

Exercise 2. Someone is entitled to a pension of PLN 800 annually for 20 years; however, he wants to convert it into another pension of PLN 1000 a year; how long will he collect this second pension if the rate is 4%.

Exercise 3. Calculate the volume of the scalene cone which shortest side $b=17^m$ is inclined to the base at an angle $\alpha=82°54'30''$ and the longest side a is inclined to the base at an angle $\beta=35°40'20''$.

[21] State Archives in Przemyśl, sign. 56/387/0/1.1/19.
[22] State Archives in Przemyśl, sign. 56/387/0/1.1/19.

Example of tasks from the oral exam[23]:

Exercise 1. Calculate the length of the string from the top of the major axis at an angle of $30°$ in the ellipsis.

Exercise 2. Solve the triangle if there is a given circumference and two angles.

As before 1864, there were three tasks in the written exam in the Gymnasium in Przemyśl. Two of them were arithmetic-algebraic, and one – geometric-trigonometric. During the written exam, more attention was paid to arithmetic-algebraic problems, while during the oral exams – to geometric-trigonometric problems. The tasks of mathematics applications in everyday life were still important.

The level of difficulty of tasks, in relations to those from 1864, definitely increased. In 1864, the Matura tasks were even at the elementary level, these were schematic tasks, which were certainly discussed during maths lessons at school. In 1886, the level of task difficulty is higher, and the tasks, especially from the oral exam, require logical, independent thinking and multi-stage reasoning.

Cracow St. Anna Gymnasium[24]	**Inowrocław Gymnasium** (School reports, Inowroclaw)
Exercise 1. Solve the equation: $\sqrt{\frac{3}{2}x-5}-\frac{1}{7}\sqrt{\frac{1}{5}x+45}=\frac{1}{4}\sqrt{10x+56}$ **Exercise 2.** The population of the city, in which there were 32500 inhabitants, rose after 24 years by 35566 souls. What is the annual percentage of growth? **Exercise 3.** Find an angle between the horizontal ground and the road leading up, if for a length of 10 fathoms (Pol. *sążeń*) on the road the height above the horizontal ground increases by one fathom.	**Exercise 1.** 6 numbers are given. The first 4 numbers of those 6 numbers creates a geometric progression and the last 4 numbers creates an arithmetic progression, such that the common difference of arithmetic progression is 12 times greater than the common ratio of the geometric progression. Moreover, the second last number is 15 times bigger than the second number. Find these numbers. **Exercise 2.** Construct a right triangle, if the sum of the hypotenuse and one of its legs is given and equal to a and the sum of the hypotenuse and the second leg of this triangle is given and equal to b. **Exercise 3.** Someone has a cash, which not exceed 350 thalers. When he distributes this cash into a sequence of 10 thalers, then he is left with 1 thaler. When he distributes this cash into a sequence of 15 thalers, then he lacks 4 thalers. Calculate how much money this person has, knowing that all cash can be completely distributed into 11 terms sequence. **Exercise 4.** Truncated cone with the radii of the bases equal to $r = 5'$ and $\rho = 3'$ and the side inclined to one of its bases at an angle $\varphi = 72°45'36''$, is given. On this cone a sphere is described. Calculate the area of the sphere bounded by the peripheries of the cone bases.

Table 4.2: Matura tasks in 1868

In 1868, after obtaining the autonomy, the first Polish Matura exams were carried out

[23] State Archives in Przemyśl, sygn. 56/387/0/2.1/91.
[24] National Archives in Cracow, sign. 29/482/191, 399, 402.

in the Austrian Partition lands. We already know that in the Polish lands before 1868 and after 1875 they were similar in form, only the level of difficulty of the Matura exams increased. Now we will check what the Polish Matura exams looked like from the period of the autonomy. We do this on the example of Cracow St. Anna Gymnasium conducted in 1868 (Table 4.2). This Matura exam will also be compared to the mathematics Matura exam conducted the same year in the Inowrocław Gymnasium under the Prussian Partition.

When analysing the tasks from the written Matura exam conducted in Cracow St. Anna Gymnasium in 1868, it can be noted that it has the same form as the examinations carried out in the territories of the Austrian Partition before obtaining the autonomy and after its limitation. The Poles took over the Austrian Matura exam rules for the mathematics exams.

There is a clear difference between the mathematics Matura in Cracow St. Anna Gymnasium and the Gymnasium in Inowrocław in 1868. Above all, Inowrocław paid more attention to geometric-trigonometric tasks. The Matura set included two tasks from this field.

Making a broader comparative analysis of mathematics Matura exams in Polish territories during the partitions, it can be stated that generally, from the mid-19th century, in schools with Polish as a language of instruction there were always three tasks in each written Matura examination in mathematics. The scopes of Matura tasks weren't established obligatory there, but usually the Matura exam sets had two arithmetic-algebraic tasks and one geometrical-trigonometric task. In schools with German as a language of instruction under Prussian Partition usually there were four tasks: one arithmetic or algebraic, one planimetric, one trigonometric and one stereometric task. The most popular tasks in both types of schools were those connected with: arithmetical and geometrical progressions, solving an equation or a system of equations, it was often an exponential, logarithmic or trigonometric equation, solving triangles, calculating the area and the volume of polyhedral or solids of resolution, everyday life calculations (calculating profits, losses, interest and the amount of pensions) and the application of mathematics to physics. There were two types of tasks which were only in schools with German as a language of instruction – construction tasks and those related to arithmetic sequences of the second degree.

4.3 Conclusions

The analysis of the form of Matura exams in mathematics conducted in classical gymnasium located in the Polish lands in the second half of the 19th century allows to conclude that schools with Polish language of instruction modelled the teaching of mathematics at the Austrian schools. While from 1875 it was imposed by the invader, in 1868-1875 it was a voluntary decision of the Poles.

Although the form of Matura exams in schools with Polish language of instruction under Austrian Partition and schools with German language of instruction in the Prussian Partition was slightly different, the detailed analysis of written and oral examinations allows to conclude that both types of schools discussed similar issues and equal weight was applied to both arithmetic and algebraic issues, as well as geometric and trigonometric ones. An important part of teaching in both cases were the applications of mathematics.

The most significant difference between the Matura exams and, hence, the mathematics curriculum, was the discussion, or lack thereof, of construction tasks. In Polish-language gymnasia, no importance was attached to this type of tasks. Certainly it was a shortcoming of the curriculum, because this type of tasks, due to their multi-stage aspect, shaped the logical and independent reasoning, and through their applications they could arouse students' interest of the subject.

In the second half of the 19th century, there is the already vivid activity of Polish teachers on the reform of teaching intended for Polish secondary schools. Kamil Kraft and Stanisław Ziobrowski from the 4th Gymnasium in Krakow along with seven other members of the Polish Society of Teachers of Universities in 1906 announced a reform project entitled "Polish secondary school – criticism of its foundations and necessity of reform" (Dropiowski, Kraft, Łopuszański, Nitsch, Sobiński, Stein, Tołłoczko, Wasung & Ziobrowski, 1906a, 1906b). The basis of the project was the removal of the Austrian educational system from Polish schools and the introduction of a Polish system adapted to the needs of the Polish society. The greatest demand in the Polish nation was for people educated in the field of mathematical and natural sciences – experts in their application in everyday life, in the economy and industry. Therefore, the reform project put a special emphasis on the teaching of mathematics. The Austrian teaching system of this subject has been accused of overloading the material, that the tasks are schematic, based on standard algorithms, and the whole science of arithmetics is based on obtaining mechanical skill in the calculations. In Polish schools, the plan was to limit the scope of the discussed material. It was believed that the basics should be taught, but in such a way that the students could understand them perfectly, not only solve schematic tasks, but also those that go beyond the patterns – requiring logical and creative thinking. Great emphasis was also placed on tasks related to the needs of everyday life. An illustrative study of mathematics and gradual introduction of the difficulty (the principle of accessibility) were recommended.

The above project has not been implemented. However, its main postulates were included in the reform introduced in Poland shortly after regaining independence (the reform introduced in 1919).

5 Summary

The comparison carried out in this article makes it possible to notice that Polish-language schools operating in Polish territories during the partitions in certain matters related to teaching of mathematics introduced pioneering solutions. One of these solutions was the issue of teacher training. Polish educational authorities have decided to improve the competence of secondary school teachers and send them on academic internships in foreign secondary schools and at universities. This solution was not used in schools under the authority of Prussia and Austria.

It was Poland, Duchy of Warsaw, that was the first country in which the first Matura rules were introduced establishing mathematics as an obligatory subject of examination. It took place on February 17, 1812. In Prussia, mathematics became an obligatory Matura exam subject a few months later on June 25, 1812. Polish ordinances established a broader scope of material for the Matura exam in mathematics than in Prussia. In Poland, knowledge of higher degrees, spherical trigonometry and the applications of mathematics in physics and geography was required. These ordinances were not purely theoretical.

Schools had to implement them. An example here is the Warsaw Secondary School, which met all the curriculum requirements. Polish mathematical curricula for humanistic schools were wider than those in the Prussian and Austrian schools of that period. Nevertheless, over the years, curricula implemented in humanistic Polish-speaking schools began to resemble those in German-language schools.

Mathematical education from the mid-19th century was the domain of real schools. In these schools, the mathematics curriculum was the widest and they show the greatest differences when it comes to schools with Polish and German languages of instruction. In the 40s, Polish-language schools under the Russian Partition and German-language schools under the Austrian Partition put a lot of emphasis on teaching geometry. About half of the time reserved for mathematics classes in these schools was devoted to geometric issues. Polish schools were the first in which systematic emphasis was placed on teaching descriptive geometry, especially due to its applications.

The period of autonomy of Polish schools, 1868-1875, brought about yet another systemic innovation of Polish schools – an illustrative introduction of stereometric and planimetric concepts in the lowest grade of real schools.

Looking in general terms at the 19th century, the most important difference in the teaching of mathematics in Polish and German-speaking schools in the Polish lands is the method of teaching geometry.

It is difficult to talk about the national characteristics of teaching mathematics in Polish-speaking schools. There are few sources to assess whether the special needs of Polish society have been taken into account in the teaching of mathematics. Clear information on this subject was found only in the case of the Real Gymnasium in Warsaw. The nationalist movement in the teaching of mathematics in the Polish lands can be seen only at the end of the 19th century thanks to the efforts of Kraft, Ziobrowski and others. The effects of their work were implemented in school education only after Poland regained its independence.

In all schools functioning on the Polish territory during the partitions, comparable importance was attached to the applications of mathematics (the exception was the descriptive geometry, which in schools under Prussian rule was treated negligibly). The applications of geometry were particularly important, e.g. in geodesy, architecture or astronomy. Geometry was taught in context. This is an indication for modern educational practice. The benefits of teaching geometry in this way are currently described by, among others, Bartolini Bussi & Boero (1998). Malkevitch (1998, 23) draws attention to the need to improve the quality of education in the United States, and among the councils indicates that it is necessary to show the increasingly powerful new applications of geometry in the world that surrounds the young man. Mammana &Villani (1998a, 4) say that in modern schools, the geometry curriculum should be prepared very carefully, which will show a balance between teaching contents, methods and motivations for learning. It is important to combine new and old ideas in new curricula (Mammana & Viliani, 1998b, 9–28). This article identifies issues from the history of teaching, which show the applications of geometry in the world that surrounds a modern man, e.g. gnomonics – sundials are an element of modern architecture. The history of teaching mathematics in the Polish lands provides new fields of experience, which can help develop students' activity, i.e. discussing descriptive geometry or measuring tasks.

REFERENCES

ВѢДОМСТВО ПРОСВѢЩЕНІЯ / Wydział Oświecenia (Ed.). (1868). *СБОРНИКЪ АДМИНИСТРАТИВНЫХЪ ЦАРСТВА ПОЛЬСКАГО. ВѢДОМСТВО ПРОСВѢЩЕНІЯ / Zbiór przepisów administracyjnych Królestwa Polskiego*, vol. 3: СРЕДНІЯ ЧЕБНЫЯ ЗАВЕДЕНІЯ / Zakłady Naukowe Średnie. Warszawa: W drukarni S. Orgelbranda.

Bartolini Bussi, M., & Boero, P. (1998). Teaching and learning geometry in contexts. In C. Mammana, & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st Century. An ICMI Study* (pp. 52–62). Dordrecht: Springer Science+Business Media.

Czerniakowska, M. (2008). Matematyka i fizyka w gimnazjum Gdańskim. In E. Kotarski (Ed.), *Gdańskie Gimnazjum Akademickie*, vol. 1. *Szkice z dziejów* (pp. 155─180). Gdańsk: Wydawnictwo Uniwersytetu Gdańskiego.

Der Minister der geistlichen, Unterrichts- und Medicinal-Angelegenheiten (Ed.). (1859). *Unterrichts- und Prüfungs-Ordnung der Realschulen und der höheren Bürgerschulen*, Second edition. Berlin: Verlag von Wiegandt und Grieben.

Dickstein, S. (1889). *Gieometryja elementarna*. Warszawa: W drukarni Józefa Sikorskiego.

Domoradzki, S., & Karpińska, K. (2017). O egzaminie maturalnym z matematyki na obszarze zaboru pruskiego od XVIII do początku XX wieku. *Antiquitates Mathematicae*, *11*(1), 157–201.

Dropiowski T., Kraft, K., Łopuszański, T., Nitsch, K., Sobiński, S., Stein, I., Tołłoczko, S., Wasung, W., & Ziobrowski, S. (1906a). Nasza szkoła średnia – krytyka jej podstaw i konieczność reformy, cz. I. In *Muzeum. Czasopismo Poświęcone Sprawom Wychowania i Szkolnictwa*, *1*(3), 191–214.

Dropiowski T., Kraft, K., Łopuszański, T., Nitsch, K., Sobiński, S., Stein, I., Tołłoczko, S., Wasung, W., & Ziobrowski, S. (1906b). Nasza szkoła średnia – krytyka jej podstaw i konieczność reformy, cz. II. In *Muzeum. Czasopismo Poświęcone Sprawom Wychowania i Szkolnictwa*, *2*(2), 92–145.

Duda, R. (2019). *Historia matematyki w Polsce na tle dziejów nauki i kultury*. Warszawa: Instytut Historii Nauki PAN, Wydawnictwo ASPRA.

Fassbender, E. (1857). Abriss einer Einleitung in die beschreibende Geometrie. In *Nachricht von dem Königlichen Gymnasium zu Thorn und den mit demselben verbundenen Real-Klassen* (pp. 1─32). Thorn: Gedruckt in der Rathsbuchdruckerei.

Fassbender, E. (1872). Die Kopernikanischen Sehnen- und Dreieckberechnungen. In *Gymnasium mit Realschule I Ordnung zu Thorn* (pp. 1─12). Thorn: Gedruckt in der Buchdruckerei von J. Buszczyński.

Galizisches Provinzial-Handbuch für das jahr 1869. (1869). Lemberg: Aus der k.k. galiz. Aerarial-Staats-Druckerei.

Grunert, J.A. (1851). *Lehrbuch der ebenen Geometrie für die mittlern Classen höherer Lehranstalten*. Brandenburg: bei J. Wiesike.

Heinen, F. (1863). *Die Städtische Realschule I Ordnung zu Düsseldorf, nebst Geschichtlichen aus der Entwickelung des Realschulwesens überhaupt*. Düsseldorf: Hof-Buchdruckerei von Hermann Voß.

Karp, A. (2009). Teaching the Mathematically Gifted: an Attempt at a Historical Analysis. In R. Leikin, A. Berman, & B. Koichu, *Creativity in Mathematics and the Education of Gifted Students* (pp. 11-29). Rotterdam: SENSE Publishers.

Karp, A., & Furinghetti, F. (Eds.). (2016). *History of Mathematics Teaching and Learning. Achievements, Problems, Prospects*. Cham: Springer.

Karp, A., & Furinghetti, F. (Eds.). (2018). *Researching the History of Mathematics Education. An International Overview*. Cham: Springer.

Koppe, K. (1869). *Anfangsgründe der Reiner Mathematik für den Schul- und Selbst-Unterricht*, vol. 1: Arithmetik und Algebra. Essen: Druck und Verlag von G.D. Bädeker.

Kröger, J.C. (1837). *Denkschrift über den Gymnasial-Unterricht im Königreich-Preußen von V. Cousin*.

Altona: bei Johann Friedrich Hammerich.
Królikowski, L. (2008). *Szkolnictwo dawnej Warszawy. Od połowy XVII wieku do wybuchu drugiej wojny światowej*. Warszawa: Muzeum Historyczne m.st. Warszawy.
Kubik, K. (1972). Nauczanie matematyki w toruńskim Gimnazjum Akademickim w XVI–XVIII w. In Z. Zdrójkowski (Ed.), *Księga Pamiątkowa 400-lecia Toruńskiego Gimnazjum Akademickiego*, vol. 1: [XVI–XVIII w.] (pp. 113–138). Toruń: Komitet Obchodu 400-lecia Gimnazjum Akademickiego im. Mikołaja Kopernika w Toruniu.
Majchrowicz, F. (1869). W sprawie egzaminów dojrzałości w naszych szkołach średnich. In *Muzeum. Czasopismo Towarzystwa Nauczycieli Szkół Wyższych*, *12*(1), 14─17.
Majorek, C. (1980). *Projekty reform szkolnictwa ludowego w Galicji u progu autonomii (1860─1873)*. Wrocław-Warszawa-Kraków-Gdańsk: Zakład Narodowy imienia Ossolińskich, Wydawnictwo Polskiej Akademii Nauk.
Malkevitch, J. (1998). Finding room in the curriculum for recent geometry. In C. Mammana, & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st Century. An ICMI Study* (pp. 18–25). Dordrecht: Springer Science+Business Media.
Manteuffel, T. (1929). *Centralne władze oświatowe na terenie b. Królestwa Kongresowego (1807–1915)*. Warszawa: Nakładem Towarzystwa Naukowego Warszawskiego.
Mammana, C., & Villani, V. (1998a). Geometry and geometry-teaching through the ages. In C. Mammana, & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st Century. An ICMI Study* (pp. 1–4). Dordrecht: Springer Science+Business Media.
Mammana, C., & Villani, V. (Eds.). (1998b). *Perspectives on the Teaching of Geometry for the 21st Century. An ICMI Study*. Dordrecht: Springer Science+Business Media.
Matauschek, T.A. (1864). *Normalien-Nachschlagebuch für Lehrer und Direktoren der Oesterreichischen öffentlichen Gymnasien*, Third edition. Prag: Carl Bellmann's Verlag.
Ministerium des Cultus und Unterrichts (Ed.). (1849). *Entwurf der Organisation der Gymnasien und Realschulen in Oesterreich*. Wien: Gedruckt in der kaiserlich-königlichen Hof- und Staatsdruckerei.
Opial, Z. (1958). O pracach Jana Brożka z teorii liczb. In *Kwartalnik Historii Nuaki i Techniki*, 3/4, 537─563.
Popławski, A. (1780). *Mowa przy otwarciu Seminarium na Professorów Narodowych Stanu Akademickiego, dnia 2 października 1780. W Krakowie miana, poprzez J. X. Antoniego Popławskiego postanowionego Rektora tegoż Seminarium, od Prześwietney Kommissyi Edukacyyney*. Kraków.
Potocki, S. (1812). *Wewnętrzne urządzenia szkół departamentowych 1812 roku*. Warszawa.
Potocki, S. (1867). Wewnętrzne urządzenia szkół wojewódzkich z roku 1819. In ВЬДОМСТВО ПРОСВЬЩЕНIЯ/Wydział Oświecenia (Ed.). (1868). *СБОРНИКЪ АДМИНИСТРАТИВНЫХЪ ЦАРСТВА ПОЛЬСКАГО. ВЬДОМСТВО ПРОСВЬЩЕНIЯ / Zbiór przepisów administracyjnych Królestwa Polskiego*, vol. 2: СРЕДНIЯ ЧЕБНЫЯ ЗАВЕДЕНIЯ / Zakłady Naukowe Średnie (pp. 441─529). Warszawa: W drukarni S. Orgelbranda.
Rakoczy-Pindor, K. (2012). Wykształcenie nauczyciela gimnazjalnego matematyki w Galicji w latach 1850─1918. In W. Więsław (Ed.), *Dzieje matematyki polskiej* (pp. 163─172). Wrocław: Instytut Matematyczny Uniwersytetu Wrocławskiego.
Reichel, O. (1866). Beiträge für den Unterricht in der Geometrie. In *Königliches evangelisches Gymnasium und Realschule erster Ordnung zu Thorn* (pp. 1─17). Thorn: Gedruckt in der Rathsbuchdruckerei.
Seidl, J. G., Bonitz, H., & Mozart, J. (Eds.). (1856). *Zeitschrift für die österreichischen Gymnasien*, vol. 7. Wien: Druck und Verlag von Carl Gerold's Sohn.
Sprawozdanie c.k. Rady szkolnej krajowej o stanie szkół średnich galicyjskich w latach szkolnych 1875─1883, vol. 2 (1885). Lwów: Nakładem funduszu naukowego.

Sprawozdanie C.K. Rady Szkolnej Krajowej o stanie szkół średnich galicyjskich w roku szkolnym 1895/6 (1896). Lwów: Nakładem c.k. Rady szkolnej krajowej, Z drukarni Władysława Łozińskiego.

Sprawozdanie urzędowe Dyrektora Gimnazjum Realnego Karola Frankowskiego z upłynnionych trzech lat szkolnych, czytane na akcie publicznym dnia 26 czerwca 1844 roku. (1844). In *Biblioteka Warszawska, 3*(1844), 371–397.

Świeboda, J. (1984). *Dzieje I Gimnazjum w Rzeszowie 1786–1918*. Rzeszów: Towarzystwo Naukowe w Rzeszowie.

Tync, S. (1959). Wyższa Szkoła Braci Polskich w Rakowie. Zarys jej dziejów (1602–1638). In L. Chmaj (Ed.), *Studia nad arianizmem* (pp. 331–389). Warszawa: Państwowe Wydawnictwo Naukowe.

Ustawy Kommissyi Edukacyi Narodowej dla Stanu Akademickiego i na szkoły w kraiach Rzeczypospolitey przepisane. (1783). Warszawa: Drukarnia Nadworna.

Bericht der Handels- und Gewerbekammer in Lemberg... in den Jahren 1854, 1855 und 1856. (1859). Lemberg: Aus der Buchdruckerei von F. Winiarz.

Wiese, L. (1864). *Das höhere Schulwesen in Preußen*. *Historisch-statistische Darstellung.* Berlin: Verlag von Wiegand und Grieben.

Więsław, W. (2007). *Matematyka polska epoki oświecenia*. Warszawa: Fraszka Edukacyjna.

School reports:

Fünfter Jahresbericht des städtischen Gymnasiums zu Inowraclaw. (1868). Inowraclaw.

Gymnasium mit Realschule I Ordnung zu Thorn. (1874, 1881). Thorn.

Jahres-Bericht über das Königliche Katholische Gymnasium zu Trzemeszno. (1856–1862). Trzemeszno.

Jahresbericht des Königlichen Realgymnasiums zu Bromberg. (1891–1912, 1915). Bromberg.

Jahresbericht des städtischen Realgymnasiums zu Bromberg. (1888–1890). Bromberg.

Jahresbericht für die städtische Realschule zu Posen. (1857). Posen.

Jahresbericht über das Gymnasium zu Braunsberg. (1876–1880, 1882–1912, 1915). Braunsberg.

Jahresbericht über das Königl. Progymnasium in Dt. Crone. (1843–1848). Deutsch-Crone.

Jahresbericht über das Königliche Katholische Gymnasium zu Braunsberg. (1849–1856). Braunsberg.

Jahresbericht über die Realschule zu Krotoschin. (1849-1854). Krotoschin.

Königliches evangelisches Gymnasium und Realschule erster Ordnung zu Thorn. (1861–1873). Thorn.

Königliches evangelisches Gymnasium zu Thorn. (1860). Thorn.

Königliches Gymnasium mit Realgymnasium zu Thorn. (1885, 1889–1892, 1894–1911). Thorn.

Na popis publiczny uczniówWarszawskiego Liceum, maiący się odbywać w Pałacu Królewsko-Saskim... zaprasza Rektor Linde. (1812). Warszawa.

Programm der k.k. Staats-Oberrealschule in Bielitz. (1877, 1881, 1884, 1900). Bielitz.

Programm der Städtischen Realschule I Ordnung zu Posen. (1875). Posen.

Programm der öffentlichen evangelischen Oberrealschule in Bielitz. (1875). Bielitz.

Programm des königlichen Gymnasiums zu Bromberg. (1856–1865, 1869, 1871, 1873–1876). Bromberg.

Programm des Städtischen Real-Gymnasiums zu Posen. (1885, 1890). Posen.

Programm, womit zu der öffentlichen Prüfung der Schüler der Petrischule. (1846–1848, 1850–1854). Danzig.

Sprawozdanie Dyrekcyi C.K. Gimnazyum w Przemyślu za rok szkolny 1889. (1889). Przemyśl.

Sprawozdanie Dyrekcji C.K. Wyższej Szkoły Realnej w Krakowie. (1876, 1884, 1899, 1904). Kraków.

Sprawozdanie Dyrekcyi C.K. Wyższej Szkoły Realnej we Lwowie za rok szkolny 1874. (1874). Lwów.

Zweiter Jahresbericht der Städtischen Ober-Realschule zu Potsdam. (1884). Potsdam.

ON THE MAIN MILESTONES IN DEVELOPING MATHEMATICS IN POLAND PRIOR TO THE XIX CENTURY THROUGH THE LENS OF MATHEMATICS EDUCATION

Ewa LAKOMA
Military University of Technology,
Gen. Witolda Urbanowicza Str. 2, Warsaw, Poland
ewa.lakoma@wat.edu.pl

ABSTRACT

This paper presents a short overview of the history of mathematics education in Poland prior to the nineteenth century, discussing a number of examples of old mathematical works and problems, and pointing out the main milestones in historical development of mathematics and educational transformations accompanying them.

1 Introduction

The main aim of this article is to present a short overview of mathematics education in Poland prior to the nineteenth century, in the context of educational transformations in that times. The knowledge about mathematics in the past, which I will discuss, was taken from the old mathematical works – their original or reconstructed forms available today, by considering them from two main perspectives: as the result of learning and, on the other side, as the subject to learn. These considerations are immersed in the context of the history of the ways of acquiring an education at different levels, which were practiced by people in the past.

2 Elementary Education in Poland in the Middle Ages – From the Perspective of Mathematical Knowledge

It can be assumed that the history of mathematics education in Poland dates back to the tenth century, the origins of the Polish statehood, connected with an acceptance of Christianity. However, we are able to identify even earlier origins of Polish mathematics, the first mathematical issues, on the grounds of prehistoric or early medieval artefacts, mainly paintings, patterns, which present the sense of symmetry and reveal an interest of their authors in astronomy, astrology or geometry. Unfortunately, other testimonies of mathematical knowledge have not survived to the present day. It is certain however that people living in the territory, later belonging to Poland as a state, possessed elementary knowledge of quantitative and geometric relationships, connected with everyday life, such as counting and measuring. Even if today there are no artefacts confirming an awareness of those relationships, it is possible to recognize some traces of this knowledge when exploring etymology of the Polish language – as words, phrases used to describe numbers, measurement, payment etc.

The adoption of Christianity in the tenth century in Poland resulted in the development of feudal culture and facilitated the permeation of cultural heritage from Western Europe (Høyrup, 2014). The first schools in Poland were founded in the eleventh century at the cathedrals. Their aim was to educate young boys to become clergymen. Latin was the language of the church, feudal administration and science, thus it was also a teaching language at school. The range of knowledge that could be obtained in the cathedral school was very limited. Teaching methods were based on memory training. Even the skill of writing was founded on memorizing the shapes of letters. The content of knowledge available for students depended on how well the teacher – a clergyman – who led the school – was educated himself.

In the later period of Middle Ages, when parish administration was developed in the territory of Poland, parish schools were founded, in addition to cathedral schools. In both those types of schools it was possible to obtain education, typical in Europe of that time: at the first degree – usually to parish schools – Trivium of ‘liberal sciences’, that is: grammar, dialectic (in sense of logic) and rhetoric (the art of arranging talks and letters), and at the second degree – usually in the cathedral or collegiate schools – Quadrivium, that is: singing (music), calculus (arithmetic), geometry (including geography) and astronomy. Parish schools were rather popular and also available for young people - without becoming clergymen. Young people attending parish schools did not gain much general knowledge and almost no mathematics; however some teachers taught pupils elements of counting. Cathedral schools were much less popular than parish schools thus the number of young people who learned arithmetic and geometry in the frame of Quadrivium, was small. One of the main reasons for such a limited level of education was that the teaching sources of that time – manuscripts – were almost inaccessible because of a very high price of paper, so it was common use Latin manuscripts belonging to churches as ‘school textbooks’. For example, the Cracow cathedral library at the beginning of the twelfth century did not contain any work with strictly mathematical content. Later, in the cathedral libraries, it was possible to read works – appreciated and known in Western Europe of that time – which contained mathematics and astronomical knowledge, such as the treatise of Isidorus of Sevilla ‘Libri etymologiarum origines’, the work of English monk Bedy, and writings of Boethius on arithmetic and Euclid’s Elements (Dianni & Wachułka, 1957, Gloger, 1900-1903, Wroczyński, 1996).

Although mathematical knowledge available in the frame of the medieval educational system in Poland was rather poor, in the thirteenth century, some young people, usually coming from rich families, after completing elementary schools and getting acquainted with first sources of mathematical knowledge at that level, felt the need to continue learning mathematics either under the care of individual teachers or at universities abroad – especially in France or Italy. That time can be considered as the beginning of a new period in the history of mathematics education of young Polish people: the smartest and wealthy enough students, after finishing elementary schools were pursuing their mathematics education in universities of Western Europe.

3 Mathematics Higher Education in the Middle Ages – Polish Students’ Experience in Studies Abroad

Before the foundation of the Cracow Academy in 1364, there was no other academic centre in Poland. Despite of it mathematical thought developed, as witnessed by the works of Polish scholars who had been educated in the major European academic centres.

In the thirteenth and fourteenth century young men who were interested in furthering their education at a higher level, travelled abroad to study at universities in France (Paris, Montpellier, Avignon), Italy (Padua, Rome, Bologna) or Bohemia (Prague). One of those medieval students interested in developing mathematical knowledge after completing elementary schools was Vitellon (1220?-1280?). After graduating from elementary schools, he studied at the universities in Paris, Padua and Rome.

On the basis of the knowledge about his works as a scholar, it is possible to discern the content of his studies in mathematics and physics (Vitellon, 1535, Birkenmajer, 1936, Trzynadlowski, 1979, Dianni & Wachułka, 1963).

As a student, he explored and mastered his knowledge of geometry, by getting acquainted with the most important works on the subject of that time, especially by exploring the fundamentals of ancient geometry. He presented the results of those studies in his own work titled: "De Elementis conclusionibus" – "On conclusions from Euclid's Elements", written in Latin. That work has not survived to our times. It seems that Vitellon understood geometry as a science that grew out of the needs of everyday life in connection with the phenomena of the surrounding world. Thus, as a scholar, he tried to develop that science, to be used as an aid to solve world problems. His second work, written in Latin, was devoted to optics and titled: "Vitellonis Mathematics Doctissimi Peri Optikis, id est de natura, ratione & proiectione radiourum uisus, luminum, colorum, atq[ue] formarum, quam uulgo, Perspectiuam uocant, Libri X [...]". the as: "On Optics: it is about the essence, the cause of the incidence of rays, sight, lights, colours and shapes that are commonly called the Perspective, ten books" (ca 1270). It seems that the work of the Arabic scholar Ibn al-Haytham (965-1040), titled: "The Book of Optics" (1011-1021) was the 'point of departure' for the Vitellon's work, which was known under its short name: "Perspectiva".

The work consists of ten books, the first of which is devoted to geometry and consists of 137 theorems with the proofs that are used in the next books on optics.

The book contains the author's knowledge about proportions, the theorems about parallel straight lines on the plane or in the three-dimensional space, the knowledge of angles and triangles, properties of a circle, a sphere and conical sections, and finally conclusions on the harmonic division of a segment. In order to show the author's style of work, two examples of propositions taken from this book are quoted below:

Proposition 53:

The arcs contained between the straight parallel lines on the plane of the circle are equal regardless of whether the straight lines are secant or tangent, or whether one is tangent and the other is secant.

Proposition 98:

There is no such a cross-section of a circular cone by a plane, that does not pass through its top, which is a triangle. (Vitellon, 1535, Dianni & Wachułka, 1963).

In order to prove these propositions Vitellon used his knowledge of ancient geometry, known from Arabic works, applying in the proof Arabic term of the notion of hyperbole (Vitellon, 1535, Dianni & Wachułka, 1963).

Among the theorems included in the book it is possible to find propositions which were originally formulated by Euclid and Apollonius and also some theorems which came from works of Arab and Greek scholars living after Apollonius.

It is worth noticing that European scholars began to study the works of Apollonius more thoroughly much later than Vitellon did it – only in the sixteenth century.

The work “On Optics [...]” has had a great impact on the development of mathematics and physics for the next generations of university students and scientists in Europe. The manuscript was repeatedly transcribed and then issued in print many times, especially in the sixteenth century (1535, 1551, 1572), so it must have served as an academic textbook for many generations of university students (Vitellon, 1572). Not only French, German and Italian scholars have referred to the scientific achievements of Vitellon: Nicoalus d’Oresmus (1320-1282), Regiomontanus (1346-1476), Luca Pacioli (1445-1517), Leonardo da Vinci (1452-1519) and Johann Kepler (1571-1630) – in his work titled “Ad Vitelloni paralipomena, quibus astronomiae pars optica traditur” (1604), but also Polish scholars such as Martinus Rex de Peremislia (1422-1460?) or Nicolaus Copernicus (1473-1543).

4 Development of Mathematics Education in Poland in the Fifteenth Century – the Cracow Academy as an Educational and Scientific Centre

In the second half of the fourteenth century, the need to establish an academic centre in Poland seemed to result from significant development in the area of social, political, economic and cultural transformations which took place in Poland of that time. The quality of education in the cathedral schools became higher, mainly because of better teachers’ competencies, and the number of young people, who travelled abroad to study in foreign universities, after graduating from elementary education, was growing.

The first academic centre – the Cracow Academy – was founded in 1364. Early at the beginning of the fifteenth century it became an important European educational and scientific centre. Preliminary level studies at the Cracow Academy covered the programme of Trivium and Quadrivium which was also taught in the best cathedral schools. Students, after graduating from this level of studies, were able to continue their higher education studying such scientific disciplines as law or theology, but also astronomy and astrology, which were connected with mathematics. Although all these disciplines were called ‘mathematical sciences’, mathematical knowledge was considered as the basis necessary for investigations in astronomy (Gloger, 1901-1903, Karbowiak, 1923, Wroczyński, 1996).

‘International exchange’ of students and scholars between the Cracow Academy and many European universities was popular: after graduating from the Cracow Academy scholars often moved to some foreign universities for further studies and then, after getting deeper scientific experience, they returned to their alma mater as professors. On the other hand, foreign students or scholars came to Poland for their studies or to develop scientific research, especially combining mathematical knowledge with the secrets of astronomy. International contacts made mathematics knowledge known to scholars, irrespective of their nationality.

In order to understand what the subject of mathematical studies at the end of the fourteenth century was, one should get acquainted with the content of manuscripts which were used as ‘academic textbooks’ at that time. Some of them were just direct notes written by students or scholars when studying in foreign universities or in the Cracow Academy, some were the scholars’ own treatises. One of those manuscripts was the work titled: ‘Algorismus Anno 1397’, written in Latin by an unknown author. That manuscript contained arithmetic of the whole numbers, in particular: a knowledge about series of numbers, squaring, cubing and taking square roots of numbers. All that knowledge was necessary to consider problems in the area of astronomy. Except for arithmetic, Euclid’s plane geometry was also the basis of mathematics at that time (Dianni & Wachułka, 1957).

Among other manuscripts used for studying mathematics in the fifteenth century there are works of Martinus Rex de Peremislia (1422-1460?), one of the members of the Cracow Academy community. Martinus Rex studied at the Cracow Academy and after graduation he became a scholar at the Academy and began to develop his own scientific research. One of his most important works written in Latin: “Algorismus minutiarum” (ca.1445) was devoted to arithmetic of fractions (Dianni & Wachułka, 1963). It is worthwhile to consider the content of this work more carefully. In that extensive lecture on fractions, the author introduced modern notation of fractions using the fractional bar, the numerator and the denominator. Moreover, he discussed basic arithmetic operations on fractions, and illustrated with drawings his method of reasoning. Martinus Rex considered not only the ordinary fractions – ‘fractiones vulgares’, but also the sexagesimal positional fractions, which were characterized by a different notation system, similar to today’s decimal number system; he called them: ‘fractiones physicae’ – ‘physical fractions’. He investigated the relationship between these two types of fractions, but also explained the principles of taking the square root or the cubic root of a fraction, removing an immeasurability of the denominator of a fraction, multiplying and dividing the fractions. The style of that work was modern, in comparison to the old medieval style of presenting mathematical arguments. It differed in its methodology from the dogmatic lecture of canons supported by the authority of the ancient sages, and took into account the usefulness of mathematics. The content of this treatise became the subject of the main systematic university courses in the Cracow Academy.

After elaborating the course of arithmetic of fractions, Martinus Rex went abroad, maintaining a contact with his alma mater, and studied in Padua, where he became acquainted with the practical geometry of Prosdocimus de Beldomandis (?-1428). The second important treatise of Martinus Rex was devoted to geometry: ‘Geometria Regis’ (ca 1450) and –similarly to the previous one – became a basic ‘academic textbook’ for students and scholars (Birkenmajer, 1895, Dianni & Wachułka, 1957). That work begins with the statement: ‘Geometry has two main parts: the theory and the practice.’ It seems that Martinus Rex understood this statement according to his general philosophical attitude that ‘the theory’ – mathematics – and ‘the practice’ – nature – are in fundamental relation: ‘the theory’ should be convincingly argued by ‘the practice’. The ‘Geometria Regis’ is considered to be the first work on so called practical geometry in Poland. In this work, the author provides solutions to practical issues, practical measurements, which are an introduction to theoretical considerations and research. Issues discussed in the treatise dealt with the following mathematical concepts: proportions, similarity of triangles, measurement on the circle, the sphere, cylinders, cones, barrels – here the author used the approximate method of ‘smoothing the cross-sections’, provided the rules for calculating the area of plane figures and the volume of geometrical solids and illustrated them with examples. The lists of measures used for length measurements as well as the relationship between them are included in the treatise. The work ‘Geometria Regis’ became the basis for further studies and investigations in geometry in Poland (Birkenmajer, 1895, Dianni & Wachułka, 1957).

When we analyse the area of Martinus Rex's mathematical research we can easily deduce that he also dealt with astronomy. He noticed the shortcomings of the Ptolemaic system and tried to make corrections to the calculations of quantities in the Toledan Tables. Martinus Rex, in his astronomical considerations, investigated trigonometric relationships in a modern form –he used in his calculations the half-chords (sinus) instead of the whole chords (chord), which were the basis of Greek calculus. His research on the Toledan Tables, which he initiated, was continued by consecutive scholars of the Cracow Academy, including Wojciech (Adalbert) from Brudzewo (1446-1495). Adalbert from Brudzewo noticed the defects of the Toledan Tables and the need to improve them, but it was Nicolaus Copernicus who posited the hypothesis that the ambiguities in the calculations might have come from the faulty construction of the entire system, not from the incorrect construction of the Toledan Tables.

To summarize, Martinus Rex's works served as the basis of studies and research for subsequent generations of students and scholars of the Cracow Academy in the Renaissance period (Birkenmajer, 1895).

5 Mathematics Education in Renaissance Era in Poland

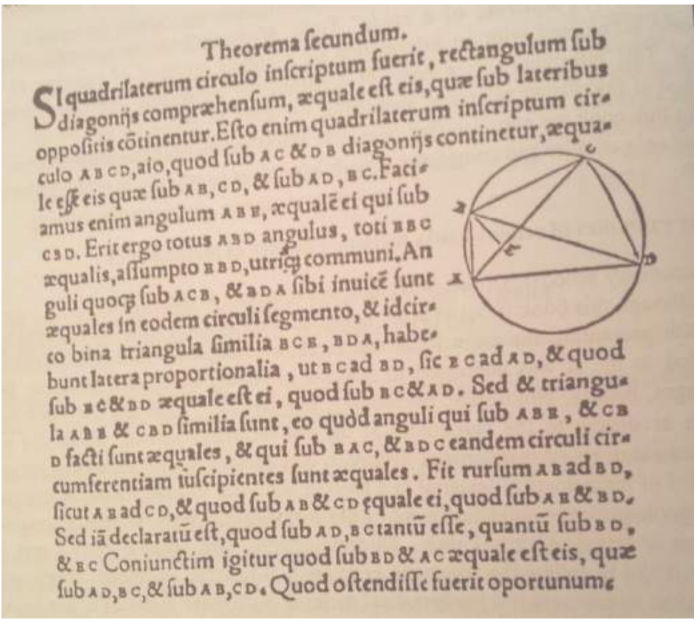
The end of the fifteenth century as well as the sixteenth century is recognized as 'the golden age' of Renaissance era in Poland. In the field of mathematics, the wide spectrum of sophistication levels in mathematical investigations can be observed: on the one side – further development of scientific research, especially connected with astronomy and its necessary mathematical tool, trigonometry, and on the other – new tendency to make elementary mathematics more available for wider groups of people, by writing textbooks in the Polish language, which was connected with creating mathematical terminology in the Polish language.

The most important representative of Renaissance science was Nicolaus Copernicus (1473-1543), who studied at the Cracow Academy from1491-1494, probably as a student of Adalbert from Brudzewo. He was especially interested in astronomy – its phenomena, and their mathematical tables. That knowledge and the strong confidence that the Ptolemaic system could not be correct, led him continuing his research, on the scientific verification of his hypothesis on the Earth motion as profoundly as possible. During his lifelong research, hecreated the scientific justification of his hypothesis on the construction of the world, and based hisarguments on mathematical considerations (Baranowski, 1854).

The most important mathematical basis for his astronomical arguments he gathered into a separate part of his treatise titled 'De Revolutionibus Orbium Coelestium' (1943, printed edition1543), (Birkenmajer (1920, 2004). That is the First Book: 'Trigonometry', devoted to considerations regarding the solution of triangles.

Chapters 12, 13 and 14 of that book were acknowledged as a kind of complementary part to Euclid's Elements. They were focused on spherical triangles and linear triangles and gave rules for calculating chords and halves of chords for given angles. Among theorems presented in those chapters there are two – the Second Theorem and the Third Theorem – which were used by Copernicus to elaborate the table of chords.

The Second Theorem: “In a quadrangle inscribed in a circle, a rectangle consisting of two diagonals [of the quadrangle] is equal to the sum of rectangles consisting of opposite sides [of the quadrangle].” (Fig. 5.1)



Theorema ſecundum.

SI quadrilaterum circulo inſcriptum fuerit, rectangulum ſub diagonijs compræhenſum, æquale eſt eis, quæ ſub lateribus oppoſitis cõtinentur. Eſto enim quadrilaterum inſcriptum circulo ABCD, aio, quod ſub AC & DB diagonijs continetur, æquale eſſe eis quæ ſub AB, CD, & ſub AD, BC. Faciamus enim angulum ABE, æqualē ei qui ſub CBD. Erit ergo totus ABD angulus, toti EBC æqualis, aſſumpto EBD, utriq; communi. Anguli quoq; ſub ACB, & BDA ſibi inuicē ſunt æquales in eodem circuli ſegmento, & idcirco bina triangula ſimilia BCE, BDA, habebunt latera proportionalia, ut BC ad BD, ſic EC ad AD, & quod ſub EC & BD æquale eſt ei, quod ſub BC & AD. Sed & triangula ABE & CBD ſimilia ſunt, eo quòd anguli qui ſub ABE, & CBD facti ſunt æquales, & qui ſub BAC, & BDC eandem circuli circumferentiam ſuſcipientes ſunt æquales. Fit rurſum AB ad BD, ſicut AE ad CD, & quod ſub AB & CD æquale ei, quod ſub AE & BD. Sed iã declaratũ eſt, quod ſub AD, BC tantũ eſſe, quantũ ſub BD, & EC Coniunctim igitur quod ſub BD & AC æquale eſt eis, quæ ſub AD, BC, & ſub AB, CD. Quod oſtendiſſe fuerit oportunum.

Figure 5.1: The Second Theorem

The Third Theorem: “If in a semicircle the chords of two unequal arcs are given, the chord of the difference of these arcs will also be known.”

The last theorem of the book: “The ratio of two arcs: larger and smaller – is greater than the ratio of chords” indicates that Copernicus had the knowledge of the fact that the chords are not directly proportional to the angles.

The First Book includes also theorems concerning the problems of solution of triangles (chapter 13) and spherical trigonometry – spherical triangles and their congruence (chapter 14), (Copernicus, 1543, Birkenmajer, 1920, Dianni & Wachułka, 1963). This book afterwards was a research inspiration for Francois Viete (1540-1603) and John Napier (1550-1617).

The main aim of Copernicus’s mathematical research was to find the appropriate models for astronomical phenomena. In the seventeenth century, even after 1616, the work of Copernicus was the main subject of scientific studies at the Cracow Academy.

The sixteenth century – the period of the Renaissance – is also characterized by the development of humanistic thought; Polish mathematical treatises were written not only in Latin, but also in mother tongues. The group of recipients of scientific knowledge was growing, the methods of teaching mathematics were changed, the scholars more often took up issues arising from everyday life and presented their solutions in a more accessible and clear way. The basis of Polish mathematical terminology was then created.
Among those works, which contain straightforward and visual methods used to solve problems as simply as possible, two textbooks can be distinguished.

The first textbook titled: ‘Algoritmus’ (1538), written in the Polish language by Tomasz Kłos, is addressed to wide group of people who must have used the science of arithmetic, merchant bills, and accounting in their daily work (Baraniecki, 1889, Wydra, 2015, Dianni & Wachułka, 1963). In order to make those calculations easy for readers, the author introduced the so-called ‘Calculus on lines’ – a specific abacus in the form of a board. Parallel lines drawn on the board indicated the places of units, tens, hundreds etc. The fields between the lines indicate the half of the higher order unit. Calculations were based on the appropriate placement and shifting of pebbles on the board. The operations of addition and subtraction, as well as the application of the 'rule of three' – 'regula detri' were presented and explained in the textbook. The same textbook included the presentation of calculations on fractions, and numerous solutions to specific problems, encountered in the reader's everyday life. The main aim of the author, as seen in this textbook, was to give to the reader the simple methods of obtaining a solution to a problem, but without justifying its correctness. The only reader's task was just to gain a mechanical skill in applying the given rule.

The author's intention for writing that book was also to address it to young people. For many years to come, despite the development in arithmetic later on in publications in Latin which used digital notation of numbers, his method of 'calculus on line' was widely known and practiced in people's everyday life.

Apart from the textbook about arithmetic and accounting, which was described above, the second important textbook written in Polish in the sixteenth century was devoted to geometry. The author of that textbook titled 'Geometry' (1566), Stanisław Grzepski (1526-1570), was a professor at the Cracow Academy (Grzepski, 1929, Dianni & Wachułka, 1957, 1963). He was interested in Euclidean geometry and its practical applications, especially in metrology. His textbook was addressed to a wide variety of people, especially to farmers, who needed to have an elementary knowledge of the plane geometry and its applications. The contents of the textbook include problems known from the first four books of the Euclid's Elements, as well as Books 5 and 6. Geometric concepts and their applications, presented in the textbook, were connected only with plane geometry. The solutions were illustrated by drawings. The author introduced not only Polish terminology for geometric concepts, but also as the first scientist in European geometry, gave the name for the concept of parallel lines: 'aequidistantes' (Grzepski, 1929, Dianni &Wachułka, 1957, 1963).

The first part of the textbook presented to a reader the axioms of Euclidean geometry, the classification of polygons, the measure of inner angles of the triangle and the area of a triangle. The author presented also the calculation of the area of a circle by considering its approximation. The second part of the textbook includes defining units of measurement of the length and the area of plane figures and applying this knowledge to solving practical problems connected with agriculture and building construction. Although many examples in the textbook are taken from Euclid's Elements and from Archimedes's works, i.e. the problem of the height of the tower or the problem of the depth of the well, their solutions gave readers a chance to become acquainted with these fundamental geometric works in a simple and mathematically accessible way.

6 Mathematics in the Seventeenth-century Educational Centres in Poland

The seventeenth century brought further development in Polish mathematics and mathematics education – the achievements of Polish scientists were known in the European scientific centres not only because of leading their research at European universities but also through giving academic lectures and writing works in Latin.

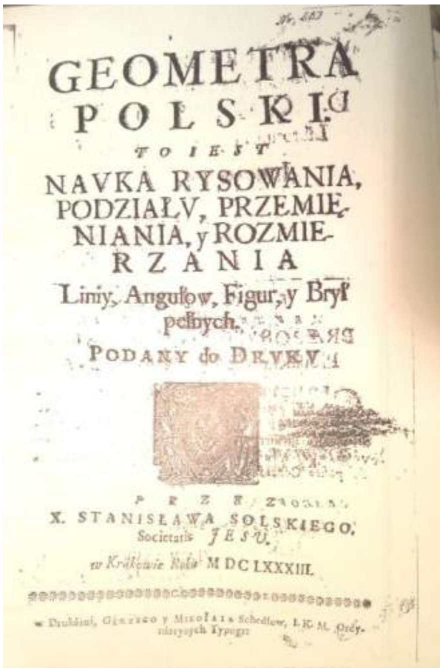
One of the most important representatives of Polish mathematics of that time was Jan Brożek (1585-1652). After completing his studies at the Cracow Academy he was a lecturer of mathematics and astrology, then he studied in Padua and after some years he became a professor in the Cracow Academy. His work titled 'Arithmetica integrorum' (1620), written in Latin, is considered as the first modern – at that time – academic textbook in Poland, addressed directly to the university students (Franke, 1884, Dianni, 1956). The book presented the complete academic knowledge of arithmetic (at that time), that means: arithmetic operations on integers, especially multiplying numbers illustrated by the method called 'the calculus on fingers', the calculus using exponents, roots of numbers, introduction to arithmetic and geometric series. One of the most interesting parts of that textbook was discussing the calculus using the exponents of the power of number 2, in which multiplication and division of powers consisted in adding and subtracting exponents. In this way calculations on numbers came down to calculating on exponents. It seems that this idea was inspired by the concept of logarithm introduced in 1614 by John Napier.

Another important academic textbook, published in the seventeenth century, was the work titled 'Arithmetica vulgaris' (1640), written by Jan Toński (?-1664), who was also a student and later a professor of the Cracow Academy (Toński, 1640, Dianni & Wachułka, 1957). That book included calculations on integers and fractions as well as on decimal fractions. The author introduced the notation of decimal fractions with a decimal colon, used in Poland even today. The second part of the book was devoted to plane and spherical trigonometry. The rules of calculations were described by the author in a general manner, without using algebraic formulas. That part of the book could be compared with Copernicus's treatise 'Trigonometry'. Although the spherical trigonometry was considered in the book as a part of the theory of mathematics rather than merely an aid in astronomy, the last part of the book included examples of applications of trigonometry in metrology and astronomy.

It can be concluded that both academic courses indicated a tendency to develop mathematics as a scientific theory rather than a tool used for solving practical problems. Both books were very popular in the academic community and were included in the supplementary list of academic textbooks for many years, even at the end of the eighteenth century.

The seventeenth century also brought a modern textbook for geometry written in Polish by Stanisław Solski (1622-1701): 'The Polish User of Geometry – learning to draw, divide, measure lines, angles, figures and solids' (1683) (Fig. 6.1 and Fig. 6.2).

This work consists of 14 chapters called 'plays', in which the author presented descriptions of geometrical figures, their properties and applications of geometry for solving practical problems (Solski, 1683-1684). The last chapters included arithmetic and combinatorics. The names of mathematical concepts introduced in that work indicated great progress in the Polish mathematical terminology.


GEOMETRA POLSKI
TO IEST
NAVKA RYSOWANIA, PODZIAŁV, PRZEMIENIANIA, y ROZMIERZANIA
Liniy, Angułow, Figur, y Brył pełnych.
PODANY do DRVKV
PRZEZ X. STANISŁAWA SOLSKIEGO, Societatis IESV.
w Krakowie Roku MDCLXXXIII.

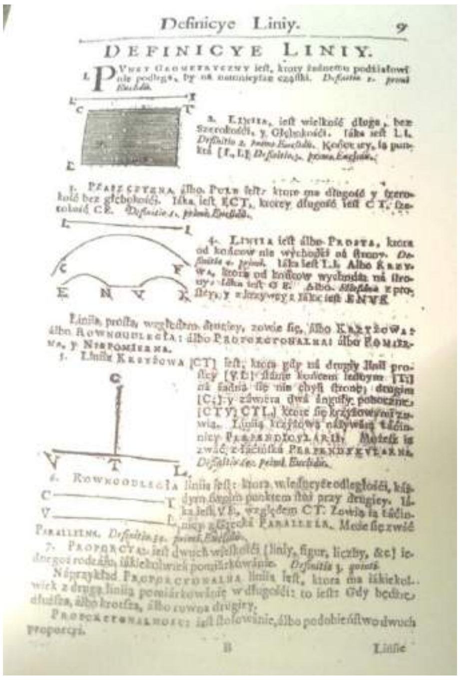

Definicye Liniy. 9
DEFINICYE LINIY.

Figure 6.1: Cover of the book Figure 6.2: Definitions of lines

The development of mathematics can also be inferred by studying certain scientific manuscripts written in the seventeenth century. One of the most interesting examples of mathematical investigations was the scientific achievements of Stanislaw Pudłowski (1597-1645). Many of his notices indicate his profound knowledge of mathematics, including some interesting proposals of solutions to problems of descriptive geometry, formulated later by Desargues and Monge (Dianni & Wachułka, 1963).

Adam Adamandus Kochański (1631-1700) was a student and later a professor of the Vilnius Academy. He was in correspondence with many outstanding European scientists of his time, such as Gottfried Leibniz (1646-1716) and Johannes Hevelius (1611-1687), thus he understood the need of sharing scientific experiences among scientists and its great impact on development of science. His works, mainly on mechanics and statics, as well as a very interesting and 'elegant' method of the approximate construction of rectification of the circle (1685), were published in thescientific journal 'Acta Eruditorium' edited in Leipzig (Barycz, 1935, Dianni & Wachułka,1963).

The Cracow Academy was not the only important scientific centre in Poland in the seventeenth century. There was also the Vilnius University founded in 1578. Apart from those universities there were academies, in which young people were educated: the Academy of Lubrański (1519-1780), the Academy of Zamoyski (1594-1784), and the Academy in Raków (1602-1638), where Jan Amos Komenski (1592-1670) was a professor (Gloger, 1900-1903). The education system included also the centres of – as we call it today –secondary education: there were colleges, especially Piarist colleges and schools (1642-) and Jesuit colleges (the period of the existence of Jesuit colleges in Poland: 1534-1773). Parish schools educated pupils at the lowest level; the teaching methods and educational programmes were rather similar to these applied in previous times. Young people belonging to nobility or rich townsmen families were educated individually by private teachers, often professors of academic centres, in order to be prepared to continue studies at the European universities (Gloger, 1900-1903, Suchodolski, 1972). The content of an education programme depended very much on the school staff. However, at the beginning of the seventeenth century, the parish schools introduced a common educational programme framework, elaborated by scholars of the Cracow Academy after analysis of educational programmes of the Italian, German and French elementary schools. The great impact which was felt with the development of teaching methods and contents was connected with the emergence of schools which were led by teachers belonging to protestant communities. In those schools it was important to educate pupils in developing humanistic thinking and learning based on understanding, gathering one's own experience rather than memorizing knowledge taken from a teacher. That style of teaching also helped students in learning school mathematics. Although Latin was still the teaching language in schools, mathematics textbooks written in the Polish language became popular among wide groups of people, especially those (i.e. merchants, farmers etc.) who had to apply mathematics to their professional activities (Gloger, 1900-1903, Suchodolski, 1972).

7 Mathematics Education in the Eighteenth Century – the Age of Modern Reforms of the Educational System in Poland

The first half of the eighteenth century was characterized by a gradual decrease in the level of Polish education. The need for reform in the whole education system as well in the political and social areas of the state was evident.

The education system of that time was dominated by the structure of parish schools and secondary schools – mainly Jesuit colleges. The old-fashioned school programmes were focused mainly on preparing students to lead philosophical disputes, thus rhetoric was still the subject of great importance and memorizing knowledge was the main teaching method. Latin was the teaching language. That style of teaching was not sufficient in the age of new European tendencies to develop humanistic thinking and gain the knowledge of natural sciences (Gloger, 1900-1903, Suchodolski, 1972, Wroczyński, 1996).

The first step towards reform was made by Piarist Stanislaw Konarski (1700-1773), who created an eight-year modern school – the Collegium Nobilium (1740). The programme of education included mathematics: algebraic expressions, proportions, algebraic fractions, roots, solving equations, geometry – plane figures, solids, and conics. Besides mathematics, economics, natural sciences, practical knowledge, foreign languages: French and German, and the Polish language as separate subjects were the components of the education programme (Gloger, 1900-1903, Suchodolski, 1972). That trend of modern reforms was expanding in a short time; other educational centers modernized their educational programmes, in which mathematics played an important role.

At the same time, in order to modernize the area of higher education, it involved Andrzej Załuski, the professor of the Cracow Academy, introducing on the basis of new foreign academic ideas, the three-year course titled 'Cursus mathematicus dogmativo-experimentalis' (1750). It consisted of three parts: the first part of the course was focused on studying arithmetic, theoretical geometry, plane and spherical trigonometry, and also algebra; the second part of the course was devoted to study mathematics applications, that is: lectures in mechanics, hydrostatics, aerometry, civil and military architectures. In the last part, at the third year of the course, students studied physics (Suchodolski, 1972, Dianni & Wachułka, 1963, Pawlikowska-Bożek, 1982).

In 1773 the Jesuit colleges were closed and that was the appropriate moment in Poland to introduce the institutional reforms concerning the system of education and the content of the teaching programmes. In the same year the Commission of National Education was established and officially took a care of all schools in Poland. The Commision of National Education was the first secular institution in Europe, similar to a modern ministry of public education. The group of modern scientists and educators, known as the "Society of Elementary Books', was responsible for drawing up modern programmes for school education and modern school textbooks. They decided to elaborate Polish translations of the best foreign textbooks of that time. These translations were published and used as the school textbooks in the Knight's School in Warsaw, founded in 1768 by Adam Czartoryski. In that school physics was taught on the basis of George Luis de Sage's and Christoph Pfleiderer's textbooks, and mathematics – on the basis of Simone L'Huillier's (1750-1840) textbook (L'Huillier, 1809). The translations of those textbooks into Polish as well as the Etienne Bezout (1730-1783) textbook's translation: 'Mathematics Education for the use of French Artillery' (1781), gave the basis for modern Polish terminology in mathematics and physics (Bezout, 1808). The textbook for mathematics included: arithmetic, geometry and algebra. It was addressed to the teachers, not to the learners, thus it included many methodical guidelines.

The Commission of National Education, working under the guidance of Hugo Kołłątaj (1750-1812), elaborated the complex and modern system of education in Poland, consisting of a three-level education. New curricula for primary and secondary schools were elaborated. The system assumed equal access to education for girls and boys. Although the idea of modern transformation of the education system was elaborated in detail, it did not gain the formal approval of the parliament. Despite that failure, the reform of higher education was carried out (1781-1788), the supervision of higher education institutions over secondary schools was introduced, and universities gained a modern organization. Also, vocational education was introduced (Gloger, 1900-1903, Suchodolski, 1972).

Although the period 1789-1794 brought a gradual loss of influence of those reforms, modern ideas in universities survived and gave the basis for developing modern branches of science, such as chemistry (i.e. works of Jędrzej Śniadecki, 1768-1838) and mathematics (i.e. works of Jan Śniadecki, 1765-1830) in the field of differential calculus, analytic geometry, and probability calculus) (Więsław, 2012-2013).

The reforms in the Polish education system and in mathematics and natural sciences that took place prior to the nineteenth century, gave the basis for development of modern mathematics and science in the nineteenth and twentieth century, becoming well known as the Polish School of Mathematics developed by the mathematicians atthe Lviv University and at the Warsaw University.

8 Final Remarks

Knowledge on the history of mathematics education provides many possibilities to apply it in today's school practice.

From the point of view of teachers the history of mathematics education is a rich source of information on the epistemology of mathematical concepts and possible epistemological obstacles which can be observed when teaching contemporary students. It is also an interesting area of mathematical problems which can be discussed with students and solved by them.

From the point of view of students some chosen old mathematical problems and their solutions can be interesting and may be helpful in understanding today's school mathematics.

The historical mathematical problems presented in original texts can also be helpful in studies of the language – Polish or Latin – for students who are interested in learning Latin or studying mother tongue, its grammar or etymology. For example, the textbook of Tomasz Kłos, Algorismus (1538), written in the Polish language, can be interesting for students because of the 'Calculus on lines' method of adding and subtracting numbers, presented and explained in the textbook. Another such example is the proof of the Second Theorem (Fig. 5.1) of the First Book of Copernicus treatise, which was included to the mathematics textbook addressed to students studying Latin (Lakoma, 2000).Also the textbook of Stanisław Solski (Fig. 6.1, Fig. 6.2), can serve as a source of old Polish geometrical terms which can be a good point of departure to study the history of the Polish language. Studying grammar and etymology of the Polish language can be also good motivation for students to learn mathematics and to find enjoyment in following historical ways of mathematical reasoning, comparing historical ones with contemporary ones, and searching for one's own solutions.

REFERENCES

Baraniecki, M. (1889). *Algoritmus, to jest nauka liczby, polską rzeczą wydana przez Tomasza Kłosa, 1538*. Kraków: Akademia Umiejętności.

Baranowski, J. (1854). *Nicolai Copernici Torunensis De revolutionibus orbium coelestium libri sex – Mikołaja Toruńczyka O obrotach ciał niebieskich ksiąg sześć.*Warszawa: Stanislai Strąbski.

Barycz, H. (1935). *Historia Uniwersytetu Jagiellońskiego w epoce humanizmu*. Kraków: Polska Akademia Umiejętności.

Bezout, E. (1808). *Cours de mathematiques a l'usage de la marine et de l'artillerie*. Paris: F. Louis Librairie.

Birkenmajer, A. (1936). *Witelo najdawniejszy polski uczony*. Katowice: Instytut Popierania Nauki.

Birkenmajer, L. (1895). *Mistrza Marcina Króla inaczej Marcinem Królem z Przemyśla zwanego z Żórawicy Geometrya Praktyczna czyli Traktat Sztuki Mierniczej*.Warszawa: Wyd. Redakcyi Prac Matematyczno-Fizycznych.

Birkenmajer, L. (1920). *Mikołaj Kopernik – Wybór pism w przekładzie polskim*. Kraków: Krakowska Spółka Wydawnicza.

Birkenmajer, L. (2004). *Mikołaj Kopernik – O obrotach ciał niebieskich i inne pisma*. Warszawa: Zakład Narodowy im. Ossolińskich, De Agostini Polska, cop. 2004.

Copernicus, N. (1493). *De Revolutionibus Orbium Coelestium Libri VI*. Torino:Chiantore.

Copernicus, N. (1543). *Nicolai Copernici De Revolutionibus Orbium Coelestium Libri VI*. Norimberga.

Dianni, J. (1956). *Ja Brożek – wybór pism*. Warszawa:PZWS.

Dianni, J., & Wachułka A. (1957). *Z dziejów polskiej myśli matematycznej*. Warszawa: PZWS.

Dianni, J., & Wachułka A. (1963). *Tysiąc lat polskiej myśli matematycznej*. Warszawa: PZWS.

Feingold, M., & Navarro-Brotons, V. (Eds.). (2006). *Universities and Science in the Early Modern Period.* London: Springer.

Franke, J. (1884). *Jan Brożek – akademik krakowski*. Kraków: Akademia Umiejętności.

Gloger, Z. (1900-1903). *Encyklopedja staropolska ilustrowana*. Warszawa: Druk. P. Laskauera i S-ki.

Grzepski, S. (1929). Geometria. Warszawa: Przegląd Mierniczy.

Høyrup, J. (2014). Mathematics Education in the European Middle Ages. In A. Karp, & G. Schubring (Eds.), *Handbook on the History of Mathematics Education*. New York: Springer.

Iłowiecki, M. (1981). *Dzieje nauki polskiej.* Warszawa: Interpress.

Karbowiak, A. (1923). *Dzieje wychowania i szkół w Polsce*. Lwów: Zakład Narodowy im. Ossolińskich.

Lakoma, E. (2000). History of mathematics in curricula and schoolbooks: a case study of Poland In J. Fauvel, & J. van Maanen (Eds.). (2000). *History in mathematics education: The ICMI Study*, New ICMI Study Series, vol. 6 (pp. 19-29) Dordrecht: Kluwer.

L'Huillier, S. (1809). *Arytmetyka dla szkół narodowych.*Warszawa: Druk. XX Piiarow.

Pawlikowska-Bożek, Z. (1982). Matematyka w okresie reform szkolnych sprzed lat dwustu, *Wiadomości Matematyczne* 24(1), 95-105. Warszawa.

Solski S.(1683-1684). *Geometra Polski*. Kraków: Druk. Jerzego i Mikołaja Schedlow.

Suchodolski, B. (1972). *Komisja Edukacji Narodowej na tle roli oświaty w dziejowym rozwoju Polski*. Warszawa: Wiedza Powszechna.

Toński, J. (1640). *Arithmetica vulgaris et trygonometria rectilineorum: prout universiae geometriae practicae*. Ingolstadii: Gregorii Haenlin.

Trzynadlowski, J. (Ed.) (1979). *Witelo – matematyk, fizyk, filozof*. Wrocław: Zakład Narodowy im. Ossolińskich.

Vitellon. (1535). *Vitellonis Mathematici Docctissimi Peri Optikis, id est de natura, ranione, et proiectione radiorum uisus, luminum, colo rum, wt[ue] forma rum, quam uulgo Perspectiuam uocant, Libri X: Habes in hoc opere, Candide Lector, quum magnum numerum Geometricorum elementorum […]*. Norimberga: Georg Collomitius Tannstetter, apud Io. Petreium.

Vitellon. (1572). Vitellonis Thuringopoloni Optical Libri Decem, *Opticae Thesaurus. Alhaezni Arabis libri septem, nunc primum editi. Item Vitellonis Thuringopoloni Optical Libri Decem. Omni instaurati […] adiectis etiam in Alhazenum commentariis, a Federico Risnero Manuscript.* Basylea: Friederich Risner.

Więsław, W. (Ed.). (2012-2013). *Dzieje matematyki polskiej*. Wrocław: Instytut Matematyczny Uniwersytetu Wrocławskiego.

Wroczyński, R. (1996). *Dzieje oświaty polskiej, t.I do roku 1795*. Warszawa: Żak.

Wydra, W. (2015). *Algoritmus to jest nauka liczby polską Reczą wydana: Tomasz Kłos*. Poznań: Wyd. Poznańskich Studiów Polonistycznych.

THE DIDACTICAL CONTRACT, ITS EFFECTS AND CLAUSES

A historical Study

Deissy NARVÁEZ ORTIZ
Universidad Distrital Francisco José de Caldas, Bogotá, Colombia
dmnarvaez@gmail.com

ABSTRACT

The notion of didactical contract was introduced by Guy Brousseau in the field of research in didactics of mathematics since 1978, as a resource to study the selective failure in mathematics. This idea was not only received by the research community with great acceptance, but also was quickly validated. The didactical contract, once theorized, allowed understanding many phenomena in the field of didactics of mathematics and in other disciplines, where it has been introduced successfully. The celebrity of the concept and the apparent simplicity of its definition, have generated the emergence of multiple and diverse interpretations during these last 40 years, which are now part of the literature. The multiplicity of uses and interpretations has led to a broad semantic slippage of the term. As a result, the concept of the didactical contract has weakened and is at risk of losing its usefulness and concrete meaning, which is problematic given its character of fundamental concept for the discipline. On the other hand, the low level of production of empirical data to validate hypotheses about the explanatory power that has the construct in real situations of the teaching-learning process, and the limited exploration on the heuristic value that the concept may have to improve the practices of the teachers generate a problematic situation whose approach is considered necessary.

To address this problem, we propose a qualitative study, which will be addressed through three questions: 1. What aspects characterize the original idea of the didactical contract, its conceptualization process and its subsequent interpretations?, 2. Which are the tools required to observe the didactical contract in the classroom and what kind of information do they contribute to the study of the didactic situations?, and 3. What new examples of the effects and the clauses of the didactical contract can be reported through observation in different mathematics classrooms?

Methodologically, we propose to develop this research in two phases: 1) A study historical-critical-analytical about the original idea of the didactical contract and of the reported studies around this concept. The findings in this phase will lead to the characterization of two tools: the effects and the clauses of the didactical contract. Such characterizations will contribute to the study of the manifestations of the didactical contract in authentic situations of classroom and will allow understanding the heuristic and explanatory power of this key concept of the fundamental didactic. 2) A qualitative study with an ethnographic approach based on the search of empirical data on Colombian classrooms of different educational levels. In these classrooms, we will examine examples of the occurrence of the effects and the manifestation of behaviors associated to clauses of the didactical contract by a non-participant observation. The findings will demonstrate the

explanatory power of the notion of the contract on the sociology of mathematics classroom.

Here we will show only results of the first phase, it's historical phase, which are: the genesis of the didactical contract concept, its predecessor ideas, its conceptualization process, the connections with other concepts, as well as its subsequent interpretations. Also, we will present a bibliographic history of the concept, in order to document several interpretations that have emerged in the literature and we present the great differences between the approaches associated with the same name. Mainly we will demonstrate that the didactical contract not only is it a founding concept of the discipline but it is a source of current research problems and that the history of the research groups practices allows us to study the original concepts of the didactic of mathematics and preserve its original and concrete meaning.

JOSEPH ZARAGOZA'S *ARITHMETICA UNIVERSALIS* AND THE TEACHING OF ALGEBRA IN SPAIN IN THE SECOND HALF OF THE 17TH CENTURY

Luis PUIG
Universitat de València Estudi Genera, Spain
luis.puig@uv.es

ABSTRACT

Born in 1627, Joseph Zaragoza joined the Society of Jesus in 1651, and the same year he began to teach arts and theology at the Society's schools in several Spanish cities, including in Valencia where he lived from 1660 to 1670. There he also taught mathematics and astronomy privately, and published in 1669 his book *Arithmetica Universalis*, just a year before he moved to Madrid to become a full professor of mathematics at the Colegio Imperial (Imperial College), an institution of the Society of Jesus. Starting in 1625, the Imperial College was funded by Felipe IV, King of Spain, Naples, Sicily and Sardinia and Duke of Milan, with the main purpose of becoming the place to educate the nobility's children. Felipe IV established and funded two Chairs of mathematics, thus turning the Imperial College into a very important centre of study and development of mathematics throughout the 17th century in Spain. Joseph Zaragoza taught there until the end of his life in 1679, publishing a number of other books.

The full title of Zaragoza's *Arithmetica Universalis* is *Arithmetica Universalis, que comprehende el arte menor, y maior, algebra vulgar, y especiosa* [*which includes the minor and the great art, the common and the specious algebra*], showing, by the inclusion of the "specious algebra" or "algebra of species", Zaragoza's knowledge of Viète's new algebra. Furthermore, Zaragoza's *Arithmetica Universalis* is the first book published in Spain in which one can find Viète's influence.

In this talk, we will present a study of the way in which Zaragoza, having adopted Viète's ideas, reorganised the teaching of algebra in the *Arithmetica Universalis*. We will focus on the characteristics of the symbolic system of signs that he uses (closer to Descartes' than to Viète's), the way he introduces the concept of exponent, the canonical forms of equations that he establishes and studies, the algebraic and numerical procedures to solve equations that he explains and proves, the type of problems he deals with, and the way he presents the Cartesian method to solve problems.

In addition, we will compare Zaragoza's book with other textbooks that are contemporary or follow it in the immediate decades, including Andrés Puig's *Arithmetica especulativa, y practica y arte del algebra*, published in 1672 in Barcelona, Spain, the part on "specious algebra" of *Escuela de Palas, o sea Curso Mathematico*, attributed to Joseph Chafrion, published in 1693 in Milan, Italy, and the part on algebra of Pedro de Ulloa's *Elementos Mathematicos*, published in 1706 in Madrid, Spain.

REFERENCES

Chafrion, J. [attributed] (1693). *Escuela de Palas, o sea Curso Mathematico. Tomo I*. Milan: En la Emprenta Real, por Marcos Antonio Pandulpho Malatesta.

de Ulloa, P. (1706). *Elementos Mathematicos*. *Tomo I*. Madrid: Antonio Gonçalez de Reyes.
Puig, A. (1672) *Arithmetica especulativa, y practica y arte del algebra*. Barcelona: Antonio Lacavalleria.
Zaragoza, J. (1669). *Arithmetica Universalis, que comprehende el arte menor, y maior, algebra vulgar, y especiosa.* Valencia: Geronimo Vilagrasa.

EQUATIONS IN CHINA: TWO MILLENNIA OF INNOVATION, TRANSMISSION AND RE-TRANSMISSION

Man-Keung SIU
The University of Hong Kong, Department of Mathematics, Pokfulam, Hong Kong
mathsiu@hku.hk

ABSTRACT

In a rather broad sense, mathematics deals with the solving of equations, different sorts of equations which help us to handle problems and to understand the world around us. It is therefore natural that the art of solving equations was developed in very early time, from the days of ancient Egypt, Babylonia, India and China. This paper tells (part of) the story of the art of solving equations in China as recorded in mathematical texts from ancient to medieval times, reaching a "golden age" in the thirteenth century, followed by a period of stagnation or even becoming a lost art until the study was revived during the eighteenth/nineteenth centuries when European mathematics was transmitted into China, first through the Jesuits since the beginning of the seventeenth century and later through the Protestant missionaries in the nineteenth century. It is hoped that this story may enrich the pedagogical aspect of learning the topic of equations in a modern day classroom.

1 Introduction

In a video made on the occasion of the award of the Abel Prize in 2018 the recipient, Robert Langlands, said, "What is called, not by me, "Langlands Program", is about various things. Among other things it is about the solution of equations."

In a rather broad sense, mathematics deals with the solving of equations, different sorts of equations which help us to handle problems and to understand the world around us. It is therefore natural that the art of solving equations was developed in very early time, from the days of ancient Egypt, Babylonia, India and China. In those early days such problems invariably appeared as word problems for a natural reason (Swetz, 2012). Symbolic language and manipulation in dealing with equations took a few thousand years more to materialize.

In China, the art of solving different types of equations appeared in various mathematical texts from ancient to medieval times, reaching a "golden age" in the thirteenth century, followed by a period of stagnation or even becoming a lost art until the study was revived during the eighteenth/nineteenth centuries when European mathematics was transmitted into China, first through the Jesuits since the beginning of the seventeenth century and later through the Protestant missionaries in the nineteenth century.

Before the era of transmission of European mathematics into China, another kind of transmission of learning took place through the famous Silk Road. As the main trade route in Central Asia that established links between a cross-cultural mix of religions, civilizations and people of many different regions, it also naturally enabled exchanges of learning and cultures of people of different races. Transmission of mathematical knowledge, either directly or indirectly, between China and regions in Central Asia and the Middle East, India, the Islamic Empire and even Europe further to the West went on for many centuries from the Han Dynasty to the Yuan Dynasty throughout fifteen centuries. A well-known example often referred to is the Method of Double False Position, originated from the method of *ying buzu* [盈不足 excess and deficit] explained in Chapter 7 of *Jiuzhang Suanshu* [九章算術 The Nine Chapters on the Mathematical Art], the prototype Chinese mathematical classics that is believed to have been compiled between the second century BCE and the first century CE (Siu, 2016).

2 A general account on solving equations in ancient and medieval China

The art of solving equations has a long history in China. Despite what the title may have suggested this paper will not give a detailed account of this history, which has been so far covered in a large collection of books and papers, some in the nature of scholarly research in history of mathematics and some in the nature of general exposition for a wider readership. Thus, there is no dearth of reading material on this topic. Primary sources are not hard to locate either, for example, see (Guo, 1993). Instead of offering a long list of references, which definitely cannot cover even a fraction of the relevant literature, I will just give some references which I frequently consult, and among that only those written in a Western language, leaving out a rich repertoire of the literature written in the Chinese language (Chemla & Guo, 2004; Hoe, 1977; Lam, 1977, Lam & Ang, 1992/2004; Libbrecht, 1973; Shen, Crossley & Lun, 1999). Readers who can read Chinese can begin with the encyclopaedic series on the historical development of mathematics in China in a span of four millennia from the very early time to the Qing Dynasty (Wu, 1982-2004).

If this topic has been so extensively studied, what is it that this paper wishes to contribute? As will become apparent in the next three sections we have the pedagogical aspect in mind. However, it is helpful to still give a brief account of the general background.

The art of solving equations was recorded and explained (in later commentaries) throughout the different chapters of *Jiuzhang Suanshu*. This book was woven around the main notion of ratio and proportion which already appeared in Chapter 1 and was much expanded and elaborated in the next five chapters, evolving into the method of excess and deficit of Chapter 7 and systems of linear equations of Chapter 8, then on to geometric interpretation and applications in the method of *gou-gu* [勾股 right triangles] of the final Chapter 9. Problems in ratio and proportion involve linear equations, but problems in area and volume of geometric figures and problems in right triangles invariably involve simple quadratic and cubic equations of the type $x^2 = A$ or $x^3 = V$, for which the methods of extracting square roots and cube roots were developed. In Chinese mathematics, the method of extracting square roots, called *kaifangshu* [開方術] and explained in Chapter 4 of *Jiuzhang Suanshu*, is a crucial discovery that led to highly elaborated methods in stages in subsequent dynasties, reaching a high point in the thirteenth century when it developed into an algorithmic procedure for solving polynomial equations of any degree. By that time the art of solving equations of any degree was known as the *tianyuanshu* [天元術 celestial source procedure] and was extended to solving a system of polynomial equations in four unknowns known as the *siyuanshu* [四元術 four sources procedure] propounded by the mathematician ZHU Shi-jie [朱世傑 1249-1314] in his treatise *Siyuan Yujian* [四元玉鑑 Precious Mirror of the Four Sources] of 1303.

In another direction the art of solving a system of linear congruence equations was developed originating from the famous Problem 26 of Book 3 of *Sunzi Suanjing* [孫子算經 Master Sun's Mathematical Manual] of the third century: "There are an unknown number of things. Counting by threes we leave two; counting by fives we leave three; counting by sevens we leave two. Find the number of things." This was further elaborated into a general method known as the *dayan qiuyi shu* [大衍求一術 great extension art of searching for unity] propounded by the mathematician Qin Jiu-shao [秦九韶 *ca*. 1202-1261] in his treatise *Shushu Jiuzhang* [數書九章 Mathematical Treatise in Nine Sections] of 1247. This significant achievement gains recognition in the naming of an important result in abstract algebra of the modern era that is called the "Chinese Remainder Theorem".

In yet another direction the art of solving indeterminate equations was developed in the text *Zhangqiujian Suanjing* [張丘建算經 Mathematical Manual of Zhang Qiu-jian] of the fifth century with the famous "problem of the hundred coins and hundred fowls" which seems to have transmitted through the Silk Road to become part of the folklore of different mathematical cultures in the Islamic world and later in the Western world. This problem was re-transmitted into China during the seventeenth to eighteenth centuries, a period of transition between traditional Chinese mathematics and modern mathematics learnt from the Westerners. This particular type of problem aroused the interest of a group of Qing mathematicians who thereby contributed to the investigation of solving indeterminate equations.

In 1613 LI Zhi-zao [李之藻 1565-1630], a scholar-official in the Ming Court, collaborated with the Italian Jesuit Matteo Ricci [利瑪竇 1552-1610] to compile the treatise *Tongwen Suanzhi* [同文算指, literally meaning "rules of arithmetic common to cultures"] based on the 1583 European text *Epitome Arithmeticae Practicae* (literally meaning "abridgement of arithmetic in practice") of Christopher Clavius (1538-1612) and the 1592 Chinese mathematical classic *Suanfa Tongzong* [算法統宗, literally meaning "unified source of computational methods"] of CHENG Da-wei [程大位 1533-1606]. Besides being an attempt of LI Zhi-zao to synthesize European mathematics with traditional Chinese mathematics this is also the first book which transmitted into China in a systematic and comprehensive way the art of written calculation that had been in common practice in Europe since the sixteenth century (Siu, 2015). With this the road was well paved for the ensuing transmission of the art of solving equations from the Western world.

3 Solving problems not by solving equations

Many problems in ancient Chinese mathematical texts involve solving equations, such as "Chasing after the visitor to return his coat" (Problem 16 in Chapter 6 of *Jiuzhang Suanshu*), "Broken bamboo"(Problem 13 in Chapter 9 of *Jiuzhang Suanshu*), "Two mice drilling a wall" (Problem 12 of Chapter 7 of *Jiuzhang Suanshu*), "Inscribed circle in a right triangle" (Problem 16 in Chapter 9 of *Jiuzhang Suanshu*), "Pheasants and rabbits in a cage" (Problem 31 in Book 3 of *Sunzi Suanjing*). For each such problem a school pupil of today will no doubt set up an equation and solve it. But in those days methods for solving equations (other than that of solving a system of linear equations using a systematic algorithm employing counting rods on the board that is equivalent to the Gaussian Elimination Method, which is explained in detail in Chapter 8 of *Jiuzhang Suanshu*) were not yet developed either in the East or in the West, so these problems were solved by other clever means, sometimes even relying on geometric explanation.

In the same manner, similar problems come up in primary school classrooms that are amenable to mathematical reasoning without relying on the knowledge of solving equations, and they serve a pedagogical purpose. This leads one to ask, “What advantage do we gain by learning to solve equations, if we can do without it but by some other means? Why do we need to learn the topic of solving equations?” I shall get back to this point at the end of this section. To keep this paper within the preferred length, among the batch of five problems mentioned in the beginning of this section let us only look at two examples, taken from ancient Chinese mathematical texts, in more detail.

(1) In Book 3 of *Sunzi Suanjing* we find the famous problem about pheasants and rabbits: “Now there are pheasants and rabbits in the same cage. The top [of the cage] has 35 heads and the bottom has 94 legs. Find the number of pheasants and rabbits.” (Lam & Ang, 1992/2004, p. 180).

A school pupil of today would probably see it as a problem in solving a pair of simultaneous linear equations by putting r as the number of rabbits and p as the number of pheasants. If H and L are respectively the number of heads and legs, then
$r + p = H$, $4r + 2p = L$. Hence, we eliminate p to obtain

$$r = (L - 2H) / 2 = L/2 - H,$$

that is, the number of rabbits r is equal to half the number of legs minus that of heads. In the text of *Sunzi Suanjing* the explanation is given exactly as what the last sentence says. Even though ancient Chinese mathematicians several centuries before were already well versed in solving a system of linear equations as explained in Chapter 8 of *Jiuzhang Suanshu*, apparently that is not how the problem was solved in *Sunzi Suanjing*.

From the text of *Sunzi Suanjing* it appears that the author was thinking along the line of tying up the legs two by two so that each pheasant has one head and one pair of legs, while each rabbit has one head and two pairs of legs. Among these $L/2$ pairs of legs we can take off H pairs, one from a pheasant and one from a rabbit, leaving behind $L/2 - H$ pairs, one for each rabbit. Thus, $r = L/2 - H$.

As a pedagogical means one can let students make up more funny and imaginative stories to explain the same formula. For instance, suppose we ask the pheasants and rabbits to each raise two legs. After doing that all the pheasants fall flat on the ground, while all the rabbits can each still stand on both legs. Count the heads (only of the rabbits) still held up, which is half of the $L - 2H$ legs standing on the ground, so $r = L/2 - H$. Or, for a more gruesome story, first cut off all legs, then distribute back to each pheasant and each rabbit a pair of legs. A pheasant can go away with a pair of legs, while a poor rabbit cannot move with only two legs. So, $L - 2H$ legs stay put, two for each rabbit, hence $r = (L - 2H) / 2 = L/2 - H$. Instead of eliminating p we can eliminate r to arrive at

$$p = (4H - L) / 2 = 2H - L/2.$$

Again, as a pedagogical means one can let students make up a story to explain this formula.

(2) Problem 13 in Chapter 9 of *Jiuzhang Suanshu* has a rich heritage, for it also appeared (with different data) in *Lilavati* written by the Indian mathematician Bhāskara II (1114-1185) in the twelfth century as well as in a European text written by the Italian mathematician Filippo Calandri in the fifteenth century. The problem asks: “Now given a bamboo 1 *zhang* high, which is broken so that its tip touches the ground 3 *chi* away from the base. Tell: What is the height of the break?” (Shen, Crossley & Lun, 1999, p.485). In

modern day mathematical language it means that we want to calculate the side b of a right triangle given a and $c + b$, where c is the hypotenuse. The answer given in *Jiuzhang Suanshu*, expressed in modern day mathematical language, is given by the formula

$$b = \frac{1}{2}\left[(c+b) - \frac{a^2}{(c+b)}\right]$$

A school pupil of today would probably arrive at this answer by invoking Pythagoras' Theorem and solving a certain equation. However, how was it done more than two thousand years ago, when the Chinese did not yet have the facility afforded by symbolic manipulation at their disposal?

Both LIU Hui [劉徽 *ca*. 225-295] and YANG Hui [楊輝 1238-1298], who were noted Chinese mathematicians more than a thousand years apart, explained in their commentaries how the answer is obtained. Their ingenious method is an excellent example to illustrate how shapes and numbers, or geometry and arithmetic/algebra, go hand in hand in traditional Chinese mathematics. By using Pythagoras' Theorem (known as *gou-gu* [勾股] in the Chinese context) we see that the gnomon formed by the larger square of side c minus the smaller square of side b has area a^2. Fold down the gnomon to form a rectangle of side $c + b = L$ and $c - b$, and complete this rectangle to a square of side L (See Figure 3.1a, 3.1b, 3.1c). Subtracting this rectangle from the square of side L we obtain a rectangle of area $L^2 - a^2$, which is also equal to the product of the two sides L and $L - (c - b) = 2b$, that is, $2bL$. Therefore, $2bL = L^2 - a^2$, thence the answer for b (See Figure 3.1d). For an animated demonstration of this process (credited to assistance from Anthony C. M. OR) readers are invited to go to a *Geogebra* applet at the link https://ggbm.at/2772025.

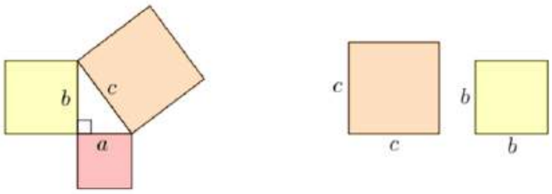


Figure 3.1a: Broken Bamboo Problem

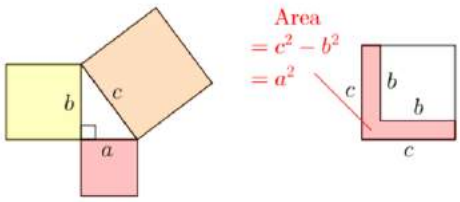


Figure 3.1b: Broken Bamboo Problem

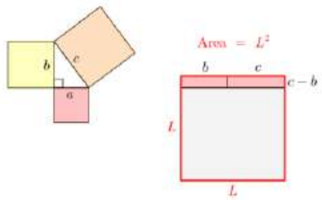


Figure 3.1c: Broken Bamboo Problem

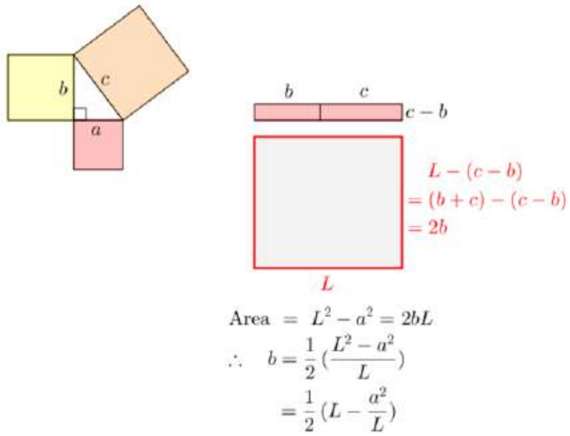


Figure 3.1d: Broken Bamboo Problem

Both examples convey the point I wish to make, namely, it may be possible to solve a problem without setting up an equation then solve it, but it may require to some extent a clever idea. Not everybody can be a clever craftsman full of bright ideas as such, but most people can become skilful workers once they have learnt the methods of setting up and solving equations reasonably well enough. In history these methods were developed since the sixteenth century and matured during the next three centuries. Today a school pupil learns it in class and can thereby solve various problems by setting up and solving equations, be they linear or quadratic or of higher degree, while in the old days only masters could accomplish the same.

4 The *Tianyuan* procedure

In the book *Yigu Yanduan*[益古演段 Development of Pieces of Area Augmenting the Ancient Knowledge] of 1259 written by LI Ye [李冶 1192-1279] Problem 8 asks: "A square field has a circular pond in the middle. The area of land off the pond is 13 *mu* and 7 plus a half *fen*. We only know that the sum of the perimeters of the square and the circle is 300 *bu*. What are the perimeter of the square and that of the circle?" The method says: "Set *tianyuan yi* as the diameter [立天元一為圓徑]**.**" In a language familiar to a school pupil of today, this means "let x be the diameter". The solution follows (if expressed in today's language): The circumference is $3x$ (the value of π is taken to be 3), so the perimeter of the square is $300 - 3x$. Sixteen times the area of the square is equal to $9x^2 - 1800x + 90000$, while sixteen times the area of the circle is equal to $12x^2$. Hence, sixteen times the area of the square minus sixteen times the area of the circle is equal to $-3x^2 - 1800x + 90000$, which, according to the given data (knowing the conversion of units in measurement), comes out to be 52800. Thus, we have set up the equation $-3x^2 - 1800x + 37200 = 0$. Solving it, we obtain $x = 20$, and so the circumference of the circle is 60 *bu* and the perimeter of the square is 240 *bu*. To solve such an equation the Chinese mathematicians developed a method called *zengcheng kaifangfa* [增乘開方法 method of extraction of roots by successive addition and multiplication], which was the same as what is known in the Western world as Horner's method, devised by William George Horner (1788-1837) in 1819.

This method of setting up and solving a polynomial equation was a significant achievement in Chinese mathematics of the thirteenth century. As early as in the Qin-Han period (third century BCE to second century CE) the famous mathematical classics *Jiuzhang Suanshu* explained the method called *daicong kaifangshu* [帶從開方術 procedure of extraction of square root with accompanying number], which, as an extension of *kaifangshu*, calculated a root of a quadratic equation to any degree of accuracy. This method was extended by later mathematicians of the twelfth/thirteenth centuries during the Song-Yuan period to solve a polynomial equation of higher degree, known by the name of *tianyuanshu* [天元術 celestial source procedure]. For instance, in the mathematical classics *Shushu Jiuzhang of* 1247 QIN Jiu-shao treated an equation of degree ten in Problem 2 (measuring a circular castle from the distance) of Chapter 8 as an illustration on how one could solve a polynomial equation of any degree by finding a root to any degree of accuracy (Siu,1995; Siu, 2009).

Ironically, this high-point of Chinese mathematics was also the beginning of its stagnation! One reason is that there was no need at the time to solve an equation of such high degree in practice. When the technical capability far exceeded the demand imposed by practical matters, motivation otherwise arising from an inner curiosity would not attract those with mainly a pragmatic attitude. For instance, the pragmatic attitude would not induce Chinese mathematicians in those days to think about existence of a root of an equation, not to mention solvability by radical, because they were sufficiently satisfied with an algorithm that could obtain a root to any degree of accuracy in terms of decimal places (Siu, 1995; Siu, 2009). One mathematician in the early Qing period, namely, WANG Lai [汪萊 1768-1813], was a rare exception. He was not satisfied with the work of mathematicians in the Song-Yuan period and wished to surpass it by posing questions such as deciding whether a quadratic equation has a real root or an imaginary root, thus touching on the theory of equations. However, he was so much ahead of his time that he was regarded at the time as a "maverick"!

Therefore, this *tianyuan* procedure was unfortunately almost lost, at best vaguely known but not well understood, after the thirteenth century owing to the turbulent era which lasted until the Ming Dynasty was established by the mid-fourteenth century. In the next section we will see the *tianyuan* procedure mentioned by Chinese mathematicians of the eighteenth century but in a different light.

5 Emperor Kangxi's study on solving equations

The story starts with the first period of transmission of European learning into China during the reign of Emperor Kangxi [康熙帝 1654-1722], which spanned the 1660s and 1670s. For a much more detailed account and discussion, interested readers are invited to read (Jami, 2011).

Following the practice of the Ming Court the Qing Court employed the service of foreign missionaries as official astronomers to make an accurate calendar, which was held by many as a major tool of legitimization of the Imperial rule in traditional Chinese thinking. The German Jesuit Johann Adam Schall von Bell [湯若望 1591-1666] convinced Emperor Shunzhi [順治帝 1638-1661] to adopt a new calendar, which was actually the fruition of a huge programme accomplished by XU Guang-qi [徐光啟 1562-1633] and his team in the Ming Dynasty but could not be implemented because of the collapse of the Ming Dynasty in 1644. When Emperor Kangxi succeeded to the throne in 1661 as a boy not yet seven-year-old, Schall lost the support of the late emperor and was accused of treason by the conservative ministers backed up by the powerful regents of the boy-emperor. Along with some other Chinese Catholic converts he was sentenced to death and would have met his tragic ending were it not for a strong earthquake that shook the capital for several days in 1665. People took this to be a warning from heaven that told them it was wrong to accuse Schall of treason. Furthermore, the Grand Empress Dowager (grandmother of Emperor Kangxi) intervened so that Schall was spared the death sentence but was expelled to Macao instead, where he died there soon afterwards in 1666. This event, known as the "calendar case", was rehabilitated in 1669 at the initiation of another foreign missionary, the Belgian Jesuit Ferdinand Verbiest [南懷仁 1623-1688].

Verbiest, who spent two years at the famous University of Coimbra before he was sent to China for missionary work, succeeded Adam Schall von Bell as the Head of the Imperial Astronomical Bureau, serving from 1669 to 1688. The way Verbiest managed to turn the tide was to challenge his opponent, the conservative minister YANG Guang-xian [楊光先 1597-1669], to compete in measuring the shadow of the sun on one December day of 1668. This event left a deep impression on the young Kangxi, as he later recounted the story:

> You all know that I am good at mathematics but do not know the reason why I study it. When I was very young there were frequent disputes between Han officials and Westerners in the Imperial Astronomical Bureau. They accused each other so badly that somebody might get beheaded! Yang Guangxian and Tang Ruowang [Adam Schall von Bell — apparently Kangxi's memory did not serve him well here!] competed in predicting the sun shadow at the Wu Gate in the presence of the nine chief ministers. However, none of them knew what the astronomers were doing. In my opinion, how can a person who lacks knowledge judge who is right or wrong? Hence I determined to study with all my might and main the subject of mathematics. Now that the methods were compiled and explained clearly in books, learners find it easy. Heaven knows how difficult it was for me to learn it in those days! (Wang, 1978, p. 69.)

Verbiest was very much respected as the teacher of Emperor Kangxi in mathematics, astronomy and science. Realizing that he was himself getting old, Verbiest wrote and asked the Society of Jesus to send more younger missionaries learned in mathematics and astronomy to China. His call was answered by King Louis XIV of France (1638-1715), who sent a group of French Jesuits to China in 1685. For a political reason of not causing problems with the Portuguese authority this group of French Jesuits was sent under a sort of disguise as the "*King's Mathematicians*". By the time the five French Jesuits settled in the Imperial Court in 1688, Verbiest had passed away, so Emperor Kangxi continued to learn assiduously Western mathematics and astronomy from these "*King's Mathematicians*" in the second period.

Two of the French Jesuits, Joachim Bouvet [白晉 1656-1730] and Jean-François Gerbillon [張誠 1654-1707], left us with their diaries which gave a detailed account of their days spent in the Imperial Court. Bouvet wrote in his diary, published in France in 1697 and soon translated into English in 1699 [*Portrait histoirique de l'empereur de la Chine: presentée au Roy* (The history of Cang-hy, the present emperour of China: presented[sic] to the most Christian King)] (Bouvet, 1697):

> His natural genius is such as can be parallel'd but by few, being endow'd with a quick and piercing Wit, a vast memory, and great Understanding; His constancy is never to be shaken by any sinister Event, which makes him the fittest Person in the World, not only to undertake, but also to accomplish Great designs.[…] But, what may seem most surprising, is, that so great a Monarch, who bears upon his shoulders the weight of so vast an Empire, should apply himself with a great deal of Assiduity to, and have a true relish of all Sorts of useful Arts and Sciences.
>
> […] so there is not any Science in Europe that ever came to his Knowledge, but he showed a great Inclination to be instructed in it. The first Occasion which had a more than ordinary Influence upon his Mind, happened (as he was pleased to tell us himself) upon a Difference arisen betwixt Yang quansien [Yang Guang-xian], the Famous Author of the last Persecution in China, and father Ferdinand Verbiest, of the Society of Iseus.
>
> […] As this Tryal of Skill in the Mathematiks was the first Occasion that introduced the Father Missionaries into the Emperor's acquaintance; so from that time, he always shew'd a great inclination to be instructed in the Mathematical Sciences, which in effect, are in great Esteem among the Chineses. During the space of two Years, Father Verbiest instructed him in the Usefulness of the best of the

Mathematical Instruments, and in what else was most Curious in Geometry, the Statique, and Astronomy; for which purpose he wrote several Treatises.[…]
He did the Honour to us four Iesuits, Missionaries then at Peking, to receive our Instructions, sometimes in the Chinese, sometimes in the Tartarian Language; […] Much about the same time, Father Anthony Thomas, did give him further Instruction concerning the Use of the best Mathematical Instruments, in the Chinese language, and the Practical part of Geometry and Arithmatik, the principles of which he had formerly been taught by Father Verbiest. He would also have us explain him the Elements of Euclid in the Tartarian Language, being desirous to be well instructed in them, as looking upon them to be the Foundation, upon which to build the rest.
After he was sufficiently instructed in the Elements of geometry, he ordered us to compile a whole System of both the Theorick and Practick of Geometry, in the Tartarian Language, which we afterwards explain'd to him in the same manner as we had done with the Elements of Euclid. At the same time, Father Thomas made a Collection of all the Calculations of geometry and Arithmaticks (in the Chinese language) containing most of the Curious Problems extant, both in the European and Chinese Books, that treat this matter. He was so much delighted in the pursuit of these Sciences, that besides betwixt two and three Hours, which were set aside every day on purpose to be spent in our Company, he bestowed most of his leisure time, both in the day and at night in his Studies. (Bouvet, 1699, p.2, 3, 47, 50, 51, 52, 55.)

The third period of transmission began with the establishment of the Office of Mathematics in *Mengyangzhai* [蒙養齋 Studio for the Cultivation of the Youth], a place situated in the garden inside the Imperial Palace. Prince Yinzhi [胤祉 1677-1732], third son of Emperor Kangxi, was made the Head of Office of Mathematics. Besides serving as a school for taking lessons in mathematics, astronomy and science, the main task of this Office was to compile the monumental treatise *Lüli Yuanyuan* [律曆淵源 Origin of Mathematical Harmonics and Astronomy], comprising three parts: *Lixiang kaocheng* [曆象考成 Compendium of Observational Computational Astronomy] in forty-two volumes, *Shuli Jingyun* [數理精蘊 Collected Basic Principles of Mathematics] in fifty-three volumes, and *Lülü Zhengyi* [律呂正義 Exact Meaning of Pitchpipes] in five volumes. The compilers were all Chinese officials and scholars, but the book mentioned the contribution of foreign missionaries at the beginning of *Shuli Jingyun*. The treatise *Shuli Jingyun* includes both traditional Chinese mathematics, the part that was still in extant and was understood at the time, as well as Western mathematics, highly likely from the "lecture notes" prepared by the missionaries for Emperor Kangxi during his first period of ardent study in the 1670s.

6 Method of Borrowed Root and Powers

Another teacher who taught Emperor Kangxi the art of solving equations was the Belgian Jesuit Antoine Thomas [安多 1644-1709], who studied at University of Coimbra from 1678 to 1680 and was ordered to go to Peking in 1685. By the time Thomas came to China he had compiled *Synopsis mathematica*, which was based on the book *De numerosa potestatum ad exegesim resolution* (On the numerical resolution of powers by exegetics) written by François Viète (1540-1603) in 1600. Thomas later revised it as *Suanfa Zuanyao Zonggang* [算法纂要總綱 Outline of the Essential Calculations] and *Jiegenfang Suanfa* [

借根方算法 Method of Borrowed Root and Powers], to be used as lecture notes for the mathematics lessons in the Imperial Court and later incorporated into Books 31-36 of *Shuli Jingyun*.

For the purpose of illustration let us look at an example in the book of Viète, namely, to solve the equation "x squared plus A times x equals B". Let x_1 be a first approximation of a root, which is $x_1 + x_2$. Substitute into the equation and neglect the comparatively much smaller term x_2 squared. We obtain x_2 in terms of x_1, A and B. So we have a better approximation $x_1 + x_2$. Keep reiterating the process to obtain a better and better approximation. Let us look at a similar problem in Book 33 of *Shuli Jingyun*: "If the cube [of root] and eight roots are equal to 1824 *che*, how much is one root?" In modern day mathematical language we want to solve $x^3 + 8x = 1824$, which is accomplished by a method called "extraction of cube root with accompanying number". The basic idea is the same as that in the previous example. If the answer is not exact, the process will give a better and better approximation to any number of decimal places. Emperor Kangxi not only studied the method in earnest but even did homework assignments which can still be read in the archive today!

The Chinese mathematician MEI Jue-cheng [梅瑴成 1681-1763] told the story on how he learnt this new method in his book *Chishui Yizhen* [赤水遺珍 Pearls Remaining in the Red River] of 1761:

> While I was serving in the Imperial Court of the late Emperor canonized as Shengzu Humane [Kangxi], I was instructed on the method of *jiegenfang* [借根方 borrowed root and powers] by the late Emperor. He issued an edict to say that the Westerners called the book *aerrebala* [阿爾熱巴拉 algebra] that means "Method from the East". I respectfully learnt the method, which is really marvellous and is the guide to mathematics. I suspected that the method resembles that of *tianyuan* [天元 celestial source], so I took up the book *Shoushi Licao* [授時曆草 Calculation Draft of the *Shoushi* Calendar] and studied it, thereby clarifying the matter. Though the terminologies are different the two methods are the same, not just a mere resemblance.
>
> Scholars in the Yuan period wrote on calendar reckoning using this method, which became a lost art for some unknown reason. It was fortunate that those [foreigners] who resided afar admired our culture and sent it back so that we retrieved it. From their naming it "Method from the East" we see that they did not forget from whom they learnt this method. (Mei, 1761/1877, p.7; Han, 2010, p.473.)

This is an indication of how the slogan of the time, "Western learning has its origin in Chinese learning", got promulgated. In making his subjects believe that Western learning originated in older Chinese learning, Emperor Kangxi knew that the Chinese would be more than willing to learn it and would not regard it as something opposing traditional value. Or, maybe he really thought that the method originated in older Chinese learning. Indeed, similar methods were explained in mathematical classics of earlier days, most of which became less known by the Ming and early Qing period. However, it is grossly wrong to say that *aerrebala* (algebra) means "Method from the East". For the Europeans the method was transmitted to them from the Islamic world — indeed "east" to them, but not from China! The word "algebra" itself does not connote anything about "Method from the East" but comes from the Arabic word *al-jabr*, meaning "restoration", which is to be understood together with another Arabic word *al-muqābala*, meaning “reduction”. These two words appear in the title of a famous Arabic treatise of the ninth century from the pen of Muhammad ibn Mūsā al-Khwārizmī (*ca*. 780-850). The two words together describe the method in solving an algebraic equation by transposing terms and simplifying the expression, something a school pupil of today would be quite familiar with.

Catherine Jami once made a very pertinent remark:

> […] the cross-cultural transmission of scientific learning cannot be read in a single way, as the transmission of immutable objects between two monolithic cultural entities. Quite the contrary: the stakes in this transmission, and the continuous reshaping of what was transmitted, can be brought to light only by situating the actors within the society in which they lived, by retrieving their motivations, strategies, and rationales within this context. (Jami, 1999, p.430.)

In the process of transmission, sometimes between several different cultures thousands of miles apart in a span of hundreds of years, the reshaping would be intricate and hard to trace. Nevertheless, what finally evolved is the result of collective wisdom that became part of human heritage for which it is actually quite unnecessary to ascertain who the originator was. I subscribe to what Joseph Needham says in the preface of his monumental treatise:

> The citizen of the world has to live with his fellow-citizens […]. He can only give them the understanding and appreciation which they deserve if he knows the achievements of the sages and precursors of their culture as well as of his own. […] Certain it is that no people or group of peoples has had a monopoly in contributing to the development of Science. Their achievements should be mutually recognized and freely celebrated with the joined hands of universal brotherhood.” (Needham, 1954, p.9.)

7 New Method of *aerrebala*

Viète wrote another book in 1591 with more important influence, namely, *In Artem Analyticem Isagoge* [Introduction to the Analytical Art]. In his book he introduced what he called “*logistica numerosa*” and “*logistica speciosa*”, that is, numerical calculation and symbolic calculation. Viète was so pleased with his idea that he concluded his book by the exclamation: “*Quod est, nullum non problema solver* [There is no problem that cannot be solved]”! It led to subsequent work of René Descartes (1596-1650), *La géométrie* [Geometry] of 1637, and that of Isaac Newton (1642-1726), *Arithmetica Universalis* [Universal Arithmetic] of 1707 (with the work actually done about forty years earlier), by which time mathematicians in Europe were familiar with the use of symbolic calculation. Again, a school pupil of today would be quite familiar with that too, but when Viète first introduced it in his book, it was a very novel idea. Let us see how Emperor Kangxi reacted to it when another French Jesuit, Jean-François Foucquet [傅聖澤 1665-1741] taught him this new method, which Foucquet called the “new method of *aerrebala*”.

In an Imperial Edict issued by Kangxi Emperor between 1712 and 1713 he said:

> Every day soon after getting up I study with the Princes the method of *aerrebala* [algebra] and find it most difficult. He [J.-F. Foucquet] says that it is easier than the old method, but it looks more difficult than the old method and has more errors as

well as many awkward features. […] Copy this Imperial Edict and issue the book in the capital to the Westerners for them to study it in details, and to delete those parts that do not make sense. It says something like *Jia* multiplies *Jia*, and *Yi* multiplies *Yi*, without any concrete number appearing. One never knows what the result of the multiplication is. It seems that this man [J.-F. Foucquet] is only mediocre in mathematical skill! (Zhongguo diyi lishi dangan guan, 2003, p.52; Han, 2010, p.468.)

In defence Foucquet said in his book *Aerrebala Xinfa* [阿爾熱巴拉新法 New Method of *Aerrebala*], the only existing copy now kept at the Biblioteca Apostolica Vaticana as document *Borgia Cinese* 319:

> The old method uses numerical values, while the new method uses symbols that are accommodating [通融記號 *tongrong jihao*] […] Using this accommodating notation, it is easy to perform calculation, and it enables one to see the situation clearly so that one can focus on the method and understand the underlying rationale of the calculation. The use of numerical value works only for a particular value, while the use of accommodating notation encompasses all values in general. (Han, 1991, pp. 24-25; Han, 2010, p. 467.)

The frustration expressed by Emperor Kangxi reminds us of the joke in the classroom in which a teacher announces routinely, "Let x be the age of the boy, then…" only to receive a matter-of-fact protest from a pupil, "What happens if it is not?" The explanation given by Foucquet reminds us of an important underlying message, which is however seldom made apparent to school pupils, namely, that we treat numerical quantities as general objects, and manipulate such general objects as if they were numerical quantities. Although we do not know (prior to solving the equation) what they are, we know that they stand for certain numbers and, as such, obey general rules. For instance we do not know what A and B are, but we know that $A \times B = B \times A$. We can therefore apply these general rules systematically to solve problems which can be formulated in terms of equations. This is in fact the underlying essence of modern day abstract algebra in which the study and vista broaden from the familiar number systems to various algebraic structures of interest.

For a teacher this phenomenon should not be unfamiliar. Many students who face some difficulty in learning a new subject, instead of putting in more effort, choose the easier way out by blaming the teacher for poor teaching and telling everybody that the teacher is incapable! In some sense, the episode between Emperor Kangxi and Father Foucquet was like that! It is important for a teacher to try to convince the pupil why symbolic manipulation is needed, why the effort to understand it is worthwhile, and to explain the meaning behind those symbols.

Of the several methods of solving algebraic equations, the "new method" gradually replaced the other methods by the eighteenth century and was further developed. However, because of this personal dislike of the subject by Emperor Kangxi, probably resulting from reluctance to admit his own inadequacy in learning it coupled with arrogance, the transmission into China of the powerful method of symbolic calculation was delayed for nearly one-and-half century! China had to wait until in 1859 LI Shan-lan [李善蘭 1811-1882] and the British missionary Alexander Wylie [偉烈亞力 1815-1887] collaborated to translate the treatise *Elements of Algebra* written by Augustus De Morgan

(1806-1871) in 1835, which was given the title *Daishuxue* [代數學 The Study of *Daishu*] (Chan & Siu, 2012). Thirteen years later HUA Heng-fang [華蘅芳 1833-1902] and another British missionary John Fryer [傅蘭雅 1839-1928] translated the book *Algebra* William Wallace (1768-1843) wrote for the *Encyclopaedia Britannica* between 1801 and 1810, which was given the title *Daishushu* [代數術 The Method of *Daishu*], in 1872. In both translated text, the term *daishu* literally means "the study of numbers represented by a character", which is no doubt motivated by what De Morgan wrote in his book:

> A letter denotes a number, which may be, according to circumstances, as will hereafter appear, either any number we please; or some particular number which is not known, and which, therefore, has a sign to represent it till it is known. (De Morgan, 1835/1837, p. vi.)

8 Conclusion

When we were in school we all learnt to solve equations. Did it ever occur to you at the time how miraculous the phrase "let x be …" is? Without knowing what this magic x stands for *a priori*, somehow at the end after certain symbolic manipulations the value for x falls out as an answer! Or, at the other extreme, did you find the routine working of solving an equation step by step boring at the time? Now that we have become teachers ourselves do we realize the difficulty most students have in understanding what the phrase "let x be …" means? Indeed, what is this x? Is it an unknown (in an equation)? Is it a variable (in a function)? Or, is it an indeterminate (in a polynomial)?

Although the Chinese had built up a rather refined kind of machinery in solving different types of equations from the ancient to medieval times, what is learnt in school today basically follows what was developed in the Islamic world since the eighth century and subsequently in Europe from the sixteenth century to the nineteenth century. This was transmitted into China in the Ming Dynasty and the Qing Dynasty. In some sense, a school pupil of today, when first encountered the topic of solving equations, might feel the same as what a Chinese in those days first encountered Western ways of solving equations. In this paper we attempted to look at the pedagogical issue from this viewpoint and hope that learners can benefit from the discussion.

REFERENCES

Bouvet, J. (1697/1699). *Portrait histoirique de l'empereur de la Chine*: *presentée au Roy*. Paris: Etienne Michallet.

Bouvet, J. (1699) The history of Cang-Hy, the present emperor of China presented to the most Christian king (English translation of Bouvet's *Portrait histoirique de l'empereur de la Chine*: *presentée au Roy*). London: F. Coggan.

Chan, Y. C. [陳葉祥], & Siu, M. K. [蕭文強] (2012). On Alexander Wylie's *Jottings on the Science of the Chinese Arithmetic*. In É. Barbin, S. Hwang, & C. Tzanakis (Eds.), *Proceedings of HPM 2012 – The HPM Satellite Meeting of ICME-12* (pp. 229-237). Daejeon: Korean Society of Mathematical Education &Korean Society for History of Mathematics.

Chemla, K., & Guo, S. C. [郭書春] (2004). *Les Neuf Chapitres: Le classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.

De Morgan, A. (1835/1837). *Elements of algebra*. London: Taylor and Walton.

Guo, S. C. [郭書春] (Ed.) (1993). *Zhongguo kexue jishu dianji tonghui* (*shuxue juan*) [中國科學技術典籍

通彙(數學卷) Collection of Chinese classics in science and technology (Mathematics), Volumes 1-5]. Zhengzhou: Henan Educational Press.

Han, Q.[韓琦] (1991) Kangxi shidai chuanru de xifang shuxeu ji qi dui zhongguo shuxue de yingxiang [康熙時代傳入的西方數學及其對中國數學的影響 Transmission of Western mathematics in the reign of Kangxi and its influence on Chinese mathematics], PhD dissertation. Beijing: Institute of the History of Natural Sciences.

Han, Q.[韓琦] (2010) Xifang shuxue de chuanru he qianjia shiqi gusuan de fuxing — yi *jiegenfang* de chuanru he *tianyuanshu* yanjiu de guanxi wei li [西方数学的传入和乾嘉时期古算的复兴 — 以借根方的传入和天元术研究的关系为例 Transmission of Western mathematics and the revival of traditional mathematics in the reign of Qian Long and Jia Qing — relationship between the transmission of *jiegenfang* and the study of *tianyuanshu* as a case study]. In P. Y. Chu [祝平一] (Ed.) *Zhongguoshi xinlun keji yu zhongguo shehui fence* [中國史新論：科技與中國社會分冊 New perspective on Chinese history : volume on science and technology and Chinese society] (pp.459-486). Taipei: Academia Sinica/Linking Publishing.

Hoe, J. (1977). *Les systèmes d'équations polynômes dans le Siyuan Yujian (1303)*. Paris: Collège de France, Institut des hautes études chinoises.

Jami, C. (1999). "European science in China" or "Western learning"? Representations of cross-cultural transmission, 1600-1800, *Science in Context*, *12* (3), 413-434.

Jami, C. (2011). *The Emperor's new mathematics: Western learning and imperial authority during the Kangxi reign* (*1662-1722*). Oxford: Oxford University Press.

Lam, L. Y [藍麗蓉] (1977). *A critical study of the Yang Hui Suan Fa: A thirteenth-century Chinese mathematical treatise*. Singapore: University of Singapore Press.

Lam, L.Y. [藍麗蓉], & Ang, T.S. [洪天賜] (1992/2004). *Fleeting footsteps: Tracing the conception of arithmetic and algebra in ancient China*. Singapore: World Scientific.

Libbrecht, U. (1973). *Chinese mathematics in the thirteenth century: The Shu-shu Chiu-chang of Ch'in Chiu-shao*. Cambridge: MIT Press.

Mei, J.C. [梅穀成] (1761/1877) *Chishui yizhen* [赤水遺珍 Pearls remaining in the red river].Suzhou: Jiangsu Publishing House.

Needham, J. (1954) *Science and civilization in China, Volume* 1. Cambridge: Cambridge University Press.

Shen, K. S.[沈康身], Crossley, J. N., & Lun, A. W. C. [倫華祥] (1999). *The nine chapters on the mathematical art: Companion and commentar*y. Oxford: Oxford University Press.

Siu, M. K. [蕭文強] (1995). Mathematics education in ancient China: What lesson do we learn from it ?, *Historia Scientiarum*, *4*(3), 223-232.

Siu, M. K. [蕭文強] (2009). Mathematics education in East Asia from antiquity to modern times, In K. Bjarnadottir, F. Furinghetti, & G. Schubring (Eds.), *Dig where you stand: Proceedings of a conference on on-going research in the history of mathematics education, Garðabær, June 20-24, 2009* (pp. 197-208). Reykjavik: University of Iceland.

Siu, M. K. [蕭文強] (2015). *Tongwen Suanzhi* (同文算指) and transmission of *bisuan* (筆算 written calculation) in China: from an HPM (History and Pedagogy of Mathematics) viewpoint, *Journal for History of Mathematics*, *28*(6), 311-320.

Siu, M. K. [蕭文強] (2016). The confluence of the Yellow River and the Mediterranean: Synthesis of European mathematics and Chinese mathematics during the seventeenth, eighteenth and nineteenth centuries. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting* (pp. 581-592). Montpellier: IREM de Montpellier.

Swetz, F. (2012). *Mathematical expeditions: Exploring word problems across the ages*. Baltimore: Johns Hopkins University Press.

Wang, Y. W. [王雲五] (Ed.) (1978) *Shiku quanshu zhenben baji* [四庫全書珍本八集 Complete Library in

Four Sections: eighth collection in 400 volumes]. Taipei: Commercial Press.
Wu,W.J. [吳文俊] *et al* (Eds.). (1998-2004). *Zhongguo shuxueshi daxi* [中國數學史大系 A Compendium of History of Chinese Mathematics], Books 1-8, Supplementary Books 1-2. Beijing: Beijing Normal University Press.
Zhongguo diyi lishi dangan guan [中國第一歷史檔案館 First office on historical documents of China].(Ed.) (2003). *Qing zhongqianqi xiyang tianzhujiao zai hua huodong dangan* [清中前期西洋天主教在華活動檔案 Archives concerning Western Catholic missions from early to mid Qing dynasty in China]. Beijing: Zhonghua Publishing.

A CAMBRIDGE CORRESPONDENCE CLASS IN ARITHMETIC FOR WOMEN

James J. TATTERSALL
Providence College, Providence RI 02918, USA
tat@providence.edu

ABSTRACT

In late nineteenth century England the Association for Promoting the Higher Education of Women in Cambridge began sponsoring a series of correspondence classes for women vying for certificates on the Cambridge Higher Local Examinations. These courses were designed for women who lived in remote areas devoid of suitable teachers and for governesses who did not have sufficient control over their time to permit them to attend classes or to receive oral instruction. The instructor of the arithmetic class sent candidates a set of mathematical problems at fortnight intervals. After forwarding their answers, students read related material and prepared for the next problem set. The teacher read over their work and returned it with comments. The teacher's correspondence illustrating his educational philosophy and tips on how to prepare for the examination is highlighted.

In November 1867, at a time when women could not earn a degree at any university in Great Britain[1], Anne Clough and Josephine Butler, both prominent in the Victorian women's suffragist movement and each with a keen interest in woman's education, founded the North of England Council for Promoting the Higher Education for Women. The organization, originally representing associations of school mistresses from several large northern English towns, established a program of lectures and courses for women. In an attempt to improve the intellectual standards prevalent in girls' schools, they petitioned Oxford and Cambridge Universities to offer an examination for women of sufficiently high standard that would serve as a test for those who wished to enter the teaching profession. Accordingly, Cambridge University initiated the Higher Local Examinations for Women in 1869 aimed at those over eighteen who wanted to be teachers, which upon completion would provide them admission to higher education.[2]

Henry Sidgwick, fellow of Trinity College Cambridge, and activist Millicent Garrett Fawcett formed a committee, which segued into the Association for the Promotion of the Higher Education of Women in Cambridge. In 1870, Henry's future wife Eleanor[3]was instrumental in establishing a series of "Lecture for Ladies" that covered a number of relevant academic topics including practical arithmetic, algebra, and geometry.[4]Cambridge at the time was a small market town with limited housing. Nevertheless, Sedgwick managed to secure lodgings, under the supervision of Anne Clough, for five women

---

[1] In 1878, the University of London became the first academic institution in the United Kingdom to offer post-secondary degrees to women.

[2] In 1858, the Cambridge University had initiated Local Examinations with the aim of raising the standards of education.

[3] Nora, as she was known to her friends, studied at Newnham Hall, precursor to Newnham College. Albeit coached by M.N. Ferrers of Christ College for the Mathematical Tripos, she declined to sit for the examination. She served as a physics researcher for her brother-in-law Lord Rayleigh. Her brother Arthur served as Prime Minister from 1902 to 1905.In 1902 she succeeded Anne Clough as Principle of Newnham College.

[4] Other topics include: mechanics, astronomy, geology, chemistry, physical geography, English and modern history, English literature, logic, political economy, moral and mental philosophy, Latin, Greek, French, German, and harmony.

interested in attending the lectures. Demand for the lectures escalated. As a result, larger accommodations and a lecture hall were procured by Sidgwick near St. John's College. In addition, a lending library was made available to women who attended the lectures.

In the fall of 1872, to alleviate the pressure to find adequate housing and for the women who were not able to reside in Cambridge to attend the lectures, the committee initiated a series of correspondence classes under the supervision of Mrs. Annette Peile that was supported by faculty from several Cambridge colleges. Walter William Skeat offered a class in English literature, Greek by John Peile, geometry and algebra by James Stuart, logic by John Venn, and political economy by Henry Sidgwick.[5] A fee of £2 was implemented,[6]sufficient to make the courses self-sufficient, high enough to keep off people who are not seriously about work, and sufficient to ensure that those who join the classes will be anxious to get an adequate return for their money.

General instructions for those interested in enrolling in a correspondence course stressed that the program was not much more than an assistance in self-education, and therefore should in no sense be taken when efficient oral teaching could be obtained. Students should be prepared to carry out honestly and perseveringly the directions of their teachers and must be capable of taking hints, for detailed explanation would be impossible. As a consequence, they should have an acquaintance that they can consult who understands the rudiments of the subject. While an examination is perhaps not to best way to identify a good teacher, correspondence courses were felt to be a novel approach to pedagogical training. Whereas in classrooms students merely listen and seldom ask questions, in correspondence courses students must answer and ask questions. In order to encourage such communication students were provided with prepaid half penny post cards (Hudson 1872a).

Thirty students enrolled in the first correspondence course in arithmetic offered by William Henry Hoar Hudson, a mathematical lecturer at St. John's and St. Catherine's Colleges, Cambridge, covering topics such as measurement, vulgar fractions, ratio, proportion, decimal fractions, and interest. In order to access their potential and avoid unintended consequences, he asked them, in an introductory letter, to send him answers to the following questions (Hudson 1872b):

- For what object are you studying arithmetic?
- Have you access to a library?
- What time do you have available for study?
- To what extent is your time occupied in teaching or otherwise?
- Can you devote 2 ½ hours at one sitting, once a fortnight, to answer a set of questions?
- Can you devote one hour a day to reading arithmetic?
- From what book or books have you learnt arithmetic?
- What books on arithmetic can you consult?

---

[5] Skeat was a philologist and later cofounder of Newnham College. Peile, also a philologist and Annette's husband, became master of Christ's College, Cambridge. Stuart was professor of mechanism and mechanics at Cambridge before becoming Lord Rector of St. Andrews University. The logician Venn, from Gonville and Caius College, Cambridge, compiled the *Alumni Cantabrigienses*, a biographical record of members of the university from earliest times to 1900. Nevertheless, he is best known for his set theoretic diagrams. Sidgwick was a philosopher and economist. The Sidgwick Site, now home to a number of arts and humanities faculties at the university, is named for him.

[6] Equivalent in purchasing power in 2017 of about $200 (Measuring worth, n.d.). Governesses were eligible for a discount.

• Enumerate parts of arithmetic you have learnt
• Have you any knowledge of algebra?

Students were advised that they must be accustomed to and be prepared for serious work, for a problem set will be sent every fortnight, usually on alternate Thursdays. A stated time, not to be exceeded, will be given at one sitting to each set, with no help from books, tables, or any source whatsoever. When done, they should send their answers with questions on any difficulties and a report of their progress stated in a clear, concise, and definite manner. Sometime during the next week, he will look over the papers and return them with comments. In the meanwhile, they should be reading and preparing for the next paper.

In his first circular letter (Hudson 1872-73), he reminded participants to be punctual in sending in their papers and letters and, at this stage, not to attach too much importance in merely answering the problems posed for their progress will depend much more on their reading, thought, and practice. They should read in the 'Cambridge sense', which entails lots of writing, per using sources critically with an aim of looking for mistakes, and read with pencil in hand. He cautioned them to aim first at understanding each problem. Never try to reproduce the words of a book in answering questions, but to give their own explanations in their own language.

He admitted that the class will, in all probability, contain members of varied degrees of knowledge and ability. Hence, his problems will usually be general ones. Nevertheless, they may contain questions that are beyond their reading. As their reading advances, more and more of the questions will come within their reach. He hoped that their reading will advance regularly and steadily, but not necessarily rapidly. Thorough understanding takes reading, practice, and thought. Through practice the basic arithmetic operations of addition and multiplication will become 'almost instinctive' for failure to implement them will be an impediment to further progress. He advised them to write clearly, never try to conceal ignorance by obscurity, and not to use rules or symbols that they do not understand. At any time, however, he would be willing to give a brief answer to a question that he can answer on a post card.

In his second letter, he discussed possible textbooks they might consult.[7]He warned students that many arithmetic books may contain valuable methods and useful examples, but lay too much stress on 'rules', neglecting to explain principles and on this account they are bad for educational purposes. They should become imbued with the principles, and familiar with the facts of arithmetic, for then they will be free from slavery to rules. On no account are they to learn or practice a 'rule,' the reason of which they do not thoroughly comprehend. He urged students to,

> Take nothing on trust. It will be far better to solve a question by a method of your own, which you do understand, even if it be a long way of arriving at the answer,

[7] Although there was no one book on arithmetic that he could give a qualified approval, Augustus De Morgan's *Elements of Arithmetic* (5th ed., London, Walton, 1969) was very good but contained too few examples. James Harris's *Graduated Exercises in Arithmetic and Mensuration with Key* (London, Longmans, 1872) has a good collection of examples but no bookwork. *The Science and Art of Arithmetic* (London, Whittaker, 1970) by Adolf Sonnenschein and Henry Arthur Nesbitt was somewhat awkwardly arranged, but contained some excellent hints for teaching. Since several of the class are actually engaged, or are about to be engaged in teaching, he felt that it will be the most convenient book on the whole.

than to allow yourself to be led blindfold by a 'rule,' by a short cut, over steps which you cannot see.

He offered them one more piece of general advice:

> Acquire the habit of concentration. In reading, especially a new part of the subject, never allow your thoughts to wander. Try and arrange to be free from interruptions, and in a silent room. If you are new to intellectual works, and find such close attention difficult, do not, at first attempt to read for too long.

In his third letter, he urged students in explaining their answers:

> To imagine that you are addressing an intelligent child, ignorant say of arithmetic, except that which comes earlier in the point in question … Never use words that you do not understand … such a child would be sure to ask you the meaning of them. Never use symbols that you do not understand …Learn the exact meaning of symbols and never use them in any other sense … Never use arithmetical terms in any other than their technical sense.

He stressed that they read each question carefully and understand it. Do not ask themselves 'what rule should I use?' Use only the rule of common sense. Attend to the logic of their work and let their solution 'tell its story.' Let the argument be clearly indicated and take care to copy down figures correctly.

In his fourth letter, he stressed the primary importance of doing their own work. He noted that it appears that some of them have not understood his directions and letters. If that be the case, they should ask him at once for explanations. He lamented that he is not getting enough questions from them, nor of the right kind. He wanted questions concerning some actual difficulty that they have felt in thinking over what they have read, and these questions must be stated so definitely that he can see how much is clear to them, and when the difficulty began. He noted that several of his students have discontinued the practice of leaving a margin by the side of their answers, and consequently he cannot criticize their solutions.

In the fifth letter, he expressed concern that some students still work by rules and not by common sense. He warned them that the mechanical rule of three does not apply to all problems. The bulk of the rest of the letter contains study hints for those taking examinations as would a Cambridge coach preparing men for a tripos examination. In particular,

- Don't omit the names of concrete quantities.
- Never use an approximation without stating that it is so.
- Don't use incomplete sentences.
- Make a clear distinction between scratching out and cancelling.
- Show clearly what you mean for the answer.
- Use words and symbols accurately.
- Nothing but practice will give you the tact to discern when it is most convenient to use vulgar and when decimal fractions.

His sixth letter (Hudson 1873), was addressed only to candidates for the Higher Local Examination in June. He stressed that any examination is a physical strain and they should prepare themselves for it by being particularly careful to be regular in sleep, exercise, and meals for some time before it. It is important that they should not do any work, either in the day, or after active exercise. It would be good, however, for them to take a short walk in the fresh air before beginning, but they should go on no account into the examination room hungry or tired.

If they have a morning paper they should take care to be up in good time, and to have their breakfast over comfortably without hurry. They shouldn't work late at night, nor for more an hour in the evening altogether and give themselves a full rather than a short allowance of sleep. They should be in good time at their place, not too early least they should become nervous by waiting, nor too late for the fear of being flustered at the start. From three to five minutes before the time is sufficient to see that they have two or three good pens and to take their own if there is any fear that defective ones will be supplied. They should arrange their seat, inkstand, blotting paper, and watch conveniently for work. They should go over again their table of weights and measures just before going in. Above all things they should keep cool, read every question completely through before commencing to answer it, for the second part of the question often throw light upon the first. They should not crowd their work or be afraid of wasting paper.

If they aim at answering every question, they should divide the time by the number of questions and try to get each question done in its portion of time, saving time on the easier questions. In the early part of the paper they should not spend more than the proper time puzzling over a doubtful question unless they have already gained some time. They should not however, let this advice prevent them from going on with a solution they are in a fair way to finish.

If they come across a question which they know they cannot answer, they should not loose time in attempting it in vain, while there are other questions in the paper which they believe that they can do. They should not refrain however from answering a part of a question because they cannot answer the whole, for sometimes the second part is easier than the first. They should be accurate and careful in their work the first time in doing it, and have confidence in their own result.

After they have done all the straight-forward questions, they should look carefully at the rest of the paper, and select the easiest remaining question for trial. They should consider how they mean to attack it, write enough to indicate the method, briefly but clearly, and send this up at all events. Imperfect solutions often get considerable credit. It is a mistake to suppose that questions are arranged according to the order of the subject and a question on an advanced part of the subject may well be easier than one on the elementary part.

They should not be deterred from trying a question because it is a long one, for it is not always difficult on that account. The length of the question gives no hint as to the length of the answer. It is a pity to neglect an easy question though not reading it through. They should not think of leaving the room until time is called; so long as there is anything on the paper that they can possibly do, do not give in. If they have finished the paper, read over their work carefully, correct and supplement it.

In writing out proofs of rules and explanations of theory, they should remember that diffuseness wastes time and omission loses marks; the point to be aimed at is to put in everything essential as briefly as possible. In revising their work just before an examination, it is seldom wise to learn anything new; it is better to make sure of whatever they have known before.

HIGHER LOCAL EXAMINATION

Tuesday, June 17, 1873, 9 to 11 P.M.

1. Multiply three hundred and seventy-two million four hundred thousand and forty-six by itself and express the results in words.

2. Reduce to tenths of a penny the difference between five guineas and eighty-nine half crowns.

3. If 17 acres 3 roods and 33 poles cost £1236. 6s., how much will 50 acres 2 roods 21 poles cast at the same rate?

4. Find, by practice, the cost of 94 lbs 10 oz. 18 cwts. 6 grs. at £3 6s. 8d. per ounce.

5. Multiply together the sum and difference of $\frac{11}{18}$ and $\frac{5}{27}$ and divide the result by the product of $5\frac{1}{7}$ and $1\frac{31}{84}$.

6. Reduce $\frac{5}{256}$ to a decimal. What decimal of £5 is £12. 16s.? Find the value of 0.01625 of a ton, and shew that when divided by 0.0175 it becomes 18 cwts. 2 qrs. 8 lbs.

7. Find the interest on £875. 10s. for 3 ½ years at 5 per cent.

8. A cubic inch of gold is made to cover an acre of metal. Find the thickness of the gilding, and the weight of one square yard of it, assuming that gold is nineteen times as heavy as water, and that a cubic foot of water weights 1000 oz. Avoirdupois.

9. Reduce $\frac{11}{56}$ to a circulating decimal and $0.631\bar{8}$ to a vulgar fraction.

10. Extract the square root of 88.604569 and of $1+\frac{1}{9}+\frac{1}{4-\frac{1}{10}}$.

11. There are three speculations which bring in respectively 5, 7, and 9 percent on the money invested. Compare the average interest on their capitals obtained by two speculators, one of whom invests and equal sum in each, and the other invests such sums that he obtains equal income from each.

12. A person buys an article and sells it at a profit of 2s. 7d. On calculating his profit he finds that if he estimates it as a percentage on his outlay, it is one-fifth as great again as it would be if he estimated it as a percentage on the selling price. What did he give for the article?

13. The cost of a ton of coal in London is four times the cost at the pit’s mouth, and half of the latter goes in the miners’ wages. If a rise of 20 per cent in miners’ wages be accompanied by a rise of 20 per cent of coals in London, what proportion of the cost of coals must have been due to carriage that the profits of those engaged in the coal trade may be only three times as great as before? (The cost of carriage is supposed to be unchanged), (University of Cambridge Higher Local Examination 1874).

In a speech given before the College of Preceptors on February 8, 1893, Hudson remarked that in teaching the fundamental law of pedagogy was “The understanding of the pupil was to be employed throughout." (Hudson 1893). The correspondence to his arithmetic students illustrates his sound educational philosophy. Much of his pedagogical advice could well apply to other disciplines and many of his suggestions about preparing for exams are relevant today as they were then. It is not evident how many of his correspondence students passed the arithmetic examination, but thanks in part to Hudson and others like him, almost 70% of the women who took the Cambridge Higher Local Examination that June passed (University of Cambridge Higher Local Examination 1874).

Hudson belonged to a talented mathematical family. He was Third Wrangler on the 1861 Cambridge Mathematical Tripos. In 1882, he was appointed professor of mathematics at King's College London. A position he held until 1903. His wife Mary Watson (née Turnbull) matriculated at Newnham Hall in 1873 but did not sit for a tripos. Their son Ronald William Henry Turnbull Hudson of St. John's College was Senior Wrangler[8]on the 1898 Mathematical Tripos.[9] After a short career as a lecturer at University College, London, he died in a climbing accident in North Wales. Their eldest daughter, Winifred Mary, attended Clapham High School and was a Winkworth Scholar at Newnham College. She was bracketed with the Eighth Wrangler on the 1900 Mathematical Tripos. For most of her life, she worked part time for the Charity Organization Society. Hilda Phoebe attended Newnham College, and was bracketed with the Seventh Wrangler on the 1903 Mathematical Tripos. She spent a year at the University of Berlin and another at Bryn Mawr, before returning to Newnham as a lecturer. She was the only woman to give a communication at the 1912 International Congress of Mathematicians. In Cambridge, England. (Semple, 1969) During WWI, She worked on stress calculations for airplane structures for the Admiralty Air Department (Barrow-Green, 2014).

Newnham College was first established as a house in which women could reside while attending the Cambridge lectures. Hudson and other supporters of the enterprise formed a limited company to raise funds, lease land, and set a course of study in preparation for examinations. In 1875, the first buildings were constructed. The Newnham Hall Company and the Newnham College Association joined in 1880 to form Newnham College.

Girton College had opened its doors in 1873, providing women two opportunities for undergraduate education in Cambridge. The lectures for ladies were subsumed by the colleges while the correspondence courses continued until the 1890s.

REFERENCES

Barrow-Green, J., (2014). Cambridge Mathematicians' Responses to the First World War. In D. Aubin, & C. Goldstein (Eds.), *The War of Guns and Mathematics*, (p. 80). Providence: American Mathematical Society.

Hudson, W. H. H. (1872a). Instruction by Correspondence, *Newham College Archives*, EC/2/3/2.

Hudson, W. H. H., (1872b). Questionnaire, *Newnham College Archives*, EC 2/3/2.

Hudson, W. H. H., (1872-73). Circular letter #1(14 October), #2 (22 October), #3 (7 November), #4 (6 January), #5 (1 May), *Newnham College Archives*, EC 2/3/2.

Hudson, W. H. H., (1873). Circular letter #6 (21 May), *St. John's College Archives* 12/15/71.

[8] The examination for an honours degree at Cambridge. The person who ranked first on the exam was the Senior Wrangler, the person next was called the Second Wrangler, and so forth. The person who ranked last for an honours degree was referred to as the Wooden Spoon.

[9] The physicist James Jeans was Third Wrangler and the mathematician G.H. Hardy was Fourth Wrangler.

Hudson, W. H. H., (1893). The teaching of mathematics. *Educational Times 46*, p. 138.
Measuring worth (n.d.). Online at www.measuingworth.com. Accessed 14 February 2019.
Semple, J. G., (1969). Hilda Phoebe Hudson, *Bulletin of the London Mathematical Society*, *1*, 357-359.
University of Cambridge Higher Local Examination 6th Annual Report (1874), *St John's College, Cambridge Archives*, 12/15/71.

NEW MATH, AN INTERNATIONAL MOVEMENT?

Geert VANPAEMEL, Dirk DE BOCK
KU Leuven, Faculty of Economics and Business, Campus Brussels, Warmoesberg 26, 1000 Brussels, Belgium
geert.vanpaemel@kuleuven.be, dirk.debock@kuleuven.be

ABSTRACT

The New Math Reform of the 1960s is commonly regarded as an international movement, in which common arguments bound together the participants of the reform. However, some authors have challenged this view, arguing instead for parallel reform movements, linked to different national school systems and different background motives. In this paper, we review some of the arguments proposed and trace the main currents which defined the New Math in Europe and the US. We conclude that European debates were mainly related to a structural Bourbakist view on mathematics. The American reform movement was stronger rooted in socio-economic and political motives and from the start driven by the Government. The European and American points of view, which originated largely independent from each other, briefly came together at the Royaumont Seminar (1959) and subsequent OECD conferences, but remained quite unrelated.

1 Introduction

One of the most striking features of the New Math Reform of the 1960s, was its widespread and international character. In many countries around the globe, from the Americas to Europe, from the Soviet Union to Nigeria and New Guinea, efforts at curriculum reform of school mathematics showed a remarkable tendency to converge on a new conception of what mathematics education should be, replacing the traditional emphasis on computational techniques and Euclidean style of reasoning with a more formal and abstract approach, based on set theory, algebraic structures and topology (Begle, 1968; Ohuche, 1978; Swetz, 1975). All of this occurred in not more than two decades. This was only possible through the orchestrated collaboration of scores of mathematicians, teachers, governments, school boards and publishers. Although comparable initiatives can be found in the reform of school physics, chemistry and biology, the revolutionary aspects of the New Math reform, its internal coherence and its public reception, including the highly mediatized controversies it raised, make it into an exceptional example of a truly international phenomenon.

This international dimension of the New Math Reform did not go unnoticed. In his book, *The "New Maths" curriculum controversy: An international story*, educationist Bob Moon observed that the New Math, "perhaps more than any other curriculum reform, caught the imagination of the world at large" (Moon, 1986, p. 8). Moon did not regard this feature of the reform as evident, and offered some explanations. Focusing on a comparison between the reform as it took shape in some selected European countries (The Netherlands, France, England & Wales, West Germany and Denmark), Moon drew attention to the role of the new information technology and the ease of international travel in furthering contacts between national educational systems. He also pointed to the influence of the media that quickly publicized the promises and successes of the New Math across national borders – reversing the trend when in the 1970s "more negative reports on the reform experience came to attract media interest" (p. 222).

More recently, Jeremy Kilpatrick has again raised the question whether the New Math reform can be regarded as an international phenomenon (Kilpatrick, 2012).Interestingly, he included in his overview the introduction of the New Math reform into non-OECD countries but this was mainly restricted to the local adoption of teaching materials imported from the US or the United Kingdom. Although the widespread dissemination of these textbooks and methods suggests some degree of international coherence in the reform movement, Kilpatrick also concedes that "the well-intentioned efforts of reformers in OECD countries to transfer their efforts to non-OECD countries did not work out as planned" (p. 567). Furthermore, he observes that "some countries seem to have missed much of the new math movement even as international contacts increased between countries" (p. 568) citing Japan and The Netherlands. The international character of the reform should therefore not hide the fact that the local appropriation of the curriculum reform could be and often was very different from the experiences in other countries. Kilpatrick even concludes his paper with the observation that "the more school mathematics is internationalized, the more clearly its national character is revealed" (p. 570). So was there indeed an international movement in the sense that it was conceived and promoted by a coherent community of scholars, following the same guidelines and discussing the same arguments at about the same time?

Once one starts to question the international coherence of the New Math reform, it also becomes possible to wonder how any form of international alignment between national reform movements could emerge. Any widespread cultural phenomenon of the extent of the New Math reform movement begs for further explanation. How did it happen that so many communities in so different political and cultural environments almost at the same moment and apparently almost in unison (or at least in close harmony) proposed similar ideas and similar actions? How did the movement spread from one place to another? And what can the New Math teach us about the identity, the structure and the politics of the mathematics community at large? Christopher Phillips has examined the New Math reform in the United States as a reflection both of the political and social environment generated by the Cold War as well as of the different views on the nature of mathematics and mathematics education among leading mathematicians (Phillips, 2015). This approach is very illuminating for understanding the strategies and ambitions of the New Math in the US, but it also tends to isolate the American reform movement from a more global understanding of what the New Math represented in other countries. Similarly, in her dissertation on precollege mathematics education reform in the United States, Emily Redman has shown that the swift and determined actions of the American mathematical community in preparing a new curriculum, can only be understood by looking at the strong "coagulation" of that community in the previous decades and at "the conscious relationship between a small, determined, and relatively coherent community of reformers with the federal structures and resources that support reform"(Redman, 2013, p. 3).Implementing a successful reform was more than engaging in intellectual debate and writing up curriculum proposals. Reformers needed funding, professional authority, government support and commercial connections in order to be able to bring their ideas into practice. But if such opportunities can be expected to exist within national communities, it is not obvious that similar resources and networks are available and effective in other countries or on an international level. Again, as it becomes clear that the American reform has to be firmly situated in the local context of American society, the analysis of the reform in other countries cannot be expected to fit into that same framework.

In this paper, we will first focus on the early development of the reform movement when contemporary curriculum drawbacks were discussed and new goals were being set. We will show that American and European debates were fundamentally different and that there was almost no contact between the two communities of reformers. Then we pay attention to the actual implementation of the reform, the dissemination of teaching plans and textbooks around the world. Here also, the available historical material suggests that each country produced material adapted to its local needs, and that there was little correlation between what happened in different countries. Finally, we will propose our own understanding of the international character of the New Math reform and its implications for its local implementation.

2 Two parallel reform movements

According to many popular accounts, the New Math reform started early in 1958, when in the wake of the Sputnik scare that captivated the American public, a School Mathematics Study Group was created that set out to produce a new mathematics curriculum for American students. The story is, however, more complicated. The New Math that emerged at the end of the decade cannot be understood without taking into account the ongoing reform efforts that had already prepared the mindset of mathematicians and teachers. Debates about the sorry state of mathematics education in the United States had started much earlier. Already in 1952, the University of Illinois Committee on School Mathematics had started to work out a new school mathematics curriculum, and in 1955 the US College Entrance Examination Board appointed a special Committee on Mathematics, headed by Albert Tucker, to propose new standards for the school mathematics curriculum in order to prepare high school graduates for university. In these early years, New Math as it came to be understood after 1960, was not on the agenda. There was a general concern for the lack of rigor in high school mathematics, and more generally for the "low levels of mathematical understanding and poor attitudes toward mathematics" among high school graduates (Kilpatrick, 2012, p. 564). One of the causes for this situation, according to some reformers, was the lack of attention for the mathematically gifted student, a concern that fitted well with the preoccupation of the American government during the Cold War with a possible shortage in scientific manpower. The aim of the early reform proposals was to make sure that high schools would be stimulated to offer high quality mathematical education that would increase student enrollments in university mathematics courses.

When in 1958 Edward Begle was appointed as director of the School Mathematics Study Group (SMSG), he conceded that "there is little agreement about what should be done" (Phillips, 2015, p. 44). With the support from massive funding from the National Science Foundation, Begle organized a nation-wide network of authors to write textbooks for each grade of high school. According to Christopher Phillips, SMSG took the occasion to redirect not only the rigor and scope of the mathematics curriculum, but also its general approach.

> SMSG's mathematicians took the charge to make the intellectual habits of American students more rigorous as an opportunity to introduce "modern" mathematics into

> the curriculum. They argued that developments over the first half of the century had fundamentally reformulated what it meant to do mathematics. […] The curriculum project was their opportunity to inscribe this view of mathematics in millions of textbooks.(Phillips, 2015, p. 47)

This new image of mathematics, one that would lead towards the New Math, was inspired by the work of the Bourbaki group in France, advocated in the US by such mathematicians as Albert Tucker from Princeton, and Marshall Stone from Chicago. To the mathematicians in SMSG, "structure, not technique, characterized the discipline" (p. 59). It was emphasized that mathematicians did not calculate, they rather pondered on logic puzzles, "that required not counting, nor measuring, but careful reasoning about particular sets of information" (p. 54). The most telling example was the "modern" treatment of arithmetic, not as a tool for calculation, but as an object to investigate the properties of numbers and operations, where the choice of notation or representation was of secondary importance.

The first SMSG textbooks were ready by September 1960. The following years SMSG would prepare similar course materials for elementary schools (1962) and kindergarten (1964). Apart from that, SMSG would write monographs to supplement the usual high school curriculum and to awaken interest in gifted students. It also prepared teacher-training materials, a series of thirty educational films, and revised texts for the "less able students". In 1968, Begle estimated that about four million texts of SMSG had been sold, although it was impossible to say how many students and teachers actually had used the textbooks (Begle, 1968).

This chronology of events suggests an early and dynamic start for the New Math in the United States. In reality, New Math was less well defined and less influential than it appears from the writings of some participants. From the start, the SMSG curriculum was contested by leading members of the academic mathematical community. Furthermore, apart from the SMSG group, other initiatives formulated different approaches. It led Robert Davis, a collaborator in the Madison Project of Syracuse University (one of many alternative reform groups), to conclude that:"There was no single thing, no single alternative to existing school programs, no agreement on how things needed to change. Indeed, the only thing these diverse projects shared in common was the firm conviction that the 'traditional' mathematics curriculum needed to be replaced by something different. Any claim that there was a well-defined 'new math' is entirely unfounded" (Davis, 2003, p. 625). Davis also contended that most schools in the US were hardly affected by the New Math reform. The fact that New Math was perceived as the major thrust of the reform may have been caused more by the public controversies it raised among mathematicians, than by its real impact on mathematics teaching in US schools.

At the same time, a similar reform movement was underway in Europe. As in the US, early initiatives dated back to the beginning of the 1950s. In April 1950 Caleb Gattegno (1911-1988), an Egyptian born mathematician and psychologist, organized the first of a series of international meetings with leading scholars in the fields of mathematics, psychology, philosophy and education (as well as teachers) to discuss the state of mathematical education. At the fourth meeting, in April 1952 the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM) / *International Commission for the Study and Improvement of Mathematics Teaching* was officially founded with an executive committee consisting of the mathematician Gustave Choquet (University of Paris) as president, the cognitive psychologist Jean Piaget (Universities of Genève and Paris) as vice-president and Gattegno (University of London) as secretary (Bernet & Jaquet, 1998). From the outset, the CIEAEM meetings had an international character with participants from eight European countries. The main goal of CIEAEM was not the actual preparation of a curriculum reform, but rather the study of learning processes, as a necessary step before any improvement in teaching methods or curriculum could be proposed. A recurrent topic of debate and investigation within the CIEAEM community during the 1950s was the use of concrete models, teaching materials and teaching aids. The work of CIEAEM did not originate in a concern for the bad state of mathematics education. Rather it focused on the improvement of the content of mathematics education, and to bring it more in line with current mathematical thinking as applied in many ways in modern science. The proposals were made with the most gifted mathematics students in mind. In this perspective the CIEAEM sought the collaboration from members of the Bourbaki group (Dieudonné, Choquet and Lichnerowicz) who directed their attention to the world of school mathematics. During the 1952 meeting at La Rochette par Melun, the Bourbakists set forth their views on the origin, meaning and "utility for discovery" of structures in modern mathematical science. Piaget, who participated in the meeting, explicitly related these mathematical structures to the mental operations through which a child interacts with the world. It is therefore within the CIEAEM community that the first conceptions of "modern mathematics" were thought through and formulated. The theoretical debates also led to practical reform strategies. In August 1958, in the margin of the 12th CIEAEM meeting in Saint Andrews (Scotland, UK), the Belgians Frédérique Lenger and Willy Servais compiled the draft of a concrete program for the teaching of modern mathematics, that was subsequently tested in two schools during the following school year, arguably the first attempt to teach "modern mathematics" in Europe (De Bock & Vanpaemel, 2018).

Compared to the American SMSG, CIEAEM was active on a much smaller scale, was not funded and not linked to any official body or government, and had no interest in implementing a grand scale reform. In spite of its impressive name, CIEAEM was and always remained a small informal group, an inside group with little outreach. Some of its members would take on leading roles in the subsequent New Math reform, but before 1958 this was not visible to outside observers. It is not likely that many American reformers were aware of CIEAEM during the 1950s. When in 1954 Howard Fehr and his colleague Myron F. Rosskopf, both from Teachers College, Columbia University New York, attended the International Congress of Mathematicians in Amsterdam, they urged their American colleagues to take notice of the reform movements going on in Europe. But at the same time they noticed the large differences in school systems.

> Naturally, European countries face in education many of the problems that we face. However, their attempted solutions are different. The problem of differences in ability and objective is met by having several types of schools rather than by having several programs in the same school, particularly at the secondary school level. At sessions of Section VII, on philosophy, history, and education, there were presented a series of papers on trends in mathematics education for students from age sixteen to twenty. It was apparent from these reports that the mathematics curriculum in the United States and the organization of the courses is quite different from any that exists in Europe. Part of the difference is due to a difference in philosophy

concerning education. We try to carry as many students as we can and as far as possible into mathematics. The Europeans try to separate by application of rigorous standards those who can do mathematics from those who cannot. (Rosskopf, 1955, p. 114)

In 1960 Fehr again observed that although "the subject matter covered [in European schools] is not vastly different from that in our schools", it is "in no case […] modern in the sense of the fine materials written by the School Mathematics Study Group and the University of Illinois School Mathematics Committee, or the program advocated by the Commission on Mathematics" (Fehr, 1960, p. 799). To Fehr, the European program of school mathematics appeared "relatively static". In all probability, Fehr did not know about CIEAEM or at least considered it of no importance. Conversely, there are little or no indications that European reformers were aware of what happened in the United States. Probably, CIEAEM did not feel the need to inquire about American developments, as they could build directly on the personal involvement of several Bourbaki mathematicians and representatives of so many European states. Georges Papy later wrote that the Lenger-Servais program "was certainly influenced by the work of the International Commission for the Improvement of Mathematical Instruction and by the work of Northrop and others at the University of Chicago" (Papy, 1966, p. 180).The statement is misleading. Papy probably hints at the Chicago mathematician Eugene Northrop, but in a later article (in which the same sentence is included) Frédérique Lenger – then Mrs. Papy – refers in a footnote to the Yale philosopher F. R. S. Northrop and his book *The logic of the sciences and the humanities* (New York, 1948) (Papy, 1968, p. 26). As the statement is published in papers by both Georges Papy and Frédérique Lenger, there may have been some truth in pointing to the American influence. On the other hand, Papyand Lenger may well have exaggerated the influence of the Americans. In 1966 Papy had taken a leading position in the European New Math reform and he may have wished to emphasize the international nature of the reform. As it stands, Papy'sand Lenger's statement is about the only evidence we found on any American influence on the European mathematicians and teachers working out the New Math curriculum in CIEAEM. We tend to believe that the interaction between American and European actors in the field of school mathematics was limited before the famous Royaumont Seminar of 1959.

3 International dissemination

The first real contact between European and American reformers was the Royaumont Seminar on New Thinking in School Mathematics, organized from November 23 to December 5, 1959 by the OEEC, as part of a larger series of similar seminars for all the sciences. The Seminar was attended by 46 participants from 18 countries, including the United States and Canada. The American delegation, consisting of Marshall Stone, Albert Tucker, Howard Fehr and Edward Begle – all of them actively involved in the ongoing New Math reform in the US –, did make a great impression on the European audience. Europeans noted the differences of the American school system with the situation in their own country, which made the American experience less relevant to their own concerns, but they were also impressed by the strong financial support of the reform movement, the widespread media campaign and even the choice of a catchy slogan, the "New Math", to further the case of reform. To underscore this point symbolically, the proceedings of the

Seminar with the official title *New thinking in school mathematics*, were translated in French with the "American" title *Mathématiques nouvelles*. But in general, the differences between the European and American reform efforts were often more obvious than the similarities.

After Royaumont, Americans, in particular Stone and Fehr, with the support from organizations as OECD, UNESCO and ICMI, started to dominate international conferences on school mathematics. It is doubtful, however, whether this had any influence on the New Math reform in terms of bringing the two traditions closer together. For all their international exposure, Stone and Fehr seem to have played only minor roles in the American New Math reform. Fehr repeatedly reported on international developments to his American fellowmen, as he was aware that American teachers had little idea of what was going on in Europe. "It is the hope of the writer", he wrote in 1965, "that the survey presented herein will serve to make the teachers in the US aware of the fact that their colleagues in foreign lands are just as concerned as they are with modernizing the mathematics curriculum" (Fehr, 1965, p. 44). None of the standard monographs on the history of New Math in the US make more than passing mention of events in Europe. Some minor reform programs may have been influenced by European ideas, such as the Secondary School Mathematics Study Group of Howard Fehr, who always remained critical towards the SMSG approach, the Madison Project of Robert Davis, which focused on the use of concrete teaching materials, and the Comprehensive School Mathematics Program of Burt Kaufman, who solicited the assistance of several European experts (Choquet, Steiner, Råde), and also convinced Frédérique Lenger to join his team in 1974 (Hayden, 1981; Phillips, 2015). For both Davis and Kaufman it can be said that they came to know European colleagues through Howard Fehr's network. Outside of this network the US-Europe connection seems to have been weak.

From the perspective of Europe, the situation looks different. Without the benefit of well-established international collaborations, the conferences sponsored by OECD, UNESCO and ICMI retained all their importance. The presence of some American researchers on these occasions added credibility and authority to the meetings. But even then, specific references to American researchers or reform groups remained vague. The main ideas of New Math in Europe were either borrowed from the Bourbaki program, or based on the Klein paradigm of transformation geometries. All of this had been convincingly prepared and disseminated by members of CIEAEM. European reformers had no need for American ideas. Possibly, the US had a lead over Europe in the early production of textbooks but European textbooks were not lagging far behind. As most European countries only implemented the reform in the second half of the 1960s, there was enough time to write textbooks adapted to the local school system. In her dissertation, Nadimi Amiri (2017) has argued that New Math was imported to Luxembourg from the US, but the textbook made available to Luxembourg schools was the French manual published already in 1960 by Camille Bréard, with a preface by Lichnerowicz. In Belgium, Georges Papy and Frédérique Lenger started their series *Mathématique moderne* in 1963. Also in Spain, original Spanish manuals were published from 1962 onwards by a Commission for the Experiments on Teaching Modern Mathematics appointed by the Ministry of Education (Ausejo, 2010). In Iceland, an American textbook was used for a pilot project in the leading school of Reykjavik, but in 1966 an Icelandic textbook was written and adopted for the whole education system (Bjarnadóttir, 2006). Begle (1968) states that SMSG materials were translated and/or adapted in many countries, including Sweden, Turkey, Taiwan, Australia, Brazil and India. But on the whole, American textbooks were not very successful on the European market, and conversely, as far as we have been able to find out, there are no examples of European textbooks introduced on the American market.

The dissemination of the New Math on other continents calls for a different interpretation. Here many local reform movements took inspiration from either the US or the United Kingdom. Best known is the African Mathematics Program (AMP), commonly known as the Entebbe Project. Headed by W. T. Martin, chairman of the Department of Mathematics at M.I.T., the AMP organized in 1962 a mathematics workshop at Entebbe, Uganda, which hosted 54 participants representing 13 countries, including 24 educators from 11 English speaking African nations. The ensuing project, financed by the Ford Foundation and the US Agency for International Development, produced over 60 volumes, which were tried in Ethiopia, Ghana, Kenya, Liberia, Malawi, Nigeria, Sierra Leone, Tanzania, Uganda, and Zambia. According to Frank Swetz (1975), the whole project "was American dominated with the writing strongly influenced by advocates of SMSG. […] The result was a black faced version of SMSG mathematics" (p. 6). Yet, he continues, "to challenge the American 'menace,' two competing British-oriented writing groups were formed. The Joint Mathematics Project was begun in West Africa and the East African School Mathematics Project in East Africa; both projects emulated the British counterpart of SMSG, i.e., the School Mathematics Project (SMP)" (pp. 6–7). In a footnote, he adds, "Unfortunately, such an argument is common between British and American educators. Much to my later embarrassment, I found myself engulfed in such a controversy while teaching in Malaysia" (p. 6).

In 1978, Hans Freudenthal edited two issues of *Educations Studies in Mathematics* on the changes in mathematics education since the 1950s. Contributions to these issues included not only papers on countries like the Netherlands, France, Great Britain and the US, but also on less publicized countries such as Sri Lanka, the West Indies, Iran and Nigeria. No countries from South America were represented. The picture that emerges from these contributions is one of local appropriations of American or European (mostly British) examples, adapted to the national school systems and to the demand for mathematically trained professionals. Local mathematicians not only studied the New Math; they were also very aware of the criticism that the New Math reform had generated in the United States and Europe. In the case of India, the criticism led to a nuanced evaluation of the advantages and disadvantages of the New Math, e.g. the "fetish of set theory". The Association of Mathematics Teachers of India adopted the position that "there is no one New Mathematics but there are many versions of it. Some features of these are desirable e.g. emphasis on concepts, discovery approach, transformation geometry approach, programming, matrices etc. and use of modern teaching aids. These must be retained. Some other feature[s] of New Mathematics are undesirable. […] These must be given up" (Kapur, 1978).

Criticism of the Western New Math had surfaced already in the early years of the reform. M. A. B. Deakin from the Papua and New Guinea Institute of Technology wrote a scathing critique of the New Math reform in his country.

> American mathematics syllabuses are exported, particularly to developing countries. In some cases this is a conscious process, in that American consultants are called in,

> or in that a university in a developing country may enter a formal relationship with an American college or university. In other cases, the export is unconscious. An overworked and underqualified administrator in an emerging nation feels impelled to “keep up with modern developments overseas”, and copies almost verbatim a course designed for (say) high schools in upper New York State. […] It is therefore disconcerting to administrators of tertiary mathematics here to find that the secondary schools have adopted a rather faddish “New Math” syllabus, loaded with set theory and its associated jargon. (Deakin, 1971, pp. 1017–1018)

In general, it is difficult to find a united, coherent view of New Math. D’Ambrosio (1991) observed that “the basis of the movement in Brazil can be described as a concoction of ideas from around the world, a synthesis of which was done by Brazilian mathematics educators themselves. […] The ideas having most impact on the Brazilian curriculum were those of the School Mathematics Study Group, George and Frédérique Papy, Zoltan Dienes, Lucienne Félix and Caleb Gattegno. Each of the programs developed by them were based on different premises and assumptions as well as having very different focuses. The fact that these programs were combined, with little or no critical analysis, was quite detrimental to the Brazilian curriculum, for it generated a curriculum based on inconsistencies of various kinds” (p. 71). De Carvalho (2014) suggests that the strong investment of the United States in Latin America for political reasons, was responsible for a preponderant American influence in mathematics education reform.

> The fear that Latin America would “go communist” had, as a result, considerable American investments in the region. As part of the “Alliance for Progress” (launched by President John F. Kennedy in 1961) or direct foreign aid programs, several cooperation agreements in the educational area were signed between the United States and specific Latin American countries. This helps to explain why, in Latin America, even though the European contribution to the new math movement was known, the major influence was American. For example, many publications of the School Mathematics Study Group (SMSG) were translated into Portuguese, including its complete secondary school mathematics textbooks, while non-American teaching materials were much less used. Exceptions were the use of Papy’s textbooks in a very prestigious school in Rio de Janeiro and the translation of a textbook of the School Mathematics Project – from England – into Spanish in Venezuela. This textbook did not follow Venezuela’s official mathematics curriculum, however, and had scant influence. (de Carvalho, 2014, p. 353)

From the examples to be found in the literature, it becomes clear that, although many countries indicate the great influence of the American textbooks and teaching material, the New Math movement was not regarded as unified or unequivocal. There was a healthy amount of criticism and a large measure of freedom to adapt the New Math message to local requirements. Many features were held in common between the main protagonists, but most of the actors were well aware of the differences between the various reform traditions. So to what extent can the New Math reform be regarded as an international phenomenon?

4 An international phenomenon?

The foregoing analysis testifies to the conclusion made by J. Kilpatrick (1997) that "all educational reform is local" (n.p.). On a more sophisticated analysis, however, the international dimension of the reform movement, which established itself after Royaumont, had important and perhaps unanticipated consequences for the implementation of the reform, and in particular for the actors involved. According to Bob Moon, the many conferences implementing, discussing and criticizing the Royaumont Seminar produced a professionalization of experts on school mathematics. This in turn, legitimized and strengthened the position of reformers in their own country. "Being international" had a positive impact on the status of national experts. Although Moon based his analysis on the implementation of New Math in primary education, his observations may have a more general validity. In the face of an international consensus (or so it was construed), governments left the details of the reform and its critical evaluation to the new group of experts, who could boast international experience. This resulted in a rapid institutional development of research institutes, teacher training programs and mathematics education centers, fostered by a general feeling in many countries "not to be left behind". From the middle of the 1960s, Bob Moon concludes, "reform became, and was initiated through, an international rather than national debate" (Moon, 1986, p. 199).If this is indeed so, one may wonder at who benefited most: University mathematicians who were already well organized on an international level, or mathematics teachers who were bound to local school systems with little chance of being heard in international conferences? Moon observes that "one interest group appears to have been particularly influential, in the early years of reform. The impact of university mathematicians, notably these advocating a "Bourbakist" reform of the school curriculum, is demonstrated in each country" (Moon, 1986, p. 216). He adds, however, that "despite all the investment of time and energy and political activity, many of the ideas advanced by those from the university world failed to become established and […] a markedly different climate existed two decades after Royaumont".

As indicated by many authors, the New Math reform was not a single, well-oriented and coherent movement but rather an amalgam of proposals and projects. Before the Royaumont Seminar, there was little interaction on an international scale. Both in the US and in Europe small, informal groups were working on some ideas of curriculum reform (often called "curriculum improvement"), but from different angles and with varying degrees of success. After Royaumont, some international collaboration between Europe and the US took shape, but its direct influence was limited. On a symbolic level, however, the international dimension (at least in Europe) favored those mathematicians who had access to the network of European and American reformers. This may have been particularly important in smaller countries, whereas it did not make so much difference in countries like the US, the UK and France. As de Carvalho (2014) concludes for Latin America: "The modern math movement had varying degrees of success in Latin American countries. Perhaps its most important result was fostering the creation and development of the community of Latin American mathematics educators" (p. 355).

The New Math reform proved to be a failure in the US. It did not survive the 1970s. Also in European countries, many of the new elements that had been introduced were either abolished or adapted. Yet, the institutional basis of mathematics education studies seemed to have survived the storm of criticism in most countries. Could it be that the same international dimension which was responsible for the amount of independence that mathematics reformers obtained in the 1960s was also instrumental in establishing mathematics education studies as an independent field?

REFERENCES

Ausejo, E. (2010).The introduction of "modern mathematics" in secondary education in Spain (1954–1970). *The International Journal for the History of Mathematics Education, 5*(2), 1–14.

Begle, E. G. (1968). SMSG: The first decade. *The Mathematics Teacher, 61*(3), 239–245.

Bernet, T., & Jaquet, F. (1998). *La CIEAEM au travers de ses 50 premières rencontres* [The CIEAEM throughits first 50 meetings]. Neuchâtel: CIEAEM.

Bjarnadóttir, K. (2006). From isolation and stagnation to "modern"mathematics in Iceland: Areform or confusion? *PaedagogicaHistorica: International Journal of the History of Education, 42*(4-5), 547–558.

D'Ambrosio, B.S. (1991). The modern mathematics reform movement in Brazil and its consequences for Brazilian mathematics education. *Educational Studies in Mathematics, 22*(1), 69–85.

Davis, R. B. (2003). Changing school mathematics. In G. M. A. Stanic & J. Kilpatrick (Eds.), *A history of school mathematics* (Vol. 1, pp. 623–646). Reston, VA: National Council of Teachers of Mathematics.

Deakin, M. A. B. (1971). Mathematics curricula for developing countries. *The American Mathematical Monthly, 78*(9), 1017–1019.

De Bock D., & Vanpaemel G. (2018). Early experiments with modern mathematics in Belgium. Advanced mathematics taught from childhood? In F. Furinghetti, & A. Karp (Eds.), *Researching the history of mathematics education* (ICME-13 Monographs) (pp. 61–77). New York: Springer.

de Carvalho, J. B. P. (2014). Mathematics education in Latin America. In A. Karp & G. Schubring (Eds.), *Handbook on the history of mathematics education* (pp. 335–359). New York: Springer.

Fehr, H. F. (1960). What mathematics is taught in European schools. *The American Mathematical Monthly, 67*(8), 797–802.

Fehr, H. F. (1965). Reform of mathematics education around the world. *The Mathematics Teacher, 58*(1), 37–44.

Hayden, R. W. (1981). *A history of the "new math" movement in the United States.* (Dissertation Iowa State University). Iowa State University Digital Repository. Retrospective Theses and Dissertations. 7427. http://lib.dr.iastate.edu/rtd/7427. Accessed: 11 July 2018.

Kapur, J. (1978). Change in mathematics education since the late 1950's: Ideas and realization.India. *Educational Studies in Mathematics, 9*(2), 245–253.

Kilpatrick, J. (1997). *Five lessons from the new math era. Reflecting on Sputnik: Linking the past, present, and future of educational reform*. (Unpublished paper). http://www.nas.edu/sputnik/kilpatri.doc. Accessed: 10 October 2018.

Kilpatrick, J. (2012). The new math as an international phenomenon. *ZDM–The International Journal on Mathematics Education, 44,* 563–571.

Moon, B. (1986). *The "New Maths" curriculum controversy: An international story.* London: The Falmer Press.

Nadimi Amiri, S. (2017). *Historicizing the journey of the New Math reform from the United States to Luxembourg in the 1960s and 1970s*. (Unpublished doctoral dissertation).The Faculty of Language and Literature, Humanities, Arts and Education, University of Luxembourg, Luxembourg.

Ohuche, R. O. (1978). Change in mathematics education since the late 1950's: Ideas and realization. Nigeria. *Educational Studies in Mathematics, 9*(3), 271–281.

Papy, F. (1968). Une expérience pédagogique de dix ans en Belgique [A pedagogical experiment of ten years in Belgium]. In G. Papy (in collaboration with P. R. Burgraeve, R. Holvoet, F. Papy, & A. Terfve)

(Ed.), *Arlon 10* (pp. 25–89). Brussels: Centre Belge de Pédagogie de la Mathématique.

Papy, G. (1966). The state of reform of mathematics instruction in Belgium, 1966. In H. F. Fehr (Ed.), *Mathematical education in the Americas II: A report of the second inter-American conference on mathematical education, Lima, Peru, December 4-12, 1966* (pp. 179–198). New York: Bureau of Publications, Teachers' College, Columbia University.

Phillips, C. J. (2015). *The New Math: A political history*. Chicago: University of Chicago Press.

Redman, E. T. H. (2013). *The National Science Foundation and a comparative study of precollege mathematics and science education reform in the United States, 1950-2000.* (Unpublished doctoral dissertation).University of California, Berkeley, US. https://cloudfront.escholarship.org/dist/prd/content/qt56c2m73k/qt56c2m73k.pdf. Accessed: 13 August 2018.

Rosskopf, M. F. (1955). Some reflections on a hurried trip to Europe. *The Mathematics Teacher, 48*(2), 113–114.

Swetz, F. J. (1975). Mathematics curricular reform in less-developed nations: An issue of concern. *The Journal of Developing Areas, 10*(1), 3–14.

History of mathematics in the Nordic countries

THE FIRST NORWEGIAN TEXTBOOKS IN MATHEMATICS

A story of independence and controversy

Andreas CHRISTIANSEN
Western Norway University of Applied Sciences,
Klingenbergvegen 4, 5414 Stord, NORWAY
e-mail: andreas.christiansen@hsh.no

ABSTRACT

Norway got its Constitution in 1814 after being subject to Denmark for 400 years. As a result of the Napoleonic wars, we had to go into a union with Sweden that lasted until 1905. However, the years after 1814 were a time for national awakening, where Norway had to rely on its own resources.

The first Norwegian university was functional from 1813, and there were learned schools that prepared students for the university. These schools had mostly been using Danish textbooks, but in 1825, Bernt Michael Holmboe wrote the first Norwegian textbook in mathematics for the learned schools. Holmboe wrote textbooks in arithmetic, geometry, stereometry and trigonometry, and most of them came in several editions.

Holmboe became a very influential person in the development of school policy in Norway, and he was a close friend of the mathematician Niels Henrik Abel. The learned community was small in Norway in the beginning of the 19th century, but that did not prevent Holmboe's textbooks from meeting opposition. His textbook in geometry caused a long and bitter controversy with the only other Norwegian professor in mathematics at that time.

1 Introduction

Towards the end of the 18th century, a great effort was done to establish mathematics as a school subject in the higher education in Norway, and a school reform that was introduced around year 1800 lead to proper teaching in mathematics. The mathematical community in Norway at that time was small, and all the participants did necessarily become significant members of the society. This was a time of considerable development in the subject of mathematics, and this also influenced the debate about mathematics education.

My focus in this paper is the textbooks in mathematics written by Bernt Michael Holmboe (1795–1850), and my motivation is to look at the development of mathematics education and the didactic debate in the early 19th century, and what significance Holmboe played. The topics in school textbooks for the learned schools during this period were arithmetic and algebra, geometry, trigonometry and stereometry, and there was a growing demand for rigour in mathematics. An intuitive understanding of numbers was, as an example, not sufficient anymore, and this reflected on the textbooks. The teaching of mathematics was also a motivation for this demand for rigour.

I will in this paper try to understand these textbooks based on the historical period, and the political and social conditions in which they were written. I will describe the content of the textbooks, I will illuminate issues like how updated they where with regards to the development of mathematics, where the inspiration came from, and a relevant question will be what we may learn today from these textbooks and the didactical debate in early 19th century.

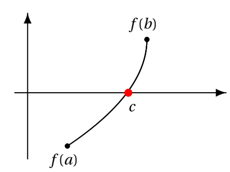
Issues addressed in this paper, and earlier research about the textbooks of Bernt Michael Holmboe, may be found in Christiansen (2009, 2010, 2012a,b, 2015b,a) and in Bjarnadóttir et al. (2013). All translations from Norwegian and Danish-Norwegian to English are made by the author.

1.1 The demand for rigour – an example

An intuitive and geometrical interpretation of real numbers did not satisfy the 19th century mathematicians' demand for purity in methodology, and the need for a new understanding of real numbers did arise in connection with the proof of the *intermediate value theorem*. This theorem was proven by Bolzano (1817), and both continuity of functions and convergence of infinite series are defined and used correctly in this paper, as it is understood in a modern sense. Bolzano's comprehensive paper was later translated into English (Russ 1980, 2004), and the latest of these two translations forms the basis of the following short description.

According to Russ (1980: 157), Bolzano's paper includes "*the criterion for the (pointwise) convergence of an infinite series, although the proof of its sufficiency, prior to any definition or construction of the real numbers, is inevitably inadequate*". This criterion is, however, *not* concerning the definition of convergence, but the Bolzano-Weierstrass theorem, which states that every bounded sequence has a convergent subsequence.

Bolzano's only strict requirement is "*that examples never be put forward instead of* proofs *and that the essence* [Wesenheit] *of a deduction never be based on the merely figurative use of phrases or on associated ideas, so that the deduction itself becomes void as soon as these are changed*"[1] (Russ 2004: 256).



In modern words, Bolzano's explains his point by considering two formulations. First we have a geometric theorem in intuitive form, which states that if a continuous line, where the ordinates are first negative and then positive – or conversely – then this line must necessarily intersect the axis of abscissas somewhere between those ordinates. Next we have a purely analytical formula – the intermediate value theorem – which states that if we have a function f, continuous on $[a, b]$ such that $f(a) < 0$, $f(b) > 0$ then it follows that $\exists c \in (a, b)$ such that $f(c) = 0$. Bolzano then states that we *cannot* conclude the intermediate value theorem from the geometric theorem in intuitive form. We may, on the contrary, conclude the mentioned geometric theorem from the intermediate value theorem. (Russ 2004: 254–255)

2 A short historical background

Some historical events had a significant impact on how Norway developed as a nation.

The black plague played a very significant role in the development of the Norwegian society. It reached Norway in the summer of 1349, and it is believed that it killed 2/3 of the country's population. This was one of several reasons why Norway entered a union with Denmark in 1380, a union that throughout the ages had many different forms. The Danish era lasted 434 years, until 1814.

[1]Bolzano also criticised Gauss's original proof of the fundamental theorem of algebra of 1799, because Gauss here used geometrical considerations to prove an algebraic theorem (Otte 2009: 53). Bolzano did not doubt the validity of the theorem, but he criticised the "impurity" of the method.

The Reformation started in Germany in 1517, and was literally forced on Norway by the Danish King in 1536. Norway had been in a *union* with Denmark, sharing the same king, but in 1536 the Norwegian *National Council* [Riksråd] was dissolved. This meant that Norway was no longer a nation, but became a part of Denmark, and this had of course a profound importance on the development of the country. The reformation had, however, a positive effect on the development of the educational system in Norway.

Much happened in Norway in and around 1814. Denmark-Norway was on the losing side of the Napoleonic Wars, and at the peace treaty in Kiel in January 1814, it was agreed between the Danish King and the Swedish Crown Prince that Norway should enter into a union with Sweden. In Norway there was great dissatisfaction with this agreement, and representatives of the Norwegian elite gathered at Eidsvoll in February in an attempt to take control of country's own destiny. It was agreed that the provision in Kiel should be rejected and that Norway should be declared an independent state. The Norwegian Constitution was drafted and was signed on May 17, 1814. On the same day, a Danish Prince, Christian Fredrik, was chosen as king. As a reaction to this, the Swedish Crown Prince went to war against Norway in the summer of 1814, a war that lasted no more than three weeks. The new king had to abdicate and left the country in October. Norway had to adjust the Constitution, and went into a union with Sweden that lasted till 1905.
(Kunnskapsforlaget 2006)

3 Educational system

Since the Middle Ages, there have been schools for the children of the elite. There was a growing religious pietism in Norway after the Reformation, and in the 18th century there was a demand to educate children of common people, mainly to make them able to read the Bible.

3.1 Schools for regular people

In the 1730s there were several laws that regulated children's confirmation in the church, and the teaching that led to it. Schools for common people were introduced where children could learn Christianity, reading, writing and calculations. Schools were established in cities and towns, but in the rural areas the situation was a bit different.

The system that developed in rural areas is something we call "*omgangsskole*", where the teacher moved from area to area, stayed at one farm, and taught all the children in the neighbourhood. After a while, the teacher moved on. In a year, the children could get around two months of schooling.
(Kunnskapsforlaget 2006)

3.2 Learned schools

Cathedral schools [Kathedralskoler] were schools from the medieval time that were connected to cathedrals, and were meant to give a theologic education to the future priests.

Table 1: Some Cathedral schools from the Middle Ages

SCHOOL	ESTABLISHED	REMARKS
Oslo	1152	
Bergen	1152	
Trondhjem	1152	
Hamar	1152	Merged with *Oslo Kathedralskole* in 1602. Re-esablished in Hamar in 1876
Stavanger	13th century	Moved to Christiansand in 1686. Stavanger had a smaller latin school that was closed in 1739, and re-established in 1824
Christiansand	1686	

All cathedral schools were turned into *Latin schools*, or *grammar schools* [latinskoler], when the reformation was introduced in Norway in 1536, and it was mandatory for every town to have a one. The Latin schools, together with the old cathedral schools, constituted the so-called *learned schools*. Some of the schools did, however, keep their old names. Many of these Latin schools were of a poor quality, so in reality, the higher education preceding the university in 1814 was only *four* cathedral schools.

Table 2: Some learned schools after 1814

SCHOOL	ESTABLISHED
Christiania Cathedralskole	1152
Bergens Cathedralskole	1152
Trondhjems Cathedralskole	1152
Christiansands Cathedralskole	1686
Drammens Latinskole	1817
Fredrikshalds Latinskole	1822
Skiens Latinskole	1822
Stavangers Latinskole	1824

By a governmental decree in 1809, the pupils started at the learned schools at the age of 9 or 10 years, and the duration was normally eight years consisting of four two-year grades, and each day at school was seven hours – four before noon and three after. The *university qualifying examination* [examen artium] were arranged by the university.

The learned schools gave a classic education, and a higher education in scientific subjects at the same standard as the learned schools could be achieved at the *Military Academy*.[2] This school admitted pupils from the age of 12–14.

Several *intermediate schools* [middelskoler] were established in smaller towns after 1814, and they were learned schools without the upper two-year grade.
(Andersen 1914; Kunnskapsforlaget 2006)

[2] Den frie mathematiske Skole (1750–1798), Det norske militære Institut(1798–1804), Det kongelige Norske Landcadetcorps' skole (1804–1820) and Den Kongelige Norske Krigsskole after 1820.

The kingdom Denmark-Norway introduced a new school reform around the year 1800 which in many ways strengthened the position of the discipline of mathematics, and from now on may we talk about proper teaching in mathematics in the higher education (Piene 1937). There was much work done in the last decades of the 18th century to reintroduce mathematics as a subject in school.

3.3 University

The first Norwegian University was established in Christiania in 1811, and came into function in 1813. The name was *The Royal Fredriks University* [Det Kongelige Frederiks Universitet (Universitas Regia Fredericiana)], named after the Danish King Fredrik the 6th, and it was changed to *The University of Oslo* [Universitetet i Oslo] in 1939. The only use of mathematics the first years was for the *examen philologico-philosophicum* – a preparatory exam for other subjects. The lectures in mathematics were on trigonometry, stereometry, basic algebra, and later applied mathematics after Christopher Hansteen's appointment. (Holst 1911; Kunnskapsforlaget 2006)

3.4 Niels Treschow (1751–1833)

The philosopher and politician Niels Treschow was an important person in the learned society of Norway, and he was professor in philosophy at the new university at the time when Holmboe was a student. Treschow published several books, among them a textbook in *Common Logic* (Treschow 1813), clearly influenced by Immanuel Kant. Treschow has a rigorous classification of statements – in *direct* and *indirect* statements, and in *analytic* and *synthetic* statements – based on the relation between subject and predicate in the statements, and on the nature of the statements (Treschow 1813: 161–162). Exactly the same classification is described in the introduction chapter of Holmboe's textbook in arithmetic (Holmboe 1825: 1–3).

A *statement* [sætning] is connection of two concepts. The first thought of, in the connected concepts in a statement is called *subject* [Subject], and the second is called *predicate* [Prædicat]. Statements are called *direct* [umiddelbar] if one perceives the subject's connection to the predicate *without* regarding other statements, and *indirect* [middelbar] if one uses other statements when considering the connection between subject and predicate. In the latter case, the indirect statement will then be a consequence of the statements used to elucidate it. The presentation of the conclusions used to elucidate the subject's connection to the predicate in indirect statements is called a *proof* [Beviis] (Holmboe 1825: 1–2).

The statements are divided in *synthetic* or *practical* statements, expressing that a connection between concepts shall be made, and *analytic* or *theoretic* statements, expressing a connection between concepts that already exists. Both synthetic and analytic statements may be direct as well as indirect, and Holmboe is therefore introducing four types of statements (Holmboe 1825: 2–3):[3]

Direct synthetic statements which he calls *Postulates* [Fordringssætninger]. A postulate expresses that two concepts shall be connected.

Direct analytic statements which he calls *Fundamental Statements* [Grundsætninger] or *Axioms*. An axiom expresses the relations between two connected concepts.

[3] All these definitions are derived from Treschow (1813: 161pp), but it seems like Holmboe has introduced the synonyms *practical* for *synthetic*, and *theoretic* for *analytic*.

Indirect synthetic statements which he calls *Problems* [Opgaver]. A problem expresses that two concepts shall be connected using already existing connections.

Indirect analytical statements which he calls *Theorems* [Læresætninger]. A theorem expresses a connection between two concepts which is proven to be a consequence of preceding connections.

There are in Holmboe's textbook in arithmetic 10 direct statements called *Fundamental Statements*, there are 27 indirect synthetic statements, or problems, and there are 30 theorems and 163 corollaries. The preface of Holmboe (1825) may seem unnecessarily complicated and difficult for a textbook meant for young students, but Holmboe clearly states that his textbook is not meant for self-study, but requires a skilled teacher.

4 Bernt Michael Holmboe (1795–1850)

Bernt Michael Holmboe was born on the 23rd of March 1795 in Vang in Valdres, centrally situated in Southern Norway, and he died on the 28th of March 1850, at the age of 55 years and 5 days. Holmboe was a teacher at Christiania Kathedralskole from 1818 till 1826, and after that he was lecturer at the university until 1834, when he was appointed professor in mathematics, a position he had until his untimely death. Holmboe's home burnt down shortly after his death, and some of his works and letters were lost, in addition to some of the works and letters from Niels Henrik Abel (1802–1829). (Bjerknes 1925: 56,79)

Among Holmboe's students we find mathematicians like Niels Henrik Abel, Ole Jacob Broch and Carl Anton Bjerknes. Holmboe proclaimed that in no other subject did novices complain more than in mathematics (Piene 1937), and his aim was to make the students familiar with mathematical signs *before* a more methodical study. He further stated that unless pupils engaged in "uninterrupted practice" ["idelig øvelse"] even persons with several years of education would find that mathematics is "something of a mind consuming and boring matter" ["noget åndsfortærende og kjedsommelig tøi"]. The lecture notes of Carl Anton Bjerknes shows, however, that Holmboe's teaching was characterized by *pre-abelian* times, in spite of his knowledge of Abel and his works (Bjerknes 1925).

Holmboe wrote in a letter to the then 24 years old Carl Anton Bjerknes (Holmboe 1849) that he is inspired by the great French mathematician Joseph-Louis Lagrange (1736–1813). Bjerknes had asked Holmboe to advise him about studies in mathematics, and Holmboe wrote "*The best I have to state in this respect is to inform you about some notes from Lagrange and some rules and remarks by him regarding the study of mathematics, which I found in Lindemanns and Bohnebergers* Zeitschrift für Astronomie *about 30 years ago ... Those who really want, should read Euler, because in his works all is clear, well said, well calculated, because there is an abundance of good examples, and because one should always study the sources*".

4.1 Holmboe's textbooks

The following table shows an overview over Holmboe's textbooks, their Norwegian titles and their various editions.

Table 3: An Overview over Holmboe's textbooks

TITLE	EDITION	YEAR	EDITED BY	PUBLISHER
Lærebog i Mathematiken	1st	1825		Jacob Lehmann
Første Deel, Inneholdende Indledning	2nd	1844		J. Lehmanns Enke
til Mathematiken samt Begyndelses-	3rd	1850		J. Chr. Abelsted
grundene til Arithmetiken	4th	1855		R. Hviids Enke
	5th	1860		R. Hviids Enke
Lærebog i Mathematiken	1st	1827		Jacob Lehmann
Anden Deel, Inneholdende	2nd	1833		Jacob C. Abelsted
Begyndelsesgrundene til Geometrien	3rd	1851	Jens Odén	R. Hviids Enke
	4th	1857	Jens Odén	J. W. Cappelen
Stereometrie	1st	1833		C. L. Rosbaum
	2nd	1859	C. A. Bjerknes	J. Chr. Abelsted
Plan og sphærisk Trigonometrie	1st	1834		C. L. Rosbaum
Lærebog i den høiere Mathematik	1st	1849		Chr. Grøndahl
Første Deel				

As a general rule throughout the first four textbooks, theorems and proofs have this structure, as described in Holmboe (1825: 3):

Theorem [Læresætning *or* Theorem] — A verbal description without any mathematical notation.

Condition [Betingelse *or* Hypothesis] — A small number of theorems have conditions set in algebraic notation. The condition is a theorem showing a preceding connection.

Algebraic statement [Sats *or* Thesis] — The theorem represented in pure algebraic notation.

Proof [Beviis] — A regular proof of the theorem using algebraic notation and/or written text.

A theorem that is a direct consequence of a preceding theorem, is called a **Corollary** [Tillæg], or a supplement. Some of the *corollaries* are followed by a proof, but I have not found any instances where a corollary is written as an algebraic statement.

4.2 Arithmetic

Volume one of Holmboe's textbook in mathematics contains the introduction to arithmetic and algebra. The book came in a total of five editions from 1825 through 1860, but only the first three (1825; 1844; 1850) were published in Holmboe's lifetime, and these three editions are the attention of this description.

In the first three editions, there is an interesting development in the definition and use of irrational numbers, which is in accordance with the development of mathematical analysis.

Table 4: Definitions of irrational numbers

Edition	Definition
1825	Any number, that cannot be expressed either as a whole number or as a fraction, whose numerator and denominator are whole and finite numbers, is called an *irrational* number.
1844 1850	Any magnitude, that cannot be expressed either as a whole number or as a fraction, whose numerator and denominator are whole and finite numbers, but whose value always falls between two fractions $\frac{t}{n}$ and $\frac{t+1}{n}$, where t and n are whole numbers, and where one can make n larger than any given number, is called an *irrational* number.

The definition from 1825 tells us what an irrational number *isn't*, it doesn't tell us what it *is*. Opposed to irrational numbers, all whole numbers and fractions, whose numerator and denominator are whole and finite numbers, are called *rational* numbers. The definition of 1844 and 1850 may indicate an influence by Bernard Bolzano and his definition of *measurable numbers* (Russ 2004: 347–349,360–361), and Holmboe now specifies *magnitudes*, and not numbers.

The following table illustrates some of the use of irrational numbers in corollaries and in the definition of the sum where at least one of the addends is irrational.

Table 5: Use of irrational numbers

Edition	Corollary / Definition
1825	One can always find a rational number, whose value approaches the value of a given irrational root, so that the difference between them is less than any given unit fraction.
1844 1850	If two irrational, [a] positive magnitudes P and Q, both independent of n and between boundaries of the form r and $r+\frac{a}{n}$, are in such a way that $r < P < r+\frac{a}{n}$ and $r < Q < r+\frac{a}{n}$, where one can make n larger than any given number and a is finite. Then $P = Q$.
1850	If one or both of two magnitudes, x and y, are irrational, and where $\frac{t}{n} \leq x < \frac{t+1}{n}$ and $\frac{p}{n} \leq y < \frac{p+1}{n}$, and one can make n larger than any given number. The sum $x+y$ is then to be understood as the common boundary for the sums $\frac{t}{n}+\frac{p}{n}$ and $\frac{t+1}{n}+\frac{p+1}{n}$, whose difference is $\frac{2}{n}$, which disappears when n grows infinitely.

[a] In the 1850 edition, the specification of *irrational* is taken out. The statement is now valid for *real* numbers or magnitudes – rational and irrational.

Bernard Bolzano introduces *measurable numbers* in his *Pure Theory of Numbers* ["Reine Zahlenlehre"][4] (Russ 2004: 347–49, 360–61). According to the definition, S is *measurable*

[4] Was not published until late 20th century (Russ 2004: 681). Bolzano wrote in a letter, dated 5th of April, 1835, to one of his former students, that he had "*one book near completion with the title* Pure Theory of Numbers *consisting of two volumes: an* Introduction to Mathematics*, the first concepts of the general theory of quantity, and then the* Theory of Numbers *itself*" (Russ 2004: 347).

when

$$\forall q \in \mathbb{N} \quad \exists p \in \mathbb{Z} \quad \text{such that} \quad S = \frac{p}{q} + p_1 = \frac{p+1}{q} - p_2 \quad \text{where} \quad p_1 \geq 0, \quad p_2 > 0$$

In other words

$$\frac{p}{q} \leq S < \frac{p+1}{q}$$

Bolzano explains that p_1 and p_2 denotes a pair of strictly positive number expressions, the former possibly denoting zero (Russ 2004: 361).[5]

The measurable number may be used to measure, or determine by approximation, the magnitude or quantity. Bolzano called the fraction $\frac{p}{q}$ the *measuring fraction*, and the fraction $\frac{p+1}{q}$ the *next greater fraction*. p_1 is called the *completion* of the measuring fraction since $S = \frac{p}{q} + p_1$. Every *rational number* is a measurable number where $p_1 = 0$, and indeed a complete measure.

Abel is the only known reference that Bolzano was known already in the 1820s, as he mentions Bolzano in his Paris notes (Schubring 1993: 45). Schubring (1993: 50) writes that during the four months Niels Henrik Abel stayed in Berlin in 1825, he was in close contact with August Leopold Crelle and his mathematical circle, where he was engaged in intensive conversations on all mathematical issues. Crelle had Bolzano's three booklets in his personal library, and Abel's reading of Bolzano was part of this process of communication.

4.3 Geometry

The textbook in basic geometry (Holmboe 1827) starts with several definitions of basic concepts. The very first definition describes *geometry* as a science about the *coherent magnitudes*. Coherent magnitudes are the space with all available dimensions and time. According to Solvang (2001), Holmboe's way of organizing the subject matter was influenced by Adrien-Marie Legendre's (1752–1833) introduction to geometry (Legendre 1819). The geometry of Legendre is constructed mainly the same way as Euclid, and starts with a long list of what he calls *explanations*, similar to what Euclid calls *definitions*.

The first definition in Legendre (1819) defines *geometry* as a *science which has for its objects the measure of extension. Extension has three dimensions, length, breadth, and thickness.* With reference to classification of coherent magnitudes in space and time, Holmboe classifies geometry in two parts:

1. *The real geometry* defined by the relations between the various magnitudes in space, without considering their changes in time.
2. *Mechanics*, defined by the changes the magnitudes goes through in time. All changes on a magnitude through time are called motion, and it is conditioned by force.

It is postulated that the space stretches indefinitely.[6]

[5] Bolzano denoted them p^1 and p^2, but the superscripts only distinguished, they where not powers (Russ 2004: 349). I have chosen to use subscripts to avoid confusion.

[6] Geometrie er en Videnskab om de sammenhængende Størrelser. Sammenhengende Størrelser er ere Rummet med enhver deri forekommende Udstrækning og Tiden. Med Hensyn til de sammenhængende Størrelsers Inddeling i Rum og Tid, inddeles Geometrien i 2 Dele. (1) Den egentlige Geometri, der bestemmer de i Rummet forekommende Størrelsers Forhold til hinanden uden Hensyn til deres Forandring i Tiden. (2) Mekanik, der bestemmer de Forandringer, som Størrelserne undergaae i Tiden. Anm. Enhver Forandring, som en Størrelse i Tiden undergaaer, kalles Bevægelse, hvis betingelse kaldes Kraft. Fordringssætning. Rummet maa tænkes udstrakt i det Uendelige. (Holmboe 1827: 1)

Holmboe advises the teacher to show moderation in the review of proofs, and to show examples using numbers before the examination of the proof. This practical advice contradicts the structure of his textbooks, which is strictly Euclidean. There are few exercises and numerical examples, and the notion of construction means to elucidate the concept, and not to use compass and ruler. Holmboe does not give any detailed instructions on how to use ruler and compass in this book, nor does he mention geometric locus, but writes about elucidative [anskueliggjørende] objects, magnitudes and concepts. His idea may be that the mathematics teaching shall educate the students with respect to formal logic, by encouraging them to think and conclude.

Holmboe's first definition is a description of a classification – any bounded part of the space is called a *body* [Legeme], the boundary of the body is called a *face* [Flade], the boundary of a face is called a *line* [Linie], and the boundary of a line is called a *point* [Punkt]. A body has three extensions, called *length* [Længde], *breadth* [Brede] and *height* [Høide], a face has two extensions, called *length* and *breadth*, a line has one extension, called *length*, and a point has no extensions. Holmboe then states without any explanation, but with an illustration,

> "*Fundamental concept. A straight line.*" [Grundbegreb. En ret Linie.]

A *curved line* [krum Linie] is a line of which no part is a straight line, and a *straight plane* [ret Flade] is a plane where one, between two arbitrary points may draw a straight line. The fundamental statements of the straight line is that a straight line may be prolonged infinitely, one may always draw one straight line between two points, and one may never draw more than one straight line between two points. The part of the straight line that lays between the two points is the shortest of all lines drawn between the points, and it is called the *distance* [Affstanden] between the points. (Holmboe 1827: 2–4)

Two of the chapters are called "*About two straight lines intersected by a transversal*",[7] and "*About parallel lines*".[8] The first of these chapters gives a thorough description of all pairs of angles this situation produces. This is followed by the consequences of two corresponding angles being equal, and vice versa, the situations which have the consequence that the corresponding angles are equal (Holmboe 1827: 11–16).

The chapter "*About parallel lines*" has a theorem with proof which states that when two straight lines are intersected by a transversal, such that an outside angle is equal to its corresponding interior angle, that is $\angle r = \angle p$ on the first figure on page 11, then the two straight lines cannot intersect no matter how far they are prolonged in both directions (Holmboe 1827: 45). The structure of the proof is that if the two lines cross on one side of the transversal, then the two lines and the transversal form a triangle, where $\angle r$ is an outside angle. Holmboe has already demonstrated that an outside angle of a triangle is always grater than any of its interior angles (Holmboe 1827: 34), so therefore $\angle r > \angle p$, which contradicts the condition.

This is followed by Holmboe's definition of *parallel lines*:

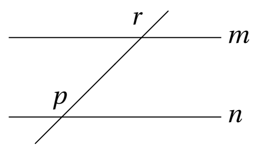
> "*Two straight lines in the same plane that do not intersect when prolonged indefinitely to both sides, are* parallel *to each other, or the one is parallel to the other*"[9] (Holmboe 1827: 46)

[7] Om to rette Linier, som overskjæres af en tredie

[8] Om parallele Linier

[9] To rette Linier i samme Plan, som til begge Sider forlængede i det Uendelige ikke skjære hinanden, siges at være *parallele* med hinanden, eller den ene at være parallel med den anden.

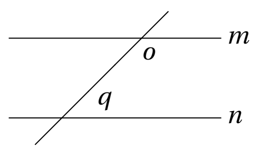
In two following theorems, using the same situation of two straight lines intersected by a transversal, he demonstrates first that if the outside angle is greater than the interior, $\angle r > \angle p$, then the two straight lines are *not* parallel. He next proves that *if* the two lines are parallel, then $\angle r = \angle p$. This last proof is done by assuming that $\angle r \neq \angle p$, and showing that the lines then are *not* parallel. (Holmboe 1827: 50)



$$\begin{aligned} \angle r = \angle p &\implies m \parallel n \\ \angle r > \angle p &\implies m \nparallel n \\ m \parallel n &\implies \angle r = \angle p \end{aligned}$$

Figure 1: Theorems about parallel lines

In a following corollary he then states that if two lines are parallel, and intersected by a transversal, then the sum of the two interior angles equals 2R. This is a consequence of the previous theorem that proves that $\angle r = \angle p$. This is followed by another corollary stating that if the sum of the two interior angles is *not* equal to 2R, then the two lines are *not* parallel. (Holmboe 1827: 51–52)



$$\begin{aligned} m \parallel n &\implies \angle o + \angle q = 2R \\ \angle o + \angle q \neq 2R &\implies m \nparallel n \end{aligned}$$

Figure 2: Corollaries about parallel lines

These two corollaries carry many characteristics of corresponding angles in the original text. It is the last one mentioned here that has the same wording as Euclid's parallel postulate, but it is not emphasized in any way. Holmboe is in his textbook very true to the ideas of the *Elements* in the way of introducing and presenting the subject matter, but without ever referring to or even mentioning Euclid.

Holmboe's textbook in geometry came in a total of four editions, but only the two first were published in Holmboe's lifetime. There are very few differences from the first edition to the second, and none concerning the concepts discussed in this paper.

4.4 Trigonometry

Most of Holmboe's textbooks came in several editions, but the textbook in plane and spherical trigonometry (Holmboe 1834: 3) was his only textbook in basic mathematics that came in one edition only, in 1834. Holmboe starts by defining *trigonometric lines* to an arc, and with this definition, lines in spherical geometry also have trigonometric lines. Trigonometric values calculated by ratios of sides in right angled triangles are results, deducted from the definitions, and presented in his textbook as a theorem with proof.

Holmboe defines *plane trigonometry* (Holmboe 1834: 3) to be that part of geometry that may be used for "solving the triangle". By this Holmboe means that there are six magnitudes in a triangle, three sides and three angles, and when a necessary and sufficient number of these six magnitudes are known to unambiguously define the triangle. The task is to find the remaining magnitudes. There is a similar definition where Holmboe defines *spheric trigonometry* (Holmboe 1834: 45) to be the task to unambiguously define a spherical triangle.

Trigonometric lines

"The dependencies between sides and angles in a triangle, or in general between straight lines and angles, or the circular arcs that measure the angles, may be expressed by certain lines, called trigonometric lines" (Holmboe 1834: 3). These lines are geometric objects shown on the constructions in figures 3 and 4, and the numerical values of their lengths are what a modern reader understand by sines, cosines, etc. With a current understanding of straight lines, we would say that the trigonometric lines were *line segments*. It was, however, not un-common in 19th century and earlier to use the word *line* (Linie or linje) for a line segment.

There is also another important definition that implies that Holmboe is assuming Euclidean geometry. §2 defines that if the sum of two angles equals 90°, they are called *complementary*, and if the sum of two angles equals 180°, they are called *supplementary*. A corollary to §2 states that each of the acute angles in a right angled triangle is the complement of the other acute angle, and that each of the angles in a right angled triangle is the supplement of the sum of the other two angles (Holmboe 1834: 3–4). This corollary refers to §44, with corollaries 1 and 2, in his textbook in geometry (Holmboe 1827: 55–56).

Figure 3 is presented in the textbook (Holmboe 1834: Final page, Figure 1). It is interesting to note that in Holmboe's figure, quadrant 1 is ACH, quadrant 2 is HCK, and so on – mirrored from what we use today.

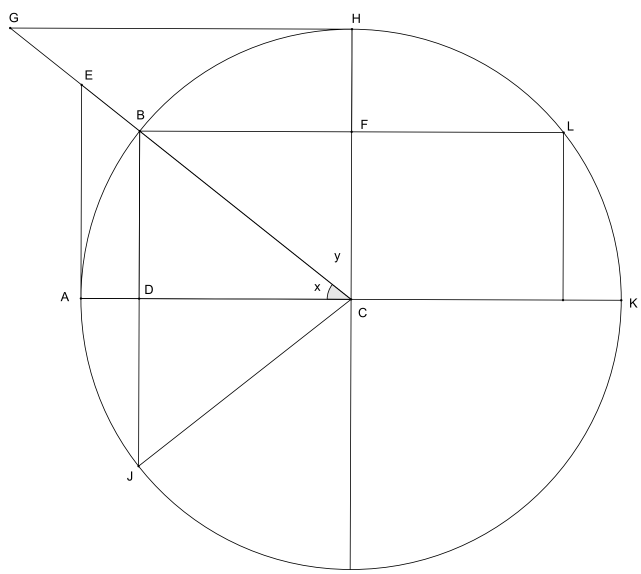


Figure 3: Construction of trigonometric lines

The trigonometric lines to the arc AB are:

Sine – BD The sine of an arc, or of the central angle when the radius is 1, is the perpendicular from the end point of the arc (B) to the diameter through the other end point of the arc (D).

Cosine – DC The cosine of an arc, or of the central angle when the radius is 1, is the sine of the complementary angle.

$BF = DC = \sin y = \cos x = DC$
A consequence of this is that $\cos x = \sin(90° - x)$

Tangent – AE The tangent of an arc, or of the central angle when the radius is 1, is the perpendicular on the diameter through one of the end points of the arc (A), to the point of intersection with the prolonged diameter through the other end point of the arc (E).

Cotangent – GH The cotangent of an arc, or of the central angle when the radius is 1, is the tangent of the complementary angle.
$GH = \tan y = \cot x$

Secant – EC The secant of an arc, or of the central angle when the radius is 1, is the distance between the centre (C), which is the angle vertex, and the end point of the tangent line of the arc, outside the periphery of the circle (E).

Cosecant – GC The cosecant of an arc, or of the central angle when the radius is 1, is the secant of the complementary angle. The distance between the centre (C) and the cotangent line outside the periphery of the circle (G).
$GH = \tan y = \cot x \implies GC = \sec y = \csc x$

Versed sine – AD The versed sine of an arc, or of the central angle when the radius is 1, equals $1 - \cos x$

Versed cosine – HF The versed cosine of an arc, or of the central angle when the radius is 1, equals $1 - \sin x$

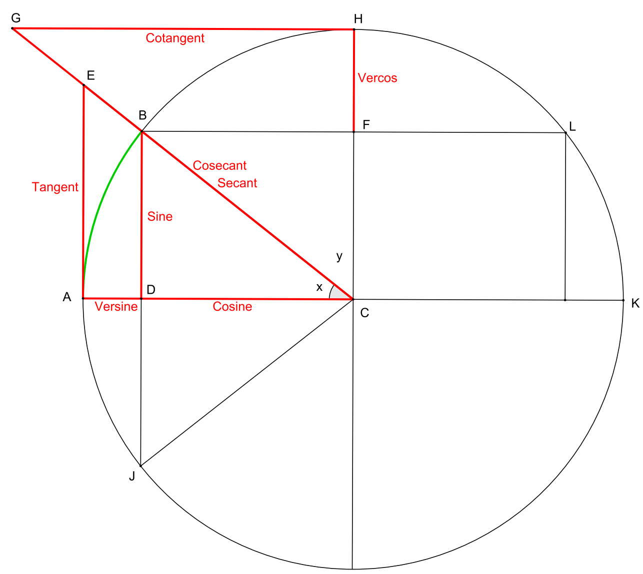


Figure 4: Trigonometric lines

Holmboe (1834: 24–25) demonstrates that $\sin x < AB < \tan x$, when $0 < x < \frac{\pi}{2}$, a point that is used later in calculating the trigonometric values. The sine line to a specific arc is half the chord of the double arc, and an arc is always longer that its chord, therefore $AB > \sin x$. The triangle CAE is greater than the segment CAB, therefore $AB < \tan x$.

Right-angled triangles

Holmboe states a theorem (Holmboe 1834: 15–16) where the trigonometric lines may be found as ratios between the sides in a right-angled triangle, that is the magnitude of one side divided by the magnitude of another side. In today's textbooks in trigonometry these ratios are the definitions of trigonometric values.

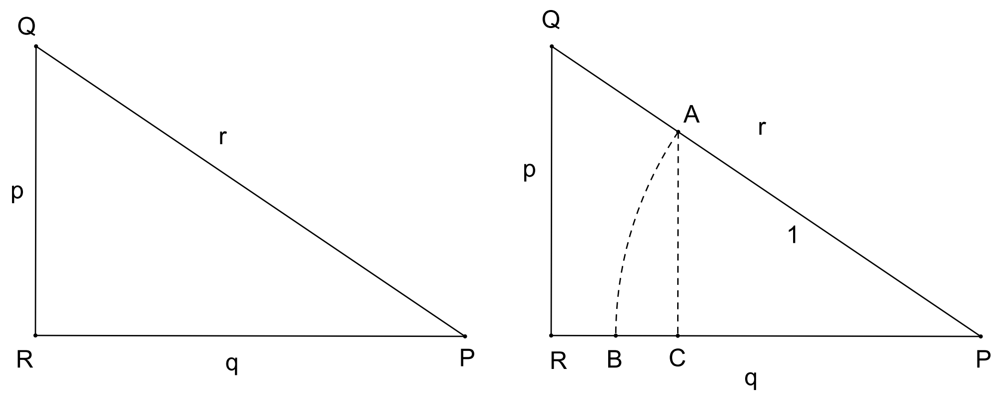


Figure 5: Trigonometry in right-angled triangles

Theorem 1
Consider the triangle △ PQR, ∠R is a right angle, and the sides are lower-case p, q, and r, as shown in Figure 5.

$$\begin{aligned} p/r &= \sin P, & q/r &= \cos P \\ p/q &= \tan P, & q/p &= \cot P \\ r/q &= \sec P, & r/p &= \csc P \end{aligned}$$

Proof With radius $PA = 1$, draw arc AB. Construct AC perpendicular on PB, as shown in Figure 5.

$$\implies \quad AC = \sin P \quad \text{and} \quad CP = \cos P$$

The ratios between the corresponding sides of the triangles $\triangle PQR$ and $\triangle PAC$ are the same, so we have

$$\begin{aligned} & r : 1 = p : \sin P \\ \implies \quad & \sin P = p/r \end{aligned}$$

and correspondingly for all the other trigonometric values. □

Calculation of trigonometric values

Holmboe (1834: 25–27) describes how to calculate trigonometric values with the precision required. In a unit circle, the length of the full periphery equals 2π, and the problem is to find a value for $\sin x$ with an arbitrary x. He has already established in a theorem that every arc x between 0 and $\pi/2$ is larger than its sine and smaller than its tangent (Holmboe

1834: 24–25), as shown in figure 4 on page 13. Holmboe states in a theorem (Holmboe 1834: 25–26) that for every arc x between 0 and $\pi/2$, the sine must be greater than $x/\sqrt{1+x^2}$. The proof for this is purely algebraic, but my geometric understanding of $x/\sqrt{1+x^2}$ is shown in Figure 6.

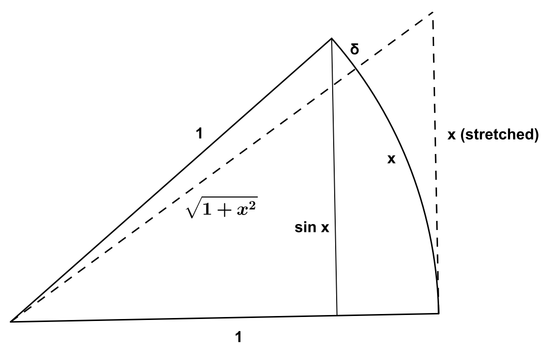


Figure 6: A geometric understanding of $x/\sqrt{1+x^2}$

The magnitude δ is varying continuously and increasing between $\delta_{\text{min}} = 0$ when $x = 0$, and δ_{max} when $x = \frac{\pi}{2}$. From the figure we then see that $\sin(x-\delta) = x/\sqrt{1+x^2}$, and since $x > x-\delta$ we have that $\sin x > \sin(x-\delta)$. We may therefore conclude that

$$0 < x < \frac{\pi}{2} \quad \Longrightarrow \quad \sin x \ > \ \frac{x}{\sqrt{1+x^2}}$$

The following assignment is then presented.

Assignment 2

> *To calculate the value of trigonometric lines to any given arc with a certain number of decimals.*

Solution 3

$$x - \frac{x}{\sqrt{1+x^2}} \ < \ 1 \text{ decimal unit}$$

$$x \ > \ \sin x \ > \ \frac{x}{\sqrt{1+x^2}}$$

By *decimal unit* is here meant "decimal unit of a certain order after the decimal point", and Holmboe gives the following example.

Example

$$
\begin{aligned}
x &= 1 \text{ arcminute} \\
&= \frac{\pi}{180 \cdot 60} \\
&= 0.00029088
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{1+x^2} &= 1.0000000423 \\
\frac{x}{\sqrt{1+x^2}} &= 0.00029087 \\
\sin 1' &= 0.00029088
\end{aligned}
$$

The value of π was known with a sufficient number of correct decimals to make these calculations accurate, and $\sin 1'$ has the first eight decimal places in common with $1'$.

The solution is that the difference $x - x/\sqrt{1+x^2}$ will be smaller than 1 decimal unit of the least value that one requires. Then $\sin x$ will be between x and $x/\sqrt{1+x^2}$ where x is the length of the arc.

Holmboe's motivation for calculating sine for arcs as small as 1' is that *Sine of every arc less than 1' has the first eight decimals common with the arc; the smaller the arc is, the smaller is the difference between x and* $x/\sqrt{1+x^2}$, *and* $\sin x$ *is always between these two magnitudes* (Holmboe 1834: 27). If one needs an accuracy of eight correct decimals, and the arc is less than 1', one may calculate the length of the arc instead of finding the sine value. To calculate larger angles with the same number of accurate decimals, one may use angles where the sine and cosine values are known, and one may use known formulas.

5 Christopher Hansteen (1784–1873)

Christopher Hansteen (1784–1873) was born in Christiania in Norway. He was first a law student in Copenhagen, but became interested in the natural sciences after he met the physicist H. C. Ørsted. He became a teacher in applied mathematics at the university in Christiania in 1814, and he was professor from 1816 till 1861. Hansteen was very productive, and wrote about terrestrial magnetism, northern lights, meteorology, astronomy, mechanics, etc. He was a well-known scientist, and received further international recognition after an expedition to Siberia in 1828–30 to study the geomagnetism. In 1835, Hansteen wrote a textbook in geometry where he challenged the traditional Euclidean geometry.

5.1 Plane geometry

In 1835, Hansteen published a textbook in basic geometry (Hansteen 1835), which in many ways challenged Holmboe's textbooks. Hansteen's book was 278 pages, which is a lot more than what is expected of a textbook in elementary geometry. The author is intentionally trying to tear down the walls that existed between the classical geometry on one side, and the newer analytical geometry and the infinitesimal geometry on the other. The basis of the textbook is real life, with references to artifacts like corkscrews, stove pipes and hourglasses. The presentation of the subject matter is very unlike Euclid's Elements. The style is narrative and written in the first person, sometimes very lengthy, and there are many numerical examples. Hansteen tried to expand Euclid's definition of straight lines and of parallel lines, and Euclid's parallel postulate (Euclid 1956).

Hansteen's textbook contains a comprehensive preface which also contains definitions of fundamental concepts. The first concept to be defined is the straight line (Hansteen 1835: III–IV), which is also, according to Hansteen, "*the foundation of geometry*" [Geometriens Grundvold]. It is of great importance that this concept is clearly defined, especially in a science that demands a consistent and logic practice. Hansteen presents five different ways a straight line may be defined

- "*A straight line is a line which lies evenly with the points on itself*" from Euclid (1956). Close to this is also Baron Wolff's definition stating that "*a line is straight when a part is similar to the whole*" [*Linae recta est, cujus pars quæcunque est toti similis*].
- Archimedes, and most French geometers after him, defined the straight line as "*the shortest trajectory between two points*".
- Some geometers regard the straight line as a hereditary concept that only needs to be mentioned to be understood, and defines a straight line as "*those things which is known to be a straight line*" [*Qvæ linea recta dicatur notum est*].
- Abraham Kästner says that "*a straight line is that whose points all bear against one trace*" [en ret Linie er den, hvis Punkter alle ligge hen mod een Egn], and he adds that "*no one will learn the straight line to know from an explanation, and no one needs to; but one may say something about it, that guides the attention to a closer attention to what makes it a straight line*".
- Finally, others say that "*when a point moves continually in the same direction, then its trajectory is a straight line*".

According to Hansteen, after such definitions, all geometers introduces a postulate which states that "*one may create a straight line between two given points, and prolong such a given straight line in any direction in both directions as one pleases*". Hansteen makes noteworthy objections to such a postulate by asking with what tool such a prolonging shall be made, and how to make sure that the line made by such a tool is homogeneous, or that is satisfies the demands made in the various definitions of a straight line (Hansteen 1835: VI–V).

Hansteen elaborates towards a definition where he let lines be produced by the movement of a point, and there are two kinds. One kind has the quality that when two points of a part of the line are placed on two arbitrary points on the whole line, then all points of the part of the line will coincide with points in the whole line – analogous, if we let a part of a line move along the whole line, and the part always fits with the whole line. Such lines are called *homogeneous lines* [eensartede Linier], and there are two types – *straight* and *curved* lines. A homogeneous line has all over the same curvature, and all perpendiculars of any plane homogeneous line will, when duly prolonged, either intersect in one point, or not intersect at all. There are in other words only two types of homogeneous lines in a plane – the straight line and the circle. When a point moves from one place to another in a space, then it describes a line. If this line is straight, it is called the direction of the motion (Hansteen 1835: IV–V,7,9,35–38). From the concept of the straight line we may derive the concept of the plane, and from these definitions we may prove that a line is straight when all the points in the line remains unchanged in the same position is the line is rotated around two arbitrary points on the line (Hansteen 1835: 9), and that a straight line between two points is shorter than any curved or broken lines between these two points (Hansteen 1835: 40).These two statements are not axioms, but theorems.

It is more proper that a "mechanical artist" derives rules for his practice from the definitions and theorems of the geometry, than that the theoretical geometer shall direct his concepts and definitions towards this practice. The carpenter's planer and the metalworker's file are tools that are suitable for producing homogeneous planes and lines, and the geometer should not neglect to acquire the theoretical principles on which these methods are based. A *ruler* is described as a tool – made by wood or metal – by which one may produce straight lines in a plane. (Hansteen 1835: VIII,13)

The cause for the much discussed controversy Hansteen's textbook made was the handling of parallel lines. Hansteen states very clearly that the Euclidean definition of parallel straight lines, embraced by nearly all geometers, has all the logic errors a definition may have. He states correctly that parallel lines are defined, according to Euclid, by a *negative* quality, and not a *positive*. He continues by stating that the quality by which the parallel lines are defined, is *outside all experience and test*, as it points towards the infinite. Euclid's definition may also not be used on curved lines, which may also be parallel – according to Hansteen. "*No one will hesitate in declaring two concentric circles reciprocal parallel*". There is a definition stating that if two lines in a plane never intersect, no matter how far they are prolonged in any direction, does not make an angle (Hansteen 1835: 28). There is, however, no mentioning that these lines are parallel.

Hansteen argued for an understanding of parallel lines where one let a perpendicular to any kind of line move along this line, in such a way that it always is a perpendicular. Any point on this perpendicular then describes a line, where any point's smallest distance to the original line all over is the same (Hansteen 1835: IX). Consequently, Hansteen has a definition of parallel lines

> "*Any line that is being described by a point on the perpendicular to a given line, when it moves along the same with an unaltered angle, is said to be parallel to the directrix*[10]" (Hansteen 1835: 59)

where the characteristics of a line, parallel to another, are

- it always cuts off equal parts of all its perpendiculars
- any perpendicular to one of these lines is also a perpendicular to the other

and a parallel to a straight line has in addition the following characteristics

- the parallel is also a straight line
- as these straight lines never intersect, they form no angle with each other
- if the parallel lines are intersected by a transversal, then the alternate interior angles are equal, the corresponding angles are equal, and the consecutive interior angles equals 2R

Figure 7: Hansteen's definition of parallel lines

[10] a fixed line used in describing a curve

Hansteen's textbook was published in one edition only, but one reason may be that it contained much subject matter outside the school curriculum. He explains that because of a limited production of textbooks in Norway, he has added subject matter that is outside the curriculum of the learned schools, but should be of interest for students that want to prepare themselves for a study of the higher mathematics (Hansteen 1835: XVIII). It is also worthwhile to mention, as a curiosity, that Hansteen in his textbook introduces and describes *Metre* as a new unit of length (Hansteen 1835: 81).

6 Disagreement and controversy

Holmboe's textbooks were more or less controlling the market regarding textbooks in mathematics in the first half of the 19th century. Hansteen's textbook challenged Holmboe's textbooks, and was the cause of a bitter controversy between the two professors in mathematics.

A newspaper polemic between Holmboe and Hansteen about Hansteen's textbook in geometry took place in *Morgenbladet* from December 1835 till January 1836, and in *Den Constitutionelle* from June till September 1836.[11]

The core of the debate that followed was whether one in mathematics education should let utilitarian considerations overrule logical deduction and theoretical thinking. Hansteen declared that proofs should not be used in the elementary teaching before it was necessary for the students. This, he said, invited the students to memorizing without understanding. To this, Holmboe replied that you either have to prove all or nothing, as half a proof is worse than no proof.

The polemics between Holmboe and Hansteen has later been called the "dispute about parallelism" and they both published booklets where they justified their views (Holmboe 1836; Hansteen 1836).

The main article on the 5th of December, 1835, was written by Holmboe and called "On Professor Hansteen's new understanding of parallel lines" [Om Professor Hansteens nye Parallellære]. It was a review of Hansteen's textbook and it was very critical to Hansteen's definitions of straight and parallel lines. Ten days later there was an unsigned article titled "Concerning Professor B. Holmboe's article in Morgenbladet: 'On Professor Hansteen's new understanding of parallel lines'".[12] The author praised Holmboe for his "touch of thoroughness", but he continued that it was too much to expect from a man who had too long occupied himself with obsolete knowledge, to be an impartial judge of new knowledge. Hansteen's signed reply to Holmboe's article was published on the 18th of December. He stated that Holmboe had reviewed his textbook in a very unseemly manner, and that Holmboe considered Hansteen's textbook dangerous, and teachers should be warned against it, so that young people not are led astray from the rigour of pure and orthodox geometry, into heresy and delusion. A short declaration from Hansteen appeared a week

[11] See Morgenbladet (1835) and Den Constitutionelle (1836).
The newspaper *Morgenbladet* was established in 1819, and was until 1857 a substantial voice for the opposition against the establishment, both literary and political. It was also the first daily newspaper in Norway, and it exists now as a weekly newspaper with a liberal, radical and intellectual profile. *Den Constitutionelle* existed as a daily newspaper in Norway between 1836 and 1847. The idea was to establish a newspaper on a considerably higher intellectual level than Morgenbladet. Den Constitutionelle made high demands on the journalistic content, and it introduced daily editorials. (Kunnskapsforlaget 2006)

[12] Angående Professor B. Holmboes i Morgenbladet No. 339, 1835, indrykkede Stykke: "Om Professor Hansteens nye Parallellære".

later, where he admitted that he probably never would agree with Holmboe about how a good mathematics textbook should be, and that he will publish a booklet the following week. Then there was a short notice signed by Hansteen, dated 18th of January 1836, titled "To the owners of my textbook in geometry" [Til Eierne af min Lærebog i Geometrie], where he admitted that some explanations in his textbook may be simplified. He had therefore made some new pages that by the end of the week would be available at the publisher, free of charge, to the owners of the book.

There was an unsigned paragraph in Den Constitutionelle on the 15th of June, 1836, informing that a professor Jürgensen of Copenhagen had written a review of Hansteen's textbook in the *Monthly Journal for Literature* [Maanedsskrift for Literatur]. This review took no part in the controversy, but asserted the intention of making Hansteen's textbook known in Denmark. Three weeks later there was an article signed Hansteen, titled "On the teaching of mathematics in the schools" [Om den mathematiske Underviisning i Skolerne], where he indicated that the reviewer had been unfortunate with his review. Holmboe now rejoined the fray. In an article he opposed Hansteen by asserting that Hansteen claimed that the only controversies that had been proposed against his textbook in ge-ometry was mainly the question if it is allowed to define a concept before one can prove its possibilities. Holmboe wrote that this is not the case. Hansteen now wrote a long and final article, titled "Farewell to Professor Holmboe" [Afsked til Professor Holmboe]. Hansteen ended his article with an anecdotal remark about Frederick II of Prussia, complaining over the difficulties of being at war with the Russians – "not only did you have to shoot them, you also had to knock them over with your rifle butt", meaning that you not only had to kill them once, you had to kill them twice. Hansteen concluded that he would leave Holmboe standing upright until he got tired – he would not have the trouble of knocking him down. This was the last newspaper article from Hansteen in this matter. Holmboe replied that he was surprised that Hansteen continued this polemics, even though he long time ago said that he would not. Holmboe also asserted that Hansteen had not read his booklet published after the controversy in Morgenbladet. (Morgenbladet 1835; Den Constitutionelle 1836)

6.1 Summary of the controversy

Both Holmboe and Hansteen published booklets where they justified their views. Hansteen wrote a booklet (Hansteen 1836) titled "Investigation of Mr. Professor B. Holmboe's review of my Plane Geometry, Morgenbladet no. 339, 5th of Dec. 1835",[13] dated 26th of December 1835, which means that it was written towards the end of the period the polemics was active in Morgenbladet. Holmboe's name appears only in the title, later he is only referred to "the reviewer". In addition to defending his own textbook, Hansteen also criticizes Holmboe's arguments in the review, and he attacks Holmboe's textbook in geometry (Holmboe 1827).

Hansteen's booklet is organized in five sections, labeled A till E where he focuses on five complaints from Holmboe's review.

A. *Absence of contingency proof* [Forsømmelse af Muelighetsbeviset]. Holmboe's complaint is that Hansteen uses the attributes of lines and surfaces *before* he defines them. Hansteen starts his textbook by classifying lines as *homogeneous* [eensartede] and *heterogeneous* [ueensartede]. Hansteen blames Holmboe for

[13] Belysning af Hr. Professor B. Holmboes Anmeldelse af min Plangeometrie, Morgenbladet No. 339, 5 Dec. 1835.

not respecting authorities like Newton and Laplace, and he is attacking the definitions of basic concepts in Holmbo's textbook in geometry. Hansteen justifies his presentation of the subject matter by the fact that his book has been used for half a year at Christiania Kathedralskole.

B. *Definition of a straight line.* Hansteen is accused of not using accurate descriptions and terms, and he argues by the fact that the textbook is written for children, and their only previous knowledge is their language, and names of concepts from their everyday life. Therefore one has to use a language that stimulates the imagination.

C. *"A circle is a circle".* A vital error, according to Holmboe, is that he states that there exist only two homogeneous lines in a plane, *the straight line* and *the circle*, at a stage where it is unfounded.

D. *Theory of parallelism.* The definition of parallel lines in Hansteen's textbook states that *a line parallel to another has the characteristics that it cuts equal parts of its perpendiculars.* This relates to straight as well as curved lines, and it follows that they will never cross no matter how long you extend them.[14] This definition is, according to Holmboe, not generally correct, as parallel curved lines may cross one another.

E. *Euclidean definition of parallel lines.* Hansteen states that it is better for a concept to be defined by a positive property than by a negative one, and parallel lines are by Euclid defined by a property that lies beyond our experience, and it refers our minds towards the infinite. He also attacks Holmboe's statement that "to construct is to elucidate the specified concepts of the definition of a magnitude",[15] and he finds it paradoxical that thorough knowledge of geometry does not assume the use of compass and ruler. How may such a mental construction elucidate the shape of a curved line, if it is defined by an equation between its coordinates, he asks. He also claims to have met students that didn't know one end of a compass from the other. Holmboe calls the use of compass and ruler an *insignificant requirement* [uvæsentlig fordring] which should not be included in a textbook, and he claims that he has not found these instruments mentioned in textbooks by Lacroix, Legendre, Kästner, Wolff, or Vega. Only the textbooks by Hansteen and Thomas Bugge mention the use of compass and ruler.

Towards the end of his booklet, Hansteen recommends that a new edition of Lindrup's textbook[16] should be made, if one wants easily understood textbooks in arithmetic and geometry that does not frighten students away from studies in mathematics.

Holmboe responded by writing a booklet (Holmboe 1836) titled "*Retort provoked by Mr. Professor Hansteen's enlightenment of my review of his textbook in geometry, containing: 1) Defense of the review containing proofs collected by a continued review of his textbook. 2) Refutation of his attack on my textbook in mathematics*",[17] and this was dated the 8th

[14] Den almindelige Charakter for en Linie, som er parallel med en anden, er altsaa: At den overalt affskjærer ligestore Stykker af dennes Normaler; hvoraf altsaa følger for alleslags parallele Linier, saavel rette som krumme, at de, i hvor langt de end forlænges, aldrig kunne skjære hinanden.

[15] at construere er at anskue det ved en Størrelses Definition fastsatte Begreb

[16] The Danish teacher of mathematics, Hans Christian Linderup (1763–1809) published a textbook in basic mathematics in 1807.

[17] Gjenmæle fremkaldt ved Hr. Professor Hansteens Belysning af min Anmeldelse af hans Lærebog i Geometrien, indeholdende: 1) Forsvar for anmeldelsen med Beviser hentede ved en fortsat Recention over hans Lærebog. 2) Gjendrivelse af hans Angreb paa min Lærebog i Mathematiken.

of March 1836. It was written in the period between the two polemics in Morgenbladet and Den Constitutionelle. Throughout the booklet, Hansteen is referred to as "the author". Holmboe's booklet is structured in the same five sections as Hansteen's, and section D is – not surprisingly – the most comprehensive. Holmboe shows a wide knowledge of the subject matter by quoting Klügel's definition of curved parallel lines from 1763, in addition to the textbook "*Theorie des lignes courbes*" by Lacroix. The latter does not call curved lines parallel. Holmboe admits that Hansteen is correct in his objection against Euclid's definition of parallel lines, that it declares a property that is beyond all experience, in the sense that the definition appears before it is proven that two straight lines in a plane could have such a location that they will never cross if they are prolonged indefinitely. Holmboe is very clear in adding that Hansteen's theory of parallel lines is in obvious conflict with the existing theory, which states that a curved line at a certain point is parallel to another curved line at a certain point, only if the tangents through each of the two points are parallel. The better part of Holmboe's booklet is a defense against the attacks made by Hansteen on his textbooks, and Holmboe is constantly referring to Legendre and his definitions.

7 After the conflict

The first textbook by Holmboe that was published after the conflict with Hansteen was the second edition of the textbook in arithmetic in 1844 and on the reverse page of the title page, the following signed declaration is printed:[18]

> **No. 200**
> The second edition of the present textbook's 1st part, or the arithmetic, is printed in 1050 copies. Each copy has a specific number, in such a way that the copies are labelled with the numbers in their order from 1 to 1050. If a copy is not numbered in this manner, and if the reverse side of the title page does not contain this declaration signed by the author, then that copy is illegal, and will be dealt with in accordance with the existing legal provisions.
> *B.M. Holmboe*

and this is the only publication by Holmboe that contains such a declaration.

8 Conclutions

The first half of the 19th century was in many ways a turning point for higher education in mathematics in Norway. The position of mathematics as a school subject was strengthened through school reforms at the turn of the century, and the first university was established in Norway in 1811. Bernt Michael Holmboe's textbooks in mathematics were the ones that were predominantly used in the learned schools at that time. His textbooks were, as we have seen, not without opposition – an opposition addressing the use of proofs in elementary mathematics, and whether the introduction of geometry should be in a traditional Euclidean way, using logical deductions and theoretical thinking – as in the case

[18]**No. 200.** Af nærværende Lærebogs 1ste Deels eller Arithmetikens andet Oplag er trykt 1050 Exemplarer. Hvert Exemplar har sit særskilte Nummer, saaledes at Exemplarerne ere nummererede med Tallene efter deres Orden fra 1 til 1050. Saafremt noget ikke saaledes nummereret Exemplar, og hvis Titelblad paa Bagsiden ikke er forsynet med nærværende af Forfatteren underskrevne Erklæring, maatte forefindes, er samme ulovligt, og vil blive behandlet overeensstemmende med de for Eftertryk gjældende Lovbestemmelser. *B.M. Holmboe* (sign.)

of Holmboe – versus a more "informal" way using everyday language and terms. Holmboe is in his textbooks very true to the Euclidean tradition in presenting the subject matter, and there was an ongoing debate about the use of Euclidean ideas in textbooks in geometry. When Hansteen published his textbook in geometry, it was evidently a controversial issue and his textbook was seen as an attack on the Euclidean textbooks.

Hansteen encouraged using a language that stimulated the pupils imagination, and he also used a definition of parallel lines, which – according to Holmboe – was not generally correct. The issues addressed in the newspaper polemics were both the mathematical topics of geometry, and didactical issues – how geometry should be presented to the pupils. Today we find it impressing that a debate like this reached as far as the public press, and that it was given so much attention and space in the papers. This shows, more than anything else does, the position professors at the university had in the society. Hansteen states in the preface of his textbook that there is no lack of good geometry books in the Danish-Norwegian language, but they are all very true to Euclid. It is Hansteen's stated intension to differ from not only Euclid, but also other textbooks. The world of mathematics had been through a development towards strong demands on rigour in definitions and methods (Christiansen 2010), and Hansteen turned against this in his way of presenting the subject matter. Hansteen concretized the mathematical objects, and talked about straight lines as something one could make with a ruler, while the mathematical objects for Holmboe were something one had to elucidate in one's mind, and not to construct. For Holmboe, the mathematical correctness was the most important, while Hansteen had, what we would call today, a much more didactical approach. Hansteen's textbook was only published in one edition, but in addition to being untraditional, Hansteen's textbook also contained much subject matter outside the school curriculum. Even if Hansteen's way of presenting the subject matter would be closer to how we today view didactics, tradition was stronger and Holmboe's books were used in the learned schools until they were replaced by textbooks by Ole Jacob Broch about a decade after Holmboe's death. (Christiansen 2009)

A question was formulated in the introduction regarding what we may learn from these textbooks and the didactical debate from the early 19th century. I have in this paper described important and relevant parts of Holmboe's textbooks in arithmetic, geometry and trigonometry. I have also described parts of Hansteen's textbook in plane geometry and made comparisons between Holmboe's and Hansteen's way of presenting the subject matter. The feud between the two is also a good and well documented example of the didactical debate. The pupils in the learned schools were normally somewhere between 12 and 20 years of age, and a newspaper debate about school mathematics today would probably look completely different. The knowledge level in the textbooks used by the pupils of early 19th century is considerably higher than in modern textbooks, but Holmboe's textbooks were very encyclopaedic, and only contained the subject matter. Modern textbooks may also be characterized by a didactic angle that is missing in the old textbooks, with a possible exception of Hansteen's textbook in plane geometry.

Norwegian pupils are scoring average to low in student assessments by OECD (PISA), and one explanation may be that the content knowledge in mathematics is too low. Mathematics is a comprehensive subject, and pupils will inevitable be encouraged to memorize without understanding if their body of knowledge is too low.

References

Andersen, O. (1914). *Norges høiere skolevæsen 1814–1914. En oversigt.* Kirke- og undervisningsdepartementets jubilæumsskrifter 1914. J.M. Stenersens forlag, Kristiania.

Bjarnadóttir, K., Christiansen, A., and Lepik, M. (2013). Arithmetic textbooks in Estonia, Iceland and Norway – similarities and differences during the nineteenth century. *Nordic Studies in Mathematics Education*, 18(3):27–58.

Bjerknes, V. (1925). *C. A. Bjerknes hans liv og Arbeide: Træk av norsk kulturhistorie i det nittende aarhundre.* H. Aschehoug & Co., Oslo.

Bolzano, B. (1817). *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwei Werthen, die ein entgegensetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege.* Gottlieb Haase, Prag.

Christiansen, A. (2009). Bernt Michael Holmboe (1795–1850) and his mathematics textbooks. *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, 24:105–113.

Christiansen, A. (2010). Bernt Michael Holmboe's textbooks and the development of mathematical analysis in the 19th century. In Barbin, E., Kronfellner, M., and Tzanakis, C., editors, *History and Epistemology in Mathematics Education. Proceedings of the Sixth European Summer University*, pages 653–663, Wien, Austria. History and Pedagogy of Mathematics, Verlag Holzhausen GmbH.

Christiansen, A. (2012a). A controversy about geometry textbooks in Norway 1835–36. In Bjarnadóttir, K., Furinghetti, F., Matos, J. M., and Schubring, G., editors, *"Dig where you stand" 2. Proceedings on the second "International Conference on the History of Mathematics Education"*, pages 117–128. New University of Lisbon, Portugal, UIED.

Christiansen, A. (2012b). Geometry textbooks in Norway in the first half of the 19th century. In *HPM2012. The HPM Satellite Meeting of ICME–12*, pages 683–692, Daejeon, Korea.

Christiansen, A. (2015a). An analysis of two Norwegian geometry books from 1827 and 1835, and the reactions they caused. *Nordic Studies in Mathematics Education (NOMAD)*, 20(3–4):101–121.

Christiansen, A. (2015b). The understanding of parallel lines in early 19th century textbooks: A comparison between two Norwegian geometry books from 1827 and 1835. In Bjarnadóttir, K., Furinghetti, F., Prytz, J., and Schubring, G., editors, *"Dig where you stand" 3. Proceedings on the third International Conference on the History of Mathematics Education. September 25–28, 2013*, pages 123–135. Faculty of Education at Uppsala University, the Department of Education at Uppsala University and the Swedish Research Council, Uppsala University, Sweden.

Den Constitutionelle (1836). Reviews, replies and notices. 25 June 1836, 16 July 1836, 4 Aug. 1836, 21 Aug. 1836, 23 Aug. 1836, 24 Aug. 1836, 19 Sept. 1836 and 22 Sept. 1836. Accessible on microfilm.

Euclid (1956). *The Thirteen Books of the Elements*, volume I–III. Dover Publications, Inc., New York. Translated with Introduction and Commentary by Sir Thomas L. Heath. Unabridged and unaltered republication of the Second Edition from 1925. (First edition 1908).

Hansteen, C. (1835). *Lærebog i Plangeometrie.* Johan Dahl, Christiania.

Hansteen, C. (1836). *Belysning af Hr. Professor B. Holmboes Anmeldelse af min Plangeometrie, Morgenbladet No. 339, 5 Dec 1835.* Johan Dahl, Christiania.

Holmboe, B. M. (1825). *Lærebog i Mathematiken. Første Deel, Inneholdende Indledning til Mathematiken samt Begyndelsesgrundene til Arithmetiken.* Jacob Lehman, Christiania, First edition.

Holmboe, B. M. (1827). *Lærebog i Mathematiken. Anden Deel, Inneholdende Begyndelsesgrundene til Geometrien.* Jacob Lehman, Christiania, First edition.

Holmboe, B. M. (1834). *Plan og sphærisk Trigonometrie.* C. L. Rosbaum, Christiania.

Holmboe, B. M. (1836). *Gjenmæle fremkaldt ved Hr. Professor Hansteens Belysning af min Anmeldelse af hans Lærebog i Geometrien.* Jacob C. Abelsted, Christiania.

Holmboe, B. M. (1844). *Lærebog i Mathematiken. Første Deel, Inneholdende Indledning til Mathematiken samt Begyndelsesgrundene til Arithmetiken.* Jacob Lehman, Christiania, Second edition.

Holmboe, B. M. (1849). Holmboes brev til Bjerknes om «Mathematikens Studium». Festskrift i anledning 100 år siden Abels fødsel, 1902. pp. 236–38.

Holmboe, B. M. (1850). *Lærebog i Mathematiken. Første Deel, Inneholdende Indledning til Mathematiken samt Begyndelsesgrundene til Arithmetiken.* Jacob Lehman, Christiania, Third edition.

Holst, E. (1911). *Det Kongelige Fredriks Universitets 1811–1911. Festskrift*, volume II, chapter Matematikken. H. Aschehoug & Co, Kristiania.

Kunnskapsforlaget (2006). Store Norske Leksikon. Kunnskapsforlaget.

Legendre, A.-M. (1819). *Elements of Geometry.* Hilliard and Metcalf. Translated from the French for the use of the students of the University at Cambridge, New-England.

Morgenbladet (1835). Reviews, replies and notices. 5 Dec. 1835, 13 Dec. 1835, 15 Dec. 1835, 16 Dec. 1835, 18 Dec. 1835, 23 Dec. 1835, 25 Dec. 1835, 28 Dec. 1835, 5 Jan. 1836 and 21 Jan. 1836. Accessible from `www.nb.no/avis`.

Otte, M. (2009). The analytic/synthetic distinction in Kant and Bolzano. In Sriraman, B. and Goodchild, S., editors, *Relatively and Philosophically Earnest. Festschrift in Honor of Paul Ernest's 65'th Birthday*, The Montana Mathematics Enthusiasts Monograph Serie in Mathematics Education, chapter 5, pages 39–55. Information Age Publishing, Inc.

Piene, K. (1937). Matematikkens stilling i den høiere skole i Norge efter 1800. *Norsk Matematisk Tidsskrift*, 19(2):52–68.

Russ, S. (1980). A translation of Bolzano's paper on the intermediate value theorem. *Historia Mathematica*, 7:156–185.

Russ, S. (2004). *The Mathematical Works of Bernard Bolzano.* Oxford University Press.

Schubring, G. (1993). Bernard Bolzano—Not as Unknown to His Contemporaries as Is Commonly Believed? *Historia Mathematica*, 20:45–53.

Solvang, R. (2001). Skolematematikken i Norge på Abels tid. *Nordisk Matematisk Tidsskrift*, 49(3):111–138.

Treschow, N. (1813). *Almindelig Logik.* Fr. Brummers forlag, Kiøbenhavn.

THE NORSE TREATISE ALGORISMUS

Preserved in manuscript GKS 1812 4to

Kristín BJARNADÓTTIR[1,3], Bjarni V. HALLDÓRSSON[2,4]
[1]University of Iceland, School of Education, Stakkahlid, 105 Reykjavík, Iceland,
[2]Reykjavík University, School of Science and Engineering, Menntaveg 1,
101 Reykjavík, Iceland
[3]krisbj@hi.is, [4]bjarnivh@hr.is

ABSTRACT

The treatise *Algorismus* is a complete prose translation of the Latin hexameter *Carmen de Algorismo* into the medieval Old Norse language. *Carmen* is dated in 1202, written by the French canon Alexander de Villa-Dei. The treatise explains for the first time the Hindu-Arabic decimal place value numeral notation and calculation methods to the Norse people, Icelanders and Norwegians. *Algorismus* also relates the four Elements: Earth, Water, Air and Fire to cubic numbers and ratios. The treatise exists in four manuscripts, one of them only a fragment. The four manuscripts are compared by digital methods to show that the two oldest of them are quite similar and possibly copies of the same copy of the original translation. This paper focuses on the version in Ms GKS 1812 4to. It is a pedogogical study of the algorithms presented in the treatise, contrasting them with current day methods and the presentation in *Carmen*.

1 Introduction

The thirteenth century Old Norse arithmetic treatise *Algorismus* exists in four manuscripts: GKS 1812 4to, preserved in Reykjavík, and AM 544 4to, AM 685 d 4to, and AM 736 III 4to, preserved in Copenhagen. The bulk of the treatise is a prose translation of the Latin hexameter poem *Carmen de Algorismo*, written in France in the early thirteenth century. *Algorismus* is also related to *Algorismus Vulgaris* by Johannes de Sacrobosco, dated around 1225. In this paper we explore incidences where *Algorismus* deviates from *Carmen de Algorismo*, and we compare the four extant manuscripts of *Algorismus* with phylogenetic alignment methods.

Algorismus was published in a scientific edition 1892–1896 by Finnur Jónsson (1892–1896) based on AM 544 4to, with corrections from the other three manuscripts as applicable. The mathematician Otto B. Bekken translated *Algorismus* into modern Norwegian in 1985 and explained its text in cooperation with linguist Marit Christoffersen (Bekken and Christoffersen, 1985). Kristín Bjarnadóttir (2003, 2007) has explained the content of *Algorismus* in English and in modern Icelandic (Bjarnadóttir, 2004). Bjarnadóttir and Halldórsson (forthcoming) studied *Algorismus*, focusing the manuscript GKS 1812 4to and the phylogenetic alignment methods, as done here.

Algorismus, a treatise of nearly 3000 words, contains an explanation of the Hindu-Arabic decimal place-value notation and calculation methods in seven algorithms: addition, subtraction, doubling, halving, multiplication, division, and extraction of roots, which is further subdivided into the square root and the cubic root. These methods have been relayed to *Algorismus* via a well-known Latin hexameter, *Carmen de Algorismo*, composed by Alexander de Villa Dei (1170–1240) between 1200 and 1203 (Beaujouan, 1954). Alexander de Villa Dei was a Franciscan and a master at the University of Paris, later canon at the St. Andrew’s Cathedral in Avranches (Beaujouan, 1954). Finnur Jónsson (1892–1896, p. cxxxii) suggested that *Carmen* was translated into the Old Norse language before 1270. Jónsson referred to an analysis by Hankel (1874, p. 325) of the typeface of Arab numerals, where the typeface used in the manuscripts of *Algorismus* corresponds Hankel's examples from before 1271. According to Tropfke (1980) the typeface belongs to the West-Arab notation. Helgi Guðmundsson (1967, p. 68) deems it possible that the translation existed in the Viðey monastery in the early 14th century.

Carmen is a verbal explanation of Hindu-Arabic arithmetic, built on a translation of the work by Muhammad ibn-Mūsā al-Kwārizmī (ca. 780–850), *Kitāb al-jam΄wal tafrīq bi hisāb al-Hind* [*The Book of Bringing Together and Separating According to the Hindu Calculation*], most likely on *Liber alghoarismi de practica arismetrice*, one of its twelfth century Latin translations (Allard, 1992, p. xxxi). This conjecture is based on the order of the arithmetic operations which varies in the different translations. The cubic root is not included in the translations of Al-Khwārizmī's work and must be acquired from another source. *Algorismus* expands *Carmen* with concrete examples as well as a concluding section of unknown origin. There are also a few omissions that do not compromise the meaning.

The poem *Carmen* exists in a great number of manuscripts, preserved in libraries in France, Great Britain, the Netherlands and many other countries. It is considered to have played an even greater role in distributing Hindu-Arabic positional number notation in Northern Europe than the well-known *Liber abaci* by Leonardo da Pisa (Jacqueline Stedall, personal communication, 2009). The translation of *Carmen* into the vernacular of the Norse people in Norway and Iceland was a further effort in the distribution of knowledge.

Carmen de Algorismo, contained in the manuscript MS. Auct. F.5.29, preserved in the Bodleian Library in Oxford, dated to the thirteenth century, has been drawn on when comparing *Algorismus* in the manuscripts GKS 1812 4to and AM 544 4to. The manuscripts AM 544 4to and MS. Auct. F.5.29 have chapter headings that are neither found in the other manuscripts of *Algorismus* nor in the two known printed versions of *Carmen de algorismo* (Steele, 1988, pp. 72–80; Halliwell, 1841, pp 73–83).

In the following, chosen passages from *Algorismus* and *Carmen de algorismo* are compared and translated into English. The passages from *Algorismus* in GKS 1812 4to have been rewritten with modern Icelandic spelling.

2 Arithmetic operations in Carmen de Algorismo and Algorismus

Carmen de Algorismo is a hexameter to be recited verbally. The beginning of the poem reads as follows: [1]

> *Hec algorismus ars presens dicitur ; in qua*
> *Talibus Indorum fruimur bis quinque figuris.*
> *0. 9. 8. 7. 6. 5. 4. 3. 2. 1.*
> *Prima significat unum : duo vero secunda :*
> *Tercia significat tria : sic procede sinistre*
> *Donec ad extremam venies, que cifra vocatur ; (Steele, 1988, p. 72).*

The ten digits in the third line of the poem are the only occurrence of the then new Hindu-Arabic numerals in *Carmen*, see Fig. 2.1. Everywhere else numbers are expressed in

[1] In the following examples, the Latin texts are from "Carmen de algorismo", printed in *The Earliest Arithmetics in English* (Steele, (Ed.), 1988).

words. The poem explains algorithms that are now common without giving concrete examples. It is not known how the poem was used to aid computation but one may assume that calculations were made on tablets or a flat surface, strewn with sand, or on a wax tablet.

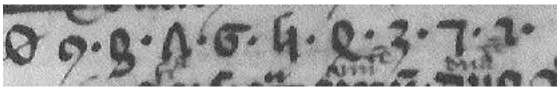


Figure 2.1: The ten digits of Hindu-Arabic numerals in *Carmen de Algorismo* in MS. Auct. F.5.29.

The initial text in *Algorismus* is a nearly literal translation of the Latin original in *Carmen*:

> *List þessi heitir Algorismus. Hana fundu fyrst indverskir menn með tíu stöfum er svo eru ritaðir 0 · 9 · 8 · 7 · 6 ·5 · 4 · 3 · 2 · 1. Hinn fyrsti stafur merkir einn í fyrsta stað. En annar tvo. En þriðji þrjá. Og hver eftir því sem skipaður er allt til hins síðasta er cifra heitir.*[2] (*GKS 1812 4to, 13v*)[3]

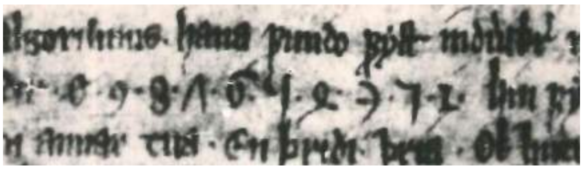


Figure 2.2: The ten digits of Hindu-Arabic numerals in *Algorismus* in GKS1812 4to.

Carmen de Algorismo is believed to be the first work where the nought (zero), *cifra*, is presented as a digit (Beaujouan, 1954). Both the Latin text of *Carmen* and that of *Algorismus* count the *cifra* as the last one in the order of numerals, and 1 as the first one in the order, which indicates that the numbers were read in the Arabic way from right to left. This is emphasised in *Carmen*: “sic procede sinistre” [proceed thus to the left].

Algorismus vulgaris by Johannis de Sacrobosco is a longer text that is related to *Carmen*. Sacrobosco’s text cites three verses from *Carmen*, e.g. on the operations:

> *Subtrahis aut addis a dextris vel mediabis ;*
> *A leua dupla, diuide, multiplicaque;*
> *Extrahe radicem semper sub parte sinistra (Steele, 1988, p. 73; Sacrobosco, 1898, p. 7).*[4]

Algorismus in GKS 1812 4to:

> *Frá hinni hægri hendi skalt þú af taka og við leggja og skipta í helminga en frá vinstri hendi skalt þú tvöfalda og skipta og margfalda og svo draga rót undan hvorutveggju (GKS 1812, 4to, 13v).*[5]

The arithmetic operations addition, subtraction and division, explained in *Carmen* and *Algorismus*, are largely similar to present day methods used in paper-and-pencil arithmetic. Multiplying two composite numbers, however, proceeds from the left, as opposed to common modern algorithms. The numbers to be multiplied are arranged so

[2] Translations from Icelandic into English were made by the author, K. B.
[3] This art we call Algorismus. It was first found by Indians and arranged by ten digits those which so are written: 0 · 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1. The first digit signifies one in the first place. But the second two. But the third three. And each as it is ordered until the last one which is called cifra [the nought].
[4] Subtract or add from the right, or halving; from left double, divide, multiply; extract roots always from the left side.
[5] From the right hand you shall deduct and add and split in halves but from the left hand you shall double and divide and multiply and so extract both roots.

that the digit farthest to the right of the multiplicand is placed below the first digit (from left) of the multiplier. The multiplicand is multiplied by this digit which then disappears under the product. *Carmen* and *Algorismus* do not illustrate their algorithms on the four arithmetic operations by examples. The following example is constructed for clarification in this paper:

Multiply 523 by 217:

First 523 is multiplied by 2, and 2 disappears under the product:

217	**1046**17
523	

Next, the multiplicand is moved one place to the right so that the rightmost digit is placed below the second digit of the multiplier in the upper row and the lower number is now multiplied by that digit in the same manner. The product is written above the multiplying digit and added to the next digits to the left. In the example, the rightmost digit of the multiplicand, 3, is now placed below the second digit of the multiplier, 1, which now is the multiplying digit. The product 1·3 = 3 replaces the multiplier 1 and products of the other digits are added to previous products. Finally, the previously mentioned digit 3 is moved one more step to the right and the procedure is repeated. In this third step, 7 is the multiplying digit.

217	**1046**17	10**983**7	10**983**7	1**13491**
523	523	523	523	523

In general, this procedure continues until all digits of the upper number have been used as multipliers. The advantage of multiplying from the left is that the products of the digits can be added to the previous product as they are found and it is not necessary to carry.

Doubling and halving were treated as distinct algorithms. Both operations are known from antiquity, even as replacements for multiplication and division (Seppala-Holzman, 2007). Doubling was done from the left as is customary in mental arithmetic, while halving was done from the right.

Square root was drawn from the left. The method is not much different from what was customary before calculators became common in schools. In extracting the square root, one digit of the number of which the root is drawn is pulled down at the time. At the beginning, the highest possible one-digit number, squared, is subtracted. Its double, the “dufl” is preserved aside, and so is the half of the double, the “subdufl”, becoming the first digit of the root. Next, as many “dufls” as possible are subtracted when the next digit has been pulled down. The multiplier is the next digit of the root. As the work progresses, the “dufls” are combined and their multiples subtracted, each multiplier constituting the next digit. The combined “subdufls”, the first found digit and the multipliers, form the desired square root. An example of drawing the square root of 119,025 was composed, see Fig. 2.3.

	119025		
3^2	9	$-(100a)^2$	subdufl 3, dufl 6
	29		
$6\cdot 4$	24	$-2\cdot 100a\cdot 10b$	
	50		
4^2	16	$-(10b)^2$	subdufl 4, dufl 8
	342		
$68\cdot 5$	320	$-(2\cdot 100a + 2\cdot 10b)\cdot c$	
	25		
5^2	25	$-c^2$	subdufl 5, dufl 10

Figure 2.3: Here a = 3, b = 4, and c = 5. The square root of 119,025 is 345

The algorithm is based on the fact that

$$(100a + 10b + c)^2 = (100a)^2 + 2\cdot 100a\cdot 10b + (10b)^2 + (2\cdot 100a + 2\cdot 10b)c + c^2$$

One notices that by this method, the digits of the square root emerge gradually in a natural way without guessing.

Extracting cubic root had also separate sections in *Carmen*, *Algorismus* and *Algorismus vulgaris*. The section in *Carmen* is considered to be the first treatment of extracting cubic root in Latin (Beaujouan, 1954). It is not contained in the Latin translations of al-Kwārizmī's work but known from the work Āryabhaṭīya by the Indian mathematician Āryabhaṭa (born 476) (Katz, 1993, p. 202).

Extracting the cubic root is done by alternatively pulling down two digits and one digit at the time. Fig. 2.4 shows the extraction of the cubic root of 15,069,223 to reach 247. A triple digit is called "tripl", while the digit itself is called "subtripl". The algorithm is based on the identity

$$(100a + 10b + c)^3 =$$
$$(100a)^3 + (100a + 10b)\cdot 3\cdot 100a\cdot 10b + (10b)^3 + (100a + 10b + c)(3\cdot 100a + 3\cdot 10b)c + c^3$$

The procedure, shown in Fig. 2.4 is less simple than that of finding the square root. In the second step, when subtracting the term (100a + 10b)·3·100a·10b, one must search the value of b by testing its most likely value, and likewise when searching for c.

	15069223		
2^3	8	$-(100a)^3$	subtripl 2, tripl 6
	706		
$24\cdot 6\cdot 4$	579	$-(100a + 10b)\cdot 3\cdot 100a\cdot 10b$	
	1309		
4^3	64	$-(10b)^3$	subtripl 4, tripl 12
	124522		
$247\cdot 72\cdot 7$	124488	$-(100a + 10b + c)(3\cdot 100a + 3\cdot 10b)\cdot c$	
	343		
7^3	343	$-c^3$	subtripl 7, tripl 21

Figure 2.4: Here, a = 2, b = 4, and c = 7. The cubic root of 15,069,223 is 247.

3 Deviations of Algorismus from Carmen de Algorismo

The poem *Carmen* contains a description of the Hindu-Arabic number notation in general terms. The treatise *Algorismus* enhances *Carmen* by demonstrating the new system's notation. It extends the first chapters, suggesting a need to clarify the text by numerical examples, while several repetitions in *Carmen* were omitted in the translation. Following *Carmen's* explanation of decimal place value notation, examples are inserted in the Old Norse translation, shown here in bold font and square brackets.

> *Ergo, proposito numero tibi scribere, primo*
> *Respicias quis sit numerus ; quia si digitus sit,*
> *Una figura satis sibi; sed si compositus sit*
> *Primo scribe loco digitum post articulum fac*
> *Articulus si sit, cifram post articulum sit. (Steele, 1988, p. 72)*

> *Ef þú vilt rita nokkra tölu þá hygg þú að ef það er fingur og rita í fyrsta stað eina hverja fígúru slíka sem þarf* ***[á þessa leið, 8].*** *En ef þú vilt lið rita þá settu cifru fyrir fígúru* ***[á þessa lund, 70].*** *Vilt þú samsetta tölu rita þá settu fígúru fyrir lið* ***[sem hér, 65].*** *(GKS 1812 4to, 13v)*[6]

Notice that the order of sentences in *Algorismus* is different from the Latin version. Next even and odd numbers are presented, where the following addition is inserted in *Algorismus*:

> *Quolibet in numero, si par sit prima figura,*
> *Par erit et totum, quicquid sibi continetur;*
> *Impar si fuerit, totum sibi fiet et impar. (Steele, 1988, p. 73)*

> *Hverja tölu er þú ritar þá er hún jöfn ef [****tigum gegnir eða]*** *jafn fingur er umfram. En öll tala er ójöfn ef ójafn fingur er umfram.* ***[Jafnir fingur eru fjórir: 2. 4. 6. 8. En ójafnir aðrir fjórir: 3. 5. 7. 9. En einn er hvorki því að hann er eigi tala heldur upphaf allrar tölu.]*** *(GKS 1812 4to, 13v)*[7]

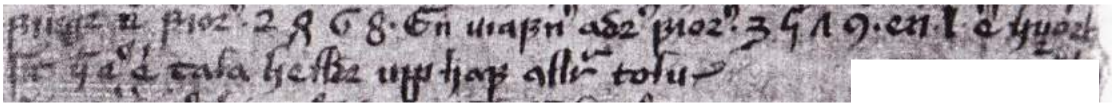


Figure 3.1: …fingur eru fjórir, 2 4 6 8. En ójafnir aðrir fjórir, 3 5 7 9, en 1 er hvorki því hann er ei tala helldur upphaf allrar tölu. (AM 685d 4to, p. 25v)

The digits inserted are written in Hindu-Arabic mode in all extant *Algorismus* manuscripts. *Algorismus* also inserts a note that one is neither even nor odd number as it is the origin of all numbers. Bekken and Christoffersen (1985, p. 27) have pointed out a likeness to the statement that one is not a number in al-Kwārizmī's *Arithmetic*, which again refers to another book on arithmetic, most likely either Euclid's *Elements*, book VII,

---

[6] *If you want to write some number then think if it is a digit and write in the first place each one figure such as is needed* ***[in this way, 8].*** *But if you want to write tens then put zero in front of the figure* ***[in this way, 70].*** *If you want to write a composite number, put a figure in front of the tens* ***[as here, 65]***. (This translation of the Latin term "post" by "fyrir", meaning "before", "in front of" (see e.g. Cleasby, 1957) is contradictory, but consistent with earlier counting from right to left where 1 was counted as the first in the row of the ten digits, and the cifra as the last one).

[7] *Each number that you write, then it is even if* ***[it is a multiple of ten or]*** *an even digit is extra; but the whole number is uneven if an uneven digit is extra.* ***[Even dig 6. 8. But uneven another four; 3. 5. 7. 9. But one is neither as it is the origin of all number.]***

definition 2, stating: “A number is a multitude composed of units” (Euclid, 1956, p. 277), or *Arithmetica* by the Neo-Pythagorean Nicomachus. The citation referred to is the following from the translation *Dixit Algorizmi* of al-Kwārizmī’s work:

> *Et iam patefeci in libro algebr et amucabalah, idest restaurationis et oppositionis, quod uniuersus numerus sit compositus et quod uniuersus numerus componatur super unum. Unum ergo inuenitur in uniuerso numero. Et hoc est quod in alio libro arithmetice dicitur quia unum est radix uniuersi numeri et est extra numerum :* (*al-Kwārizmī, 1992, p 1*).[8]

The next insertion to *Algorismus* is when *Carmen*’s text states that there are seven operations: addition, subtraction, doubling, halving, multiplication, division and root extraction:

> *Septem sunt partes, non plures, istius artis ;*
> *Addere, subtrahere, duplare, dimidiare ;*
> *Sextaque est diuidere, set quinta est multiplicare ;*
> *Radicem extrahere pars septima dicitur esse. (Steele, 1988, p. 73)*

Then *Algorismus* adds that root extraction has two branches, extracting square root and cubic root:

> *Í sjö staði er skipt greinum þessarar listar. Heitir hin fyrsta viðurlagning. Önnur afdráttur. Þriðja tvefaldan. Fjórða helminga skipti. Fimmta margfaldan. Sjötta skiptingin. Sjöunda að taka rót undan* ***[og er sú með tveimur greinum. Önnur er að taka rót undan ferskeyttri tölu. En önnur grein er það að draga rót undan átthyrndri tölu þeirri er verpils vöxt hefur]****. (GKS 1812 4to, p. 13v)*[9]

Sacrobosco’s elaboration of al-Kwārizmī’s work, *Algorismus Vulgaris*, states: “... radicem extractio, et haec dupliciter, quoniam in numeris quadratis et cubicis” [extraction of roots, which is twofold, since [it applies] to square numbers and cube numbers].” (Sacrobosco, 1897, p. 1). This quotation suggests that the translator may have known Sacrobosco’s text in addition to Villa Dei’s *Carmen*. Sacrobosco claims, however, that there are nine arithmetic operations, adding numeration and progression as operations number one and eight (Sacrobosco, 1897, p. 1).

Each arithmetic operation is explained in a separate chapter. In order to multiply, the reader is instructed to arrange the two numbers to be multiplied in columns such that the first digit (from the right) of the multiplier is placed below the last digit of the multiplicand as explained previously. However, one must first check the difference of the larger digit of the multiplicand from ten and then delete the smaller one from its tens as often as that difference:

> *In digitum cures digitum si ducere, major*
> *Per quantes distat a denis respice, debes*
> *Namque suo decuplo tociens delere minorem; (Steele, 1988, p. 75)*

---

[8] *And I have already explained in the book on algebra and almucabalah, that is on restoring and comparing, that every number is composite and every number is composed of the unit. The unit is therefore to be found in every number. And this is what is said in another book on arithmetic that the unit is the origin of all numbers and is outside numbers* (English translation by the author, KB, after André Allard’s translation from Latin to French).

[9] *In seven parts is divided this art’s branches. The first one is called addition. Second subtraction. Third doubling. Fourth halves splitting. Fifth multiplication. Sixth the division. The seventh to take root from under [****and is that one in two branches. One is taking a root from under a square number. But another branch is drawing a root from under an eight-vertex number, the one which has cubical growth].***

Þar næst skalt þú hugsa hversu mikið hina meiri fígúru skortir á tíu þá er þú vilt margfalda. Og svo margar einingar sem áskortir á tíu svo oft skalt þú hina minni töluna þá er þú vilt margfalda taka af tigum hennar. (GKS 1812, 4to, p. 14v)[10]

Algorismus adds this explanation as the last clarification example:

Og að þú skiljir þetta margfalda sjö og níu. Níu skortir einn á tíu, því tak þú eina sjö af sjötigum. Þá verða eftir þrír og sextigir það eru sjö sinnum níu. Að slíku skapi mátt þú aðrar tölur reyna. (GKS 1812 4to, 14v)[11]

In modern notation this can be written

$7 \cdot 9 = 10 \cdot 7 - 1 \cdot 7$

or more generally:

$a \cdot b = 10a - (10 - b)a \qquad (0 < a, b < 10).$

Two conclusions may be drawn from this explanation. First, the Latin text's presentation was not considered clear enough so that an example was needed. Second, the example demonstrates that the translator/transcriber was not fully confident with Hindu-Arabic digits, so he used words. Numerals do not have a consistent representation across manuscripts. The manuscript GKS 1812 4to uses words, the AM 544 4to Roman numerals, and in the youngest manuscript containing the whole treatise *Algorismus*, the AM 685 d 4to, some numbers in this particular example are written in the Hindu-Arabic numerals while others in words or Roman numerals.

4 A chapter in Algorismus on numbers related to the Elements

In addition to the calculation examples, a separate chapter is added to the translation. It is on the cubic numbers 8 and 27 and their intermediate numbers 12 and 18, and their relation to the Elements: Earth, Water, Air and Fire. This chapter does not exist in *Carmen*, and its content is unrelated to the bulk of *Algorismus* in modern understanding. Its introduction says:

Hver ferskeytt tala hefur tvær mælingar, það er lengd og breidd. En cubicus tala hefur þrenna mæling. Það er breidd og lengd og þykkt eður hæð. Og því kalla spekingar hvern sýnilegan líkama með þessi tölu saman settan að hann hefur saman þessa mæling þrenna. Með því að eilíf speki og einn guð vildi heiminn sýnilegan og líkamlegan skapa, þá setti hann fyrst tvær hinar ystu höfuðskepnur eld og jörð. Því að ekki má náttúrlega sýnilegt vera utan þær. Þar sem eldur gerir ljós og hræring. En jörð staðfesti og hald. En með því að þau hafa þrenna ójafna huiligleika og gagnstaðlega[12] *þá var náttúruleg nauðsyn að setja nokkuð milli þeirra það er samþykkti þeirra ósætti. Og sem fyrr er sagt að eldur og jörð og það allt sem líkamlegt er er með þrefaldri tölu er vér köllum cubicum saman sett þá ritum vér þessa*

---

[10] *Next you are to think how much the larger figure differs from ten, the one you want to multiply. And so many units as differ from ten so often you are to take the lesser number, the one you want to multiply, from its tens.*

[11] *So that you understand this multiply seven and nine. Nine differs by one from ten, therefore, take one seven from seventies. Then remain three and sixties, that is seven times nine. In that way you may try with other numbers.*

[12] Corrected from "gang staðlega" in GKS 1812 4to. The manuscripts AM 544 4to and AM 685 d 4to have "gagn staðlega", meaning "opposite", "contrary".

tvo cubus. Ritum vér jörðina þessa leið. Tvisvar sinnum tveir tvisvar, 2, 4, 8. En eldinn svo: þrisvar þrír þrisvar, 3, 9, 27. (GKS 1812 4to, 16r–16v)[13]

Thus Earth was assigned the numerical value $2 \cdot 2 \cdot 2 = 8$ and Fire $3 \cdot 3 \cdot 3 = 27$. As no single mediator existed between these two cubes, two proportional numbers were found by taking the square (4) of the root 2 of the smaller cube, multiplied by the root 3 of the larger cube ($4 \cdot 3$): 2, 4, 12. In the same way, the root of the smaller cube (2) was multiplied by the square (9) of the larger cube ($2 \cdot 9$): 3, 9, 18. These two numbers belong equally to the two previously mentioned cubes, 8 and 27, as 27 contains 18 and half of 18, and 18 contains 12 and half of 12.

Similarly, God arranged two elements between Fire and Earth: Air and Water. Water contains two attributes and two numbers from Earth and one attribute and one number from Fire. Air contains two attributes from Fire and two numbers, but one from Earth and one number. The four Elements are thus assigned numerical values: Earth, $2^3 = 8$; Water, $2^2 \cdot 3 = 12$; Air, $2 \cdot 3^2 = 18$; Fire, $3^3 = 27$. This puts the Elements in the correct order by lightness: Fire (27), Air (18), Water (12) and Earth (8). These numbers constitute a sesquialterate progression 8:12::12:18::18:27, or in general terms: n : (n + ½n). The text of *Algorismus* concludes by saying that this can be more perfectly understood from a figure later in the manuscript, called Cubus Perfectus (GKS 1812 4to, 16v).

The idea about the Elements has a strong relation to Plato's *Timaeus*, paragraphs 31b-32c. Calcidius translated the first part (to 53C) of *Timaeus* from Greek into Latin around the year 321 CE.

[31b] … And since it [the world] was rightly to be corporeal, visible, and tangible, and there is no perception of anything visible in the absence of fire, or of anything tangible in the absence of solidity, and no solidity without earth, god laid down fire and earth as the foundations of the world body … no two things cohere firmly and indissolubly without the binding force of a third [31c] … [32a] … if the body of the world were required to have only length and width but no solidity and were of the same sort as the surface of fully formed bodies, then one mean would suffice [32b] for the cohesion of it and its extreme parts. But as it is, since the world body required solidity, and the cohesion of solids involves never one but two means, the craftsman of the world accordingly inserted air and water between fire and earth, salubriously balancing the same elements so that the relationship between air and water would be the same as that between fire and air … And so from the four material elements here named [32c] he fabricated this splendid engine as visible, tangible, and bound together by a harmonious proportion in the equilibrium of its parts … (Calcidius, 2016, p. 49-51)

Comparing texts, originally written in different languages and brought together through translations from language to language, is intriguing and needs vigilance. The texts about

[13] *Every quadratic number has two measures, that is length and breadth. But a cubic number has three measures. That is breadth, length and thickness or height. And therefore wise men call each visible body composed by this number, that it has these three measures. As eternal wisdom and one God wanted to create the world visible and corporeal, he first set the two outmost elements, fire and earth. Because nothing can be naturally visible without them. As fire makes light and motion. But earth solidity and hold. But as they have three different and contrary sets of attributes and then there was a natural necessity to add something in between them that would agree their alienation. And as said previously that fire and earth and everything that is corporeal is combined by a triple number which we call a cube then we write these two cubes. We write the earth in this way. Twice two twice, 2, 4, 8. But the fire so: thrice three thrice, 3, 9, 27.*

the four elements in *Algorismus* and *Timaeus* bear remarkable resemblance. Here, the term “staðfesti” has been translated by “solidity”, the term used in the translation of *Timaeus*, referring to that the earth is a solid and firm body.

Calcidius provided an extensive commentary to his translation of *Timaeus*. In the commentary on the passages above, Calcidius discussed the analogy of the relations between the four Elements to continuous proportions, where the air would have “two powers of fire, its fineness and mobility, and one of earth, i.e. its compactness … for air is compact, fine and mobile”. Similarly, water would have “two powers of earth, i.e. its compactness and corporeality, and one of fire, i.e. its movement, and the substance of water will emerge, that being a body compact, corporeal and mobile. And thus between fire and earth from the coalescence of extremes air and water will arise, giving binding continuity of the world” (Calcidus, 2016, p. 153).

Calcidius (2016, pp. 139–145) had already discussed continuous proportions in relation to these items. However, he did not bring up the sequence 8 – 12 – 18 – 27. That sequence, however, appeared in a manuscript of Euclid’s *Elements*, the Vat. Gr. 190 (codex P), on p. 115v, see Fig. 4.1. The Vatican Euclid (Vat. Gr. 190, called P) is a version of a Greek text of Euclid’s *Elements* dating from the ninth century.

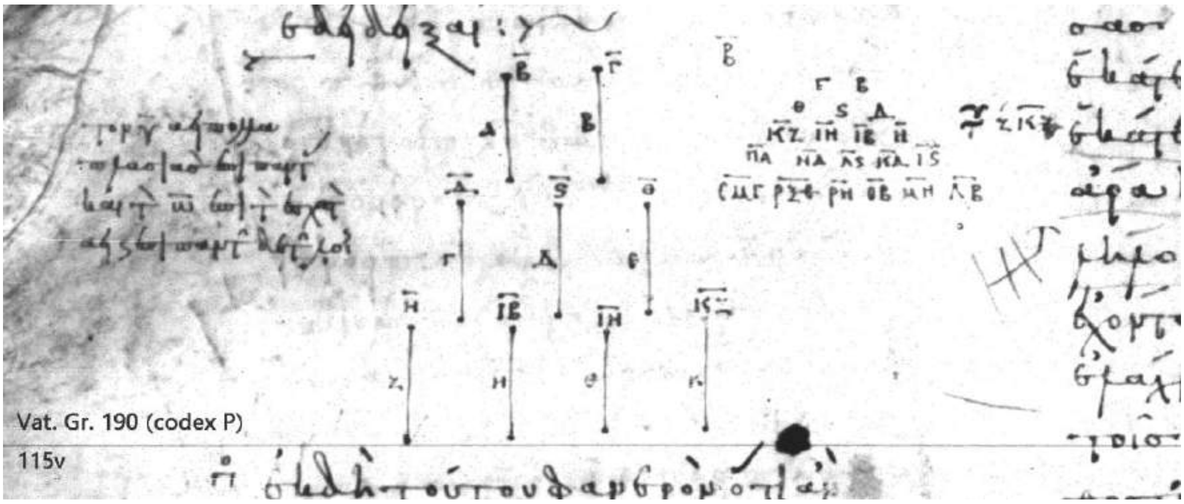


Figure 4.1: A diagram of continued proportions in the manuscript Vat. Gr. 190 (codex P), 115v.

The diagram shown in Fig. 4.1 appeared at the end of proposition VIII 2 in Euclid’s *Elements*. Proposition VIII 2 is the following:

> *To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio. (Euclid, 1956, p. 346)*

At the top of the vertical line segments in Fig. 4.1, marked A, B, Γ, Δ, etc., there are numbers, written in the ancient Greek number notation where alphabetical letters with a bar at the top denote numbers. Thus Ā = 1, and the B and Γ with bars on top of the uppermost vertical line segments denote 2 and 3. The first vertical line segment in the second row is marked Δ with a bar, denoting 4. Thus, in the top row the numbers are 2 and 3; in the middle row 4, 6, 9; and in the bottom row 8, 12, 18 and 27, the numbers appearing in *Algorismus* and in a number of medieval manuscripts, representing continuous proportions with the rate 3/2. On the right side of the diagram, another set of numbers appear, the first three rows with numbers, identical to those on the left, in a reversed order. The sequences thereafter continue up to 2 and 3 to the fifth power on each end, and the numbers in-between in continuous proportions.

5 Diagrams of the Elements

The manuscript AM 736 III 4to contains only a fragment of the treatise *Algorismus*. It does not contain the text on the Elements and their associated numbers. However, on a different leaf in the same manuscript a diagram of the four Elements is found together with their names and the texts "*bis bini tres* xii" [twice two three 12] associated to *Aqua*, Water, "*tres trium bis* xviii" [three thrice twice 18] to *Aer*, Air, and "*tres trium tres*" [three thrice three] to *Ignis*, Fire. The roundels to the right distribute three pairs of qualities: acuity (*acutus* above, *obtusus* below), density (*subtilis* above, *corpulentus* below), and capacity for motion (*mobilis* above, *immobilis* below), see Fig. 5.1. Water is associated with three qualities, being corporeal, soft and mobile. Similarly, Air is soft, mobile and light. It seems fair to conclude that we have here the Cubus Perfectus which is mentioned in the three complete copies of *Algorismus*.

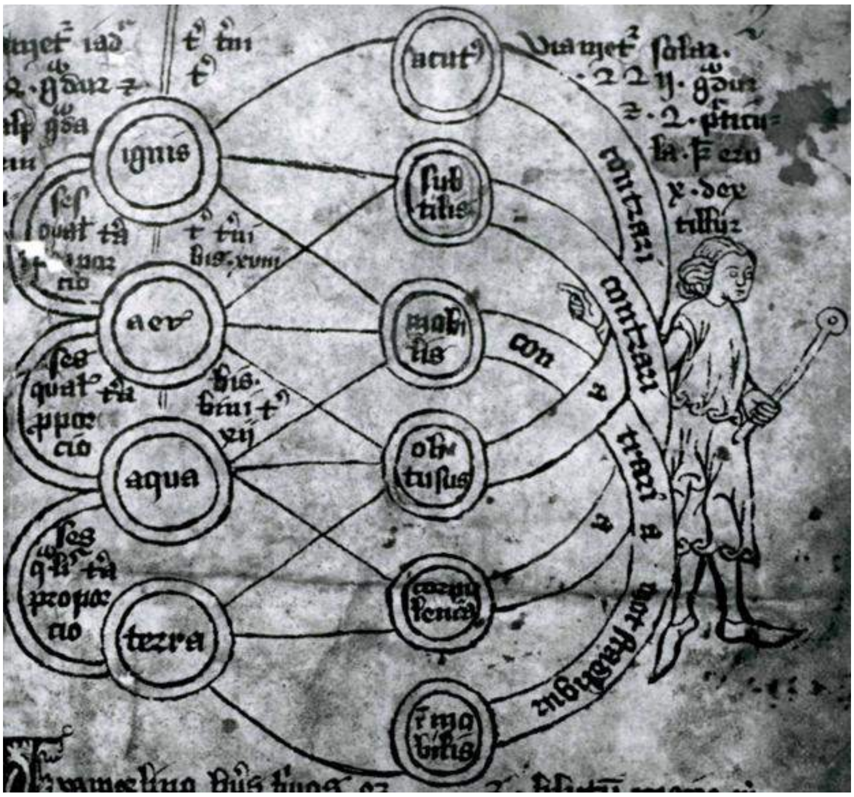


Figure 5.1: A diagram of the four Elements in the manuscript AM 736 III 4to, 2r.

Diagrams with the Elements and the four numbers exist in other foreign manuscripts on medieval cosmology, but those manuscripts are not related to *Algorismus*. For instance, the same sequence of proportions, 8, 12, 18 and 27, and Elements, *Ignis*, *Aer*, *Aqua* and *Terra*, and the same qualities appear on the right in St. John's College MS 17 (Oxford Digital Library),[14] see Fig. 5.2. A similar schema exists in an eleventh century manuscript of Boethius, Madrid Biblioteca nacional Vit. 20 fol. 54v (Bekken, 1986). It is also found in an anonymous treatise on cosmology in Bodleian Library Digby 83, fol. 3r.

[14] The Calendar and the Cloister – St. John's College MS 17, commentary.

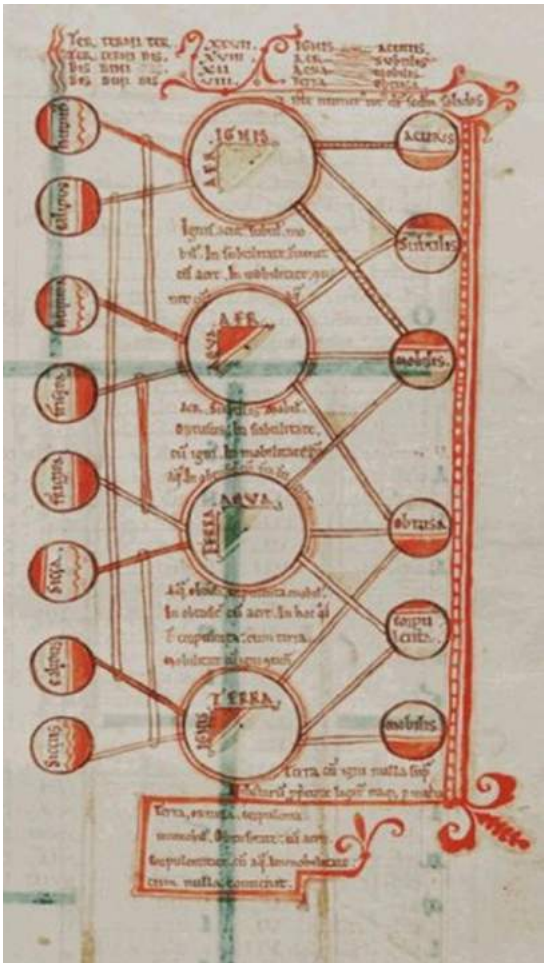


Figure 5.2: St. John's College MS 17, dated early 12th century.

6 Comparing the four manuscripts of Algorismus

The texts of *Algorismus* in the manuscripts AM 544 4to and GKS 1812 4to are identical in most respects, as is *Algorismus* in AM 685 d 4to, which however has added a 306-word long section, placed after the section on subtraction. It describes a method of halving a number, the fourth operation. This section is neither contained in *Carmen* nor in the manuscripts AM 544 4to and GKS 1812 4to, and is not discussed further in this paper. The AM 736 III 4to is only a fragment.

AM 544 4to, preserved in the manuscript collection *Hauksbók*, contains the oldest manuscript of the treatise, estimated to be written in the period 1302–1310, most likely in 1306–1308 (Karlsson, 1964). The text is divided into chapters bearing headings. Numbers are written using Hindu-Arabic numerals in the introduction and in the additions to *Carmen* with examples of place value notation and even and odd numbers, shown earlier. Numbers, however, are mainly written in Roman numerals, until in the last chapter on the Elements, which does not originate in *Carmen de Algorismo,* and where Hindu-Arabic numerals are used.

The part of **GKS 1812 4to** containing *Algorismus* is estimated to be written in 1300–1400 (*A Dictionary of Old Norse Prose – Indices*, 1989, p. 26). There are no chapter headings. Numbers are mainly written using words as in *Carmen*, exceptionally in Roman numerals. Hindu-Arabic numerals are only used in the first additions to *Carmen*, as is done in AM 544 4to, and in the chapter on the Elements.

AM 685 d 4to, is dated to 1450–1500 (*A Dictionary of Old Norse Prose – Indices*, 1989, p. 26). It has no chapter headings. Numbers are written alternately in words, Roman numerals, and Hindu-Arabic notation which is the most common. Finnur Jónsson states that the text of *Algorismus* in AM 685 d 4to is the most error free of the four texts, basing this conclusion on various spelling examples (Jónsson, 1892–1896, p. cxxxi). Furthermore, this text is the most concise of the four texts as it is often contracted, preserving a correct meaning. The text in AM 685 d 4to is also correct where other texts have an error on the origin of one half (Jónsson, 1892–1896, p. 419), called *semiss*, coming up after halving an odd number, which indicates that one of the transcribers of AM 685 d 4to understood the treatise well.

AM 736 III 4to is estimated to origin around 1550 (*A Dictionary of Old Norse Prose – Indices*, p. 26). It contains only a fragment of the text of *Algorismus*, a section on root extraction, in addition to the leaf with the diagram of the Elements in Fig. 5.1.

The adaptations made to *Carmen de Algorismo* to create *Algorismus* suggest that *Algorismus* served a role in introducing the use of Hindu-Arabic numerals in the Norse societies. In the oldest manuscript of *Algorismus*, AM 544 4to, Roman numerals are used to explain the text, or plain words are used as in *Carmen*. The use of Roman numerals indicates that the transcriber needed to shorten the text and that he was not used to Hindu-Arabic numerals.

Plain-word number notation is dominant in GKS 1812 4to. The youngest whole manuscript, AM 685 d 4to, rarely has Roman numerals, while words and Hindu-Arabic numerals are used interspersed.

6.1 Manuscript comparison - methodology

When reading the four manuscripts of *Algorismus* it is apparent that they are quite similar; sentence structure and phrasing suggests that they all derive from the same prototype. The same text insertions and deletions are made in all four manuscripts to *Carmen de Algorismo*, exemplifying that these are not different translations.

How similar are these manuscripts? Numerical methods were used to compare the manuscripts, comparable to methods used extensively in comparative linguistics (Fox, 1995) and in gene and protein comparison. In the following comparison, difference in spelling is generally not revealed as the texts of all the manuscripts have been rewritten in modern Icelandic.

The four texts were aligned using the computer programme ClustalW (Thompson, Higgins and Gibson, 1994), and a weighted number of mismatches between the manuscripts was computed. As ClustalW is designed to align protein sequences it takes as input sequences from the twenty letter alphabet of protein sequences. The Icelandic version of the Latin alphabet is larger than twenty letters, so each letter was mapped to two letters in the alphabet of protein sequences. ClustalW was then used to align the texts and the text was mapped back to the Latin alphabet. The alignment was then corrected manually, considering in particular word reorder and different forms of the imperative.

Mismatches between the manuscripts were counted and classified into three distinct classes; *single character mismatches, word reorders* and *word mismatches*.

- *Single character mismatches* were defined as:

    - Identical spelling apart from a single character difference.
    - Mismatches in writing style of the numerals; Hindu-Arabic, Roman or spelled out.
    - Mismatches in the writing of the imperative, e.g. *tak þú* - *taktu*.
- *Word reorders* were defined as parts of the manuscripts where the order of two or more words had been reordered.
- *Word mismatches* were all other types of differences such as word insertion, missing words or a different word being used.

The weighted distance between the manuscripts was used to infer the phylogeny of the manuscripts, using the assumption that it is unlikely that the same change is made more than once. One may also assume that each transcriber is equally likely to cause a distinction. Finally, a simple programme was written to count the number of differences.

6.2 Results

The manuscripts are different in length. In the following, a section in AM 685 d 4to of length 306 words, not extant in the other manuscripts, has been removed. The lengths are:

Manuscript	Words #	Characters #
GKS 1812 4to	2986	15174
AM 544 4to	2960	15110
AM 685 d 4to	2902	14772
AM 736 III 4to	630	3323

Table 6.1: No. of words and characters in the four manuscripts of *Algorismus*.

That AM 685 d 4to has fewest words of the complete manuscripts suggests that the transcriber(s) of AM 685 d 4to sometimes shortens the text.
The following weights of mismatches were used:

Word mismatches:	1.00 point
Word reorders:	0.25 point
Single character mismatches:	0.25 point

Results from counting mismatches between the three complete texts in AM 685 d 4to, AM 544 4to, and GKS 1812 4to, were:

Manuscripts	GKS 1812 4to	AM 544 4to	AM 685 d 4to
	0.00	123.25	264.50
AM 544 4to	123.25	0.00	261.00
AM 685 d 4to	264.50	261.00	0.00

Table 6.2: No. of mismatches between the three complete manuscripts of *Algorismus.*

The shortest distance between two manuscripts is between AM 544 4to, and GKS 1812 4to, 123.25 mismatches by 2986 words, or 4.1%.
The greatest distance between two manuscripts is between AM 685 4to, and GKS 1812 4to, 264.5 mismatches by 2986 words, or 8,9%.
The parts of the manuscripts that all have in common, i.e. the part also found in AM 736 III 4to, were compared separately.

The results were:

Manuscripts	GKS 1812 4to	AM 544 4to	AM 685 d 4to	AM 736 III 4to
GKS 1812 4to	0.00	26.00	55.00	57.75
AM 544 4to	26.00	0.00	52.00	57.25
AM 685 d 4to	55.00	52.00	0,00	73.25
AM 736 III 4to	57.75	57.25	73.25	0.00

Table 6.3: No. of mismatches in the part common to all manuscripts of *Algorismus*.

The distance of AM 736 III 4to is greatest from AM 685 d 4to, while it is closest to AM 544 4to, and nearly equally close to GKS 1812 4to. Clearly, AM 544 4to and GKS 1812 4to are more close to each other than the other two, which are also different from each other.

In this counting of mismatches the ratio 1 : 0,25 or 4 : 1 between *word mismatches* and other mismatches was used. Counting was also done using the ratio 3 : 1 and lead to comparable conclusions.

Fig. 6.1 exhibits the relation between the different copies of *Algorismus*. A matrix was made according to the distances between the four manuscripts, from which was constructed a phylogenetic tree with distances similar to the distances in the distance matrix. The diagram was made by the programme ATV (Zmasek and Eddy, 2001).

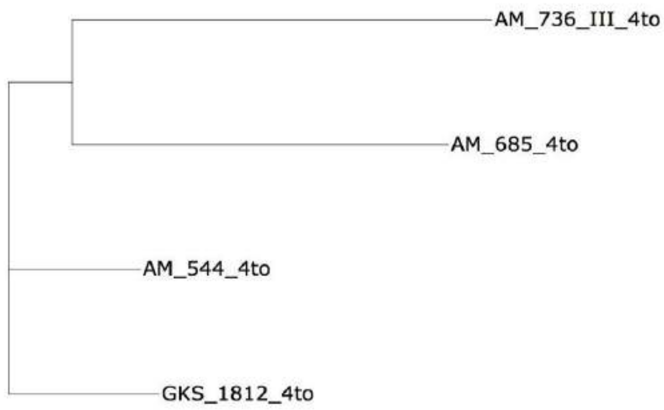


Figure 6.1. A phylogeny of the copies of the part of *Algorismus* in common to the manuscripts AM 736 III 4to, AM 685 d 4to, AM 544 4to, and GKS 1812 4to, made by the programme ATV.

The phylogeny may be interpreted such that the manuscript AM 544 4to, contains the most original copy of the treatise, and that the copy in GKS 1812 4to is closest to it. The copies in the manuscripts AM 736 III 4to and AM 685 d 4to are partly drawn from the same stem, but are further from the origin, in particular AM 736 III 4to.

Comparing copies of *Algorismus* in GKS 1812 4to and AM 544 4to we see that both copies were written about +/-50 years after Finnur Jónsson’s estimated translation date, before 1270. The difference in the number of words in the two copies are 26 words where GKS 1812 4to is the longest. Of them, 18 can be ascribed to the differently expressed imperative form of the verbs, such as *skalt þú – skaltu*. The contracted form is more common in AM 544 4to while the separated form is the norm in GKS 1812 4to.

Out of the points for mismatches, 123.25, 11 may be explained by different form of the imperative and different expression of numbers. Both may be interpreted as efforts to save the precious vellum. Most other mismatches may be ascribed to personal preferences of the scribes or simple mistakes.

It is not unreasonable to conclude that the two versions of *Algorismus* are copies of the same origin, possibly the first or second copies of the original version. The two versions of *Algorismus* are similar in length. Both contain several same errors, for example in the section about doubling:

> *... en ef semiss stendur yfir uppi í ysta stað þá legg við einn því að þar var áður <u>jöfn</u> tala er í helminga var skipt.(GKS 1812 4to, 14r)*[15]

Here, *jöfn*, *even* [number] should be replaced by *ójöfn*, *uneven*/*odd* [number]. This error is not found in the copy contained in the fifteenth century manuscript AM 685 d 4to.

7 Discussion

We have explored the thirteenth century treatise *Algorismus*, written in the Old-Norse language, its content of arithmetic studies and cosmology, and compared its manuscripts. *Algorismus* was written in the transition period when Hindu-Arabic decimal place value numeral notation and associated arithmetic methods were being introduced in Europe. The four different manuscripts of *Algorismus*, written in the time span from early fourteenth century until mid-sixteenth century, reveal that the new style of number notation gradually entrenched. We may wonder how large role *Algorismus* played in that development.

What motivated the Old Norse people to translate *Carmen de Algorismo*? Certainly, they had to count their belongings and assets, e.g. for taxation, but they could have done so with the Roman numerals they knew. Writing manuscripts was an integral part of the Christian monastic culture. The reason may have been an aspiration to belong to the European cultural world. The Old-Norse-speaking population in Iceland and Norway was never large compared to populations of millions in the centres of the Christian world in the present Italy and France. Producing writings in the vernacular was an important factor in creating a common culture for this small group of people.

The additions in *Algorismus* to its original, *Carmen de Algorismo*, bear witness to a desire for learning, to understand the text as demonstrated by insertions for clarification. Comparison of the four copies of *Algorismus* of different age reveals that people continued to work on understanding the text and gradually began to use the convenient Hindu-Arabic number notation.

But it took time. According to the phylogeny and other considerations, manuscript AM 544 4to was not the original of *Algorismus,* suggesting that *Algorismus* may originally have been written in the second half of the thirteenth century, as proposed by Finnur Jónsson, or about 200 years before AM 685 d 4to, and possibly up to 300 years before AM 736 III 4to. *Algorismus* therefore played an important role in Icelandic culture until the era of printing, when printed books began to spread much more rapidly between countries than manuscripts.

Iceland was originally an independent society in close contact to Norway, but from 1397 it belonged to the Danish realm until 1944. It lagged gradually behind other European countries in educational respect. *Algorismus* appears in history whenever mathematics education was revived, serving as a monument of the proud past, when

---

[15] *... but if a semiss* [symbol for one half] *is placed above in the farthest place then add one as there was earlier an <u>even</u> number divided into halves.*

Icelanders kept up with the latest global knowledge. Even the most distinguished Icelandic scholars continued to refer to *Algorismus* until the nineteenth century (see e.g. Gunnlaugsson, 1865, p. 4), paying respect to the time when Icelanders were familiar with the latest mathematical knowledge in the world and translated it to their own language.

REFERENCES

Printed and Electronic Sources

A Dictionary of Old Norse Prose – Indices. (1989). Copenhagen: The Arnamagnæan Commission.

al-Kwārizmī, Muhammad Ibn Mūsā. (1992). Dixit Algorizmi. In A. Allard (Ed.) *Le Calcul Indien, (Algorismus) / Muhammad Ibn Musa Al-Khwarizmi ; histoire des textes, édition critique, traduction et commentaire des plus anciennes versions latines remaniées du XIIe siècle par André Allard*, pp. 1–22. Paris: Societé des Études Classiques. Librairie scientifique et technique.

Allard, A. (1992). Les algorismes et leur auteurs, In A. Allard (Ed.) *Le Calcul Indien (Algorismus)*, pp. i-xxiii Paris: Societé des Études Classiques. Librairie scientifique et technique.

Beaujouan, G. (1954). D'Alexandre de Villedieu à Sacrobosco. *Homenaje à Millás-Vallicrosa*, Vol. 1, pp. 106–111. Barcelona: Consejo superior de investigaciones científicas.

Bekken, O. B. (1986). *On the Cubus Perfectus of the Algorismus in Hauksbok. Skrifter 1986:2*. Kristiansand: Agder Distrikthøgskole, fagseksjon for matematikk.

Bekken, O. B., & Christoffersen, M. (1985). *Algorismus i Hauksbok. Skrifter 1985:1*. Kristiansand: Agder Distrikthøgskole, fagseksjon for matematikk, fagseksjon for norsk.

Bjarnadóttir, K. (2003). Algorismus. In O. B. Bekken, & R. Mosvold (Eds.) *Study the Masters. The Abel-Fauvel Conference. Gimlekollen Mediacentre – Kristianssand, June 12–15, 2002* (pp. 99-108). Gothenburg: Nationellt Centrum för Matematikutbildning, NCM.

Bjarnadóttir, K. (2004). Fornt stærðfræðirit í íslenskum handritum. *Netla, veftímarit um uppeldi og menntun*. Available at http://netla.hi.is/greinar/2004/001/index.htm,

Bjarnadóttir, K. (2007). *Mathematical Education in Iceland in Historical Context. Socio-economic demands and influences*. IMFUFA tekst. Roskilde: Roskilde University electronic library. Available at https://rucforsk.ruc.dk/ws/files/2051540/IMFUFA_456.pdf

Bjarnadóttir. K., & Halldórsson, B. V. (forthcoming). Algorismus: Hindu-Arabic arithmetic in GKS 1812 4to. Reykjavík: The Árni Magnússon institute for Icelandic studies.

Calcidius (2016). *On Plato's Timaeus. Calcidius*. Edited and translated by John Magee. Cambridge, Massachusetts: Harward University Press.

Cleasby, R. (1957). *An Icelandic – English dictionary*, initiated by Richard Cleasby, subsequently revised, enlarged and completed by Gudbrand Vigfusson. Oxford: Clarendon Press.

Euclid (1956). *The Thirteen Books of the Elements*, vol. 2. Translated with introduction and commentary by Sir Thomas L. Heath. Second edition unabridged. New York: Dover.

Fox, A. (1995). *Linguistic reconstruction: An introduction to theory and method.* Oxford: Oxford University Press.

Guðmundsson H. (1967). Um Kjalnesingasögu: nokkrar athuganir. *Studia Islandica 26*. Reykjavík: Menningarsjóður.

Gunnlaugsson, B. (1865). *Tölvísi*. Reykjavík: Hið íslenzka bókmenntafélag.

Halliwell, J. O. (Ed.) (1841). Carmen de algorismo. In *Rara Mathematica; or a collection of treatises on the mathematics and subjects connected with them from ancient inedited manuscripts* (pp. 73–83). London: Samuel Maynard.

Hankel, H. (1874). *Zur Geschichte der Mathematik in Alterthum und Mittelalter*. Leipzig: Teubner.

Jónsson, F. (Ed.) (1892–1896). *Hauksbók. Udgiven efter de arnamagnæanske Håndskrifter no. 371, 544 og*

675, 4° samt forskellige Papirhåndskrifter. Copenhagen: Det Kongelige Nordiske Oldskrift-Selskab.
Karlsson, S. (1964). Aldur Hauksbókar. *Fróðskaparrit 13* (1964), 114-121.
Katz, V. J. (1993). *A history of mathematics. An introduction.* New York: Harper Collins.
Sacrobosco, J. (1897). *Petri Philomeni de Dacia in algorismum vulgarem Johannis de Sacrobosco commentarius*. Ed. by Maximilianus Curtze. Copenhagen: A.F. Høst & fil.
Seppala-Holzman, D. (2007). Ancient Egyptians, Russian peasants foretell the digital age. *Mathematics teacher 100*(9), 632-635.
Steele, R. (Ed.) (1988). Carmen de algorismo. In *The Earliest Arithmetics in English* (pp. 72–80). Millwood, N.Y.: Kraus Reprint.
Thompson, J., Higgins, D. G., & Gibson, T. J. (1994). CLUSTAL W: improving the sensitivity of progressive multiple sequence alignment through sequence weighting, position-specific gap penalties and weight matrix choice. *Nucleic Acids Research 22*(22), 4673-4680.
Tropfke J. (1980). *Geschichte der Elementarmathematik*. 4. Auflage. Band 1. Arithmetik und Algebra. Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmut Gericke. Berlin: Walter de Gruyter.
Zmasek, C. M., & Eddy. S. R. (2001). ATV: display and manipulation of annotated phylogenetic trees. *Bioinformatics 17*(4), 383–384.

Manuscrips

Den Arnamagnæanske Samling. Copenhagen.
AM 544 4to, 90r–93r.
AM 685 d 4to, 24v–29r.
AM 736 III 4to, 2r, 4r–4v.
The Árni Magnússon Institute for Icelandic Studies, manuscript collection, Reykjavík.
GKS 1812 4to, 13v–16v.
Bodleian Library, University of Oxford.
MS. Auct. F.5.29.
Oxford Digital Library.
The Calendar and the Cloister – St. John's College MS 17. http://digital.library.mcgill.ca/ms-17/
The Vatican library.
Vat. Gr. 190 (codex P) 115v.

PROBLEMS AND METHODS IN ELEMENTARY GEOMETRY, ACCORDING TO JULIUS PETERSEN

René GUITART
Université Paris 7 Diderot, France
rene.guitart@orange.fr

ABSTRACT

According to Julius Petersen in 1866 (Petersen, 1866/1927), to do geometry is to examine a figure, by propositions which either are theorems (a certain well constructed figure satisfies certain conditions) or are problems of construction (we require a figure so constructed, that certain given conditions are satisfied). So here theorems and problems do not mean exactly the same thing as in Euclid and Proclus: they are thought in duality.

By using straight edge and a pair of compasses, Petersen solved problems of construction of a point satisfying several conditions. Two ancient methods are usual for that: analysis by angles and lengths, with the tool of the three cases of equality of triangles taken as atomic properties to be combined; and analysis by equations on lines, with the tool of equational calculus. These methods are rather statics. In contrast with these methods, thanks to the development of geometrical transformations especially during the 19th century, Petersen proposed a more dynamical method by determination of locus of points, and by transformations of figures. His method turns on three rules: 1. To draw a figure representing the solved problem. 2. Imagine one of the given conditions for the figure in quest of removed, and seek the loci of the points of the figure thus rendered indeterminate. 3. Of the drawn figure try to form another, in which the relations between the given and sought elements are more convenient.

It is not completely new to draw a figure, to observe a locus of points as a curve, or to modify a figure: what is truly new here is the explicit determination of a method with this design of rules. And furthermore, Petersen proved that many of the problems previously solved by examinations of triangles or equations can be solve in this way. Finally, the very point of the method is that the book is not ordered as a theory, but it is ordered according to problems and as a practice of invention, according to difficulties of constructions.

According to works of É. Barbin (2001), J. Lützen *et al* (1992), G. Moussard (2015), we will consider Petersen's book among other books for teaching geometry in the 19th century, showing how Petersen's rules are in action in some problems.

This book is a splendid tool for teaching, noticeably for a self-taught-person, because it is truly a method: not only you know solutions, but also you are learned how to find them. So, it can be considered as a method to teach and learn geometry.

REFERENCES

Barbin, É. (2001). Qu'est-ce que faire de la géométrie ? *Repères-IREM*, *43*, 59-82.

Guitart, R. (1999). *La pulsation mathématique*. Paris: L'Harmattan.

Lützen, J., Sabidussi, G, & Toft, B. (1992). Julius Petersen 1839-1910: A biography. *Discrete Mathematics*, *100*(1-3), 9-82.

Moussard, G. (2015). *Les notions de problèmes et de méthodes dans les ouvrages d'enseignement de la* *géométrie en France (1794-1891).* Thèse dirigée par É. Barbin, Université de Nantes, 636p.

Petersen, J. (1866/1927). *Methods and theory for solution of problems of geometrical constructions applied to 410 problems* (first editions: 1866, 1879; reprinted 1927). New York: G. E. Strechert and Co.

INFINITE SUMS AND THE CALCULATION OF π, AS PRESENTED BY THE SWEDISH MATHEMATICIAN ANDERS GABRIEL DUHRE IN THE EARLY 18TH CENTURY

Johanna PEJLARE
Mathematical Sciences, Chalmers University of Technology and University of Gothenburg,
SE-412 96 Gothenburg, Sweden
pejlare@chalmers.se

ABSTRACT

Anders Gabriel Duhre, an important mathematician and mathematics educator in Sweden during the 18th century, contributed with two textbooks in mathematics, one in algebra and one in geometry. Among others, he treats infinitesimals based on Nieuwentijts' theories from *Analysis infinitorum* and infinite sums based on Wallis' method of induction from *Arithmetica infinitorum*. Based on these results, Duhre develops an ingenious method to determine the area enclosed by curves by constructing a corresponding curve. He applies his method to the circle in order to find an expression of π as an infinite series. The series he finds is a modified version of the Gregory-Leibniz' series. In the present paper we consider in detail Duhre's presentation in order to further investigate the influence upon him as well as his influence on the Swedish mathematical society of his time.

1 Introduction

The Swedish mathematician and mathematics educator Anders Gabriel Duhre (c.1680–1739) was an important and influential person in the Swedish mathematical society in the early 18th century (Rodhe, 2002). He studied mathematics at Uppsala University, Sweden, and for some time he was a student of the Swedish scientist, inventor and industrialist Christopher Polhem (1661–1751) at his school *Laboratorium Mechanicum* in Stjärnsund. For some years Duhre taught mathematics to engineering students at Bergskollegium (a central agency in the mining industry) and to prospective officers at the Royal Fortification Office in Stockholm. In 1723 he opened his own school, *Laboratorium Mathematico-Oeconomicum*, outside Uppsala, where theoretical and practical subjects were taught to young boys (Hebbe, 1933). Of particular interest is that mathematics was taught in this school; Duhre had knowledge of mathematics that was not yet taught at the university, and students at the university turned to him to learn more on modern mathematics. Among his students were several of the Swedish mathematicians to be established during the 1720s and 1730s (Rodhe, 2002). Duhre taught in Swedish and early on planned to write mathematical textbooks in Swedish in order to introduce the Swedish youth to new and modern mathematics.

Duhre contributed with two textbooks in mathematics – one in algebra and one in geometry. Both were based on his lecture notes from his teaching at Bergskollegium and the Royal Fortification Office. The first book, *En Grundelig Inledning til Mathesin Universalem och Algebram* ("A thorough introduction to universal mathematics and algebra"), was edited by Georg Brandt and published in 1718. In this book, modern algebra based on Descartes' notation is presented, as well as examples from Newton's, Wallis' and Nieuwentijt's theories from the end of the 17th century. For example, he treats infinitesimals based on Nieuwentijt's theory as presented in *Analysis infinitorum* (1695) and utilizes Wallis' method of induction, as presented in *Arithmetica infinitorum* (1656), to determine the quotient of infinite series. In his second book, *Första Delen af en Grundad Geometria* ("The first part of a founded geometry"), published in 1721, Duhre takes advantage of the theories he presented earlier in his book on algebra. Of particular interest is his use of algebra in the geometrical context (Pejlare, 2017).

In this paper, we will consider Duhres' utilization of infinitesimals and infinite sums to determine the quotient between the circumference and the diameter of a circle, in order to find π expressed as an infinite series. We will first give a short introduction to Nieuwentijt's *Analysis infinitorum* and his utilization of infinitesimals, before we consider Duhre's interpretation of Nieuwentijt's work. Thereafter we will consider Wallis' *Arithmetica infinitorum* and how Duhre utilizes his method of induction to determine the quotient of infinite series. Following that, we will consider Duhre's method to find the area enclosed by curves. Finally, we will consider how Duhre utilizes this method on a circle and how he determines an expression for π.

2 Infinitesimals in Nieuwentijt's *Analysis infinitorum*

The Dutch philosopher and mathematician Bernard Nieuwentijt (1654–1718) is, in particular, known for his critique on the foundations of Leibniz' infinitesimal calculus. In 1695 he published *Analysis infinitorum*, a book "written by a beginner for beginners"[1] on elementary infinitesimal calculus. This book is primarily of a didactic character; he attempted at presenting mathematics in a systematic way as a coherent unit (Vermij, 1989). In the prologue he presents three definitions and two axioms which enable him to deduce rules for calculating with the infinite and infinitesimal quantities through more than 50 lemmas. In the chapters following the introduction, these lemmas lead to the propositions on infinitesimal calculus.

For Nieuwentijt, a quantity is infinitesimal if it is smaller than any arbitrary given quantity and it is infinite if it is greater than any arbitrary given quantity. The word infinitesimal is however not used in the definitions, axioms or lemmas. Instead, Nieuwentijt uses the expression "datâ minor" which can be translated into "the given smallest". Of central importance is his first axiom:

> Anything that when multiplied, however many times, does not equal another given quantity, however small, cannot be considered a quantity, geometrically it is absolutely *nothing*.[2]

The main peculiarity of Nieuwentijt's approach to infinitesimals is represented in Lemma 10, where it is stated that if an infinitesimalquantity is multiplied by an infinitesimal quantity, then the product is zero or nothing. The product of two infinitesimal quantities, or "the infinite small of the infinite small", can be interpreted as Leibniz' second differential. However, whereas Nieuwentijt considered squares of infinitesimals to be equal to zero, this is generally not the case with Leibniz' differentials (Mancosu, 1996).

---

[1] "Tyroni scriptum tyronibus" (Nieuwentijt, 1695, præfatio).

[2] "Quicquid toties sumi, hoc est per tantum numerum multiplicari non potest, ut datam ullam quantitatem, ut ut exiguam, magnitudine suâ æquare valeat, quantitas non est, sed in re geometricâ merum *nihil*" (Nieuwentijt, 1695, p. 2).

3 Infinitely small quantities in Duhre'stextbook on algebra

In Chapter XXVI of his book on algebra, Duhre presents an interpretation of the prologue of Nieuwentijt's *Analysis infinitorum* (1695). An infinitely small quantity is defined by Duhre as:

> If a *quantity* is divided by an infinitely big number, one should consider the received *quotient* to be infinitely small; it is something that is smaller than the smallest *quantity* that can ever be given.[3]

Thus, according to Duhre, if $\mathfrak{D}$ is an infinitely big number then the quotient $\frac{a}{\mathfrak{D}}$ is infinitely smaller than the quantity a. Duhre considers the nature of an infinitely big number to be that it is bigger than every given number and that it thus can be seen as "ceaselessly growing with no return".[4]From this it follows that $\frac{a}{\mathfrak{D}}$ is smaller than the smallest quantity that can ever be given. Duhre gives a proof by contradiction that $\frac{a}{\mathfrak{D}}$ really is "smaller than the smallest": if c is a quantity that is smaller than $\frac{a}{\mathfrak{D}}$ then the given quantity a is bigger than $\mathfrak{D}c$ and the quotient $\frac{a}{c}$ is bigger than the infinitely big quantity $\mathfrak{D}$, but this "contradicts all truth".[5] Therefore,$\frac{a}{\mathfrak{D}}$must be smaller than the smallest quantity, i.e., an infinitely small quantity.

The arguments above show that handling the infinite is problematic. Duhre treats the infinite as a fixed number, but this is in conflict with his earlier statement that an infinite number grows ceaselessly. Also, it seems easier to accept the infinitely big than the infinitely small, since the existence of the infinitely small is proven with the help of a given existence of the infinitely big.

After introducing infinitely small quantities, Duhre continues with 14 lemmas with rules for calculating with them; 10 of these are also found in Nieuwentijt's *Analysis infinitorum*. Among Duhre's lemmas we find, among others, that the sum of two infinitely small quantities is an infinitely small quantity (Lemma 1) and that the product of any number and an infinitely small quantity is an infinitely small quantity (Lemma 3). Of great importance for his later presentation on infinite sums is Lemma 4, which corresponds to Nieuwentijt's Lemma 10:

> If an infinitely small part $\frac{a}{\mathfrak{D}}$ is either *multiplied* by itself or by another infinitely small part $\frac{d}{\mathfrak{D}}$; then the received *product* $\frac{aa}{\mathfrak{D}\mathfrak{D}}$ or $\frac{ad}{\mathfrak{D}\mathfrak{D}}$ is nothing or no *quantity*.[6]

Thus, Duhre, just as Nieuwentijt, considers the square of infinitely small quantities to be equal to zero. In the proof of this lemma Duhreuses Nieuwentijt's first axiom: If the product of two infinitely small quantities is multiplied by an infinite number, this will be equal to an infinitely small quantity, i.e., $\frac{\mathfrak{D}\times aa}{\mathfrak{D}\mathfrak{D}} = \frac{aa}{\mathfrak{D}}$ and $\frac{\mathfrak{D}\times ad}{\mathfrak{D}\mathfrak{D}} = \frac{ad}{\mathfrak{D}}$, and since something multiplied by an

---

[3] "Om en föreställd *quantitet* hålles före wara fördehlad utaf ett oändeligen stort tahl; bör man anse then ther af komna *quotienten* för oändeligen lijten thet är för en ting som är mindre än then allerminsta *quantitet* som någonsin kan gifwas" (Brandt, 1718, p. 212).

[4] "[…] ouphörligen växande utan någon återvända" (Brandt, 1718, p. 213).

[5] "[…] stridande emot all sanning" (Brandt, 1718, p. 213).

[6] "Om en oändeligen lijten dehl $\frac{a}{\mathfrak{D}}$, antingen warder *multiplicerad* med sig sielf eller med någon annan oändeligen lijten dehl $\frac{d}{\mathfrak{D}}$; at then ther af komna *producten*$\frac{aa}{\mathfrak{D}\mathfrak{D}}$ eller $\frac{ad}{\mathfrak{D}\mathfrak{D}}$ måtte wara alsintet eller ingen *quantitet*" (Brandt, 1718, p. 214).

infinite number is equal to an infinitely small number then this something is not a quantity and geometrically is nothing.

In this proof Duhre does not seem to have a problem handling the infinite; it is no problem for him to shorten the expression with the infinitely big number $\mathfrak{D}$. He uses Lemma 4 in Lemma 14 where he deals with how infinitely small quantities can be handled in equations. He concludes that in an equation involving infinitely small quantities, the infinitely small quantities can be omitted, since, if the equation is divided by an infinitely big number $\mathfrak{D}$, then it follows from Lemma 4 that these can be considered as nothing. Algebraically this lemma can be interpreted as $x + \frac{a}{\mathfrak{D}} = x$ since $\frac{x}{\mathfrak{D}} + \frac{a}{\mathfrak{D}\mathfrak{D}} = \frac{x}{\mathfrak{D}}$.

4 Wallis' *Arithmetica infinitorum*

After considering the introduction of Nieuwentijt's *Analysis infinitorum*, Duhre, in Chapter XXVII of his book on algebra, proceeds with studying John Wallis' (1616–1703) *Arithmetica infinitorum* from 1656. *Arithmetica infinitorum* was an important text in the 17th century, in particular regarding the transition from geometry to algebra and regarding infinite series (Stedall, 2005). For example, Isaac Newton (1642–1727) was influenced by Wallis in his work towards integral calculus. Introducing new methods and concepts, Wallis' purpose was to find a general method of quadrature, i.e., finding the area enclosed by curves, or rather the ratios of those areas to inscribed or circumscribed rectangles. He achieved this by drawing together ideas from René Descartes' (1596–1650) algebraic geometry and Bonaventura Cavalieri's (1598–1647) theory of indivisibles. Wallis' results were based on the summation of indivisibles or infinitesimal quantities, where an indivisible can be considered to have at least one dimension equal to zero, as for example a line or a plane, while an infinitesimal is considered to have an arbitrarily non-zero width or thickness. Wallis was however not concerned with the distinction between indivisibles and infinitesimals and generally spoke of infinitely small quantities.

In order to find the area enclosed by curves, Wallis reduced the geometric problem to the summation of arithmetic sequences (Stedall, 2004). Two important mathematical methods he developed were *induction* and *interpolation*. Wallis' method of induction relied on intuition; he believed that if a pattern was established for a few cases then it could be assumed to continue indefinitely. Also, in his method of interpolation he relied on intuition; for example, he assumed continuity regarding sequences of numbers in order to interpolate intermediate values. One example of this is when he used his method of interpolation between the triangular numbers 1, 3, 6, 10 … Another example of interpolation is when he, in Proposition 191, found the ratio of a square to an inscribed circle: $\frac{4}{\pi} = \frac{3\times3\times5\times5\times7\times7 etc.}{2\times4\times4\times6\times6\times8 etc.}$.

5 Infinite sums in Duhre's textbook on algebra

We now turn our attention to Duhre's textbook on algebra again. We will here only consider those parts when Duhreuses Wallis' method of induction in order to deal with infinite sums. Duhre begins Chapter XXVIIby determining that the proportion of the sum of infinitely many squares with the roots 1, 2, 3, 4, 5 et cetera to the *summan totidem terminorum maximo æqualium* equals the proportion of 1 to 3. The *summan totidem terminorum maximo æqualium* is explained to be "the sum of the greatest term as many times as there are terms in the progression"[7]. Thus, in modern notation the proportion to be determined can be interpreted as:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^2}{(n+1)n^2} = \frac{1}{3}$$

Duhre proves this proportion using Wallis' method of induction, as presented in *Arithmetica infinitorum*. To do this, he first examines the proportion when *n* equals 1, 2, 3, 4, and 5 in the expression above:

$$\frac{0+1}{1+1} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{0+1+4}{4+4+4} = \frac{1}{3} + \frac{1}{12}$$

$$\frac{0+1+4+9}{9+9+9+9} = \frac{1}{3} + \frac{1}{18}$$

$$\frac{0+1+4+9+16}{16+16+16+16+16} = \frac{1}{3} + \frac{1}{24}$$

$$\frac{0+1+4+9+16+25}{25+25+25+25+25+25} = \frac{1}{3} + \frac{1}{30}$$

Duhre examines the pattern of the partial proportions and concludes that the denominators 6, 12, 18, 24, 30 et cetera form an arithmetical sequence. As long as the number of squares is finite the proportion is bigger than $\frac{1}{3}$. However, if we have infinitely many ($\mathfrak{D}$) squares, the proportion will be $\frac{1}{3} + \frac{1}{\mathfrak{D}}$, but since $\frac{1}{3} + \frac{1}{\mathfrak{D}} = \frac{1}{3}$ according to Lemma 14 in Chapter XXVI (see Section 3), the proportion will be $\frac{1}{3}$. Therefore, he concludes, the proportion of the sum of infinitely many squares with the roots 1, 2, 3, 4, 5 et cetera to the *summan totidem terminorum maximo æqualium* equals the proportion of 1 to 3.

In this presentation, Duhre closely follows Wallis, but unlike Wallis who in his following propositions offers geometrical interpretations of this result, Duhre does not do so. According to Wallis, the above proportion 1 to 3 geometrically corresponds to the proportion of the complement of half a parabola to the parallelogram completed by the same half parabola and its complement (Wallis, 1656, Prop. XXIII). Furthermore, Wallis' method of induction would not be an accepted method of induction today, since only a limited number of cases for $n = 1, 2, 3, \ldots$ were tested and the induction step (i.e., if the property is assumed to be true for $n = k$ it should be proven to be true for $n = k + 1$) was not included.

Duhre proceeds by proving the corresponding proportion for cubes with the help of Wallis' method of induction. In modern notation, he proves the following:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^3}{(n+1)n^3} = \frac{1}{4}$$

[7] "[…] en summa innehållande then största ledamoten så ofta som progressionens ledamöter äre" (Brandt, 1718, p. 77).

After these two proofs, using Wallis method of induction, Duhre states that, again interpreted in modern notation, the following proportions are true:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^4}{(n+1)n^4} = \frac{1}{5}$$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^5}{(n+1)n^5} = \frac{1}{6}$$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^6}{(n+1)n^6} = \frac{1}{7}$$

6 Duhre's method of finding the area enclosed by curves

Let us now turn to Duhre's textbook on geometry. We will consider Duhre's method of finding the area enclosed by curves in order to see how he uses the proportions including infinite sums that he considered in his *Algebra*. In Chapter XXX Duhre formulates a proposition where he considers the curve $\boldsymbol{ABCD}$ and from it constructs the curve $\boldsymbol{AIOM}$ such that the area of the segment $\boldsymbol{ADCBA}$ is equal to half of the area $\boldsymbol{AEMOIA}$ (see Figure 6.1). The curve $\boldsymbol{AIOM}$ is constructed in the following way: Let $\boldsymbol{AS}$ be a tangent at the point $\boldsymbol{A}$, parallel to the ordinate $\boldsymbol{DE}$ and for every point $\boldsymbol{C}$ on $\boldsymbol{ABCD}$with a tangent $\boldsymbol{CG}$ where $\boldsymbol{G}$ is a point on $\boldsymbol{AS}$, the ordinate $\boldsymbol{OK}$ is equal to the line $\boldsymbol{AG}$.

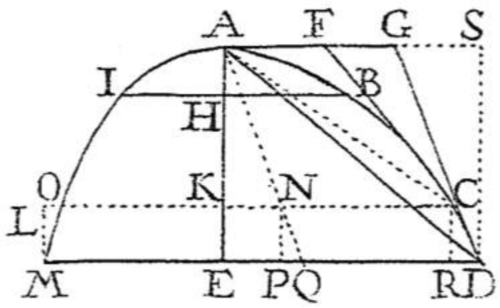


Figure 6.1: The area of the segment $ADCBA$ is equal to half of the area $AEMOIA$ (Duhre, 1721, p. 572).

Duhre proves this proposition without using algebra, only considering geometrical properties. First, he draws a few help lines. He draws the line $\boldsymbol{AQ}$ parallel to $\boldsymbol{DG}$ such that $\boldsymbol{ADGQ}$ is a parallelogram. If the point $\boldsymbol{C}$ is considered to be infinitely close to the point $\boldsymbol{D}$, he concludes that the line $\boldsymbol{CD}$ can be considered to be a straight line and thus it can be considered to be a part of the tangent $\boldsymbol{DG}$. Then he draws the line $\boldsymbol{CL}$ parallel to $\boldsymbol{DM}$ and the lines $\boldsymbol{CR}$, $\boldsymbol{LM}$ and $\boldsymbol{NP}$ parallel to $\boldsymbol{AE}$. Finally, he draws the line $\boldsymbol{AC}$. The proof of the proposition follows:

Since the two parallel lines $\boldsymbol{DM}$ and $\boldsymbol{CL}$ are infinitely close to each other, the points $\boldsymbol{L}$ and $\boldsymbol{O}$ are infinitely close to each other, and thus the mixed lines figure $\boldsymbol{EMOK}$ must be the same as the parallelogram$\boldsymbol{EMLK}$. Furthermore, the lines $\boldsymbol{EM}$, $\boldsymbol{AG}$ and $\boldsymbol{CN}$ are equal to each other and hence the parallelogram $\boldsymbol{EMLK}$ equals the parallelogram $\boldsymbol{PNCR}$, which in turn equals the parallelogram $\boldsymbol{QNCD}$. Now, if $\boldsymbol{CD}$ is considered as a base, the parallelogram $\boldsymbol{QNCD}$ is twice as big as the triangle $\boldsymbol{ACD}$, since the lines $\boldsymbol{CD}$ and $\boldsymbol{AQ}$ are parallel. This implies that also the mixed lines figure$\boldsymbol{EMOK}$ and the parallelogram$\boldsymbol{PNCR}$ are twice as big as the triangle $\boldsymbol{ACD}$.

Finally, if other lines parallel to the line $\boldsymbol{DM}$ are drawn, each of the resulting mixed lines figures are twice as big as the corresponding triangles for the same reason that the mixed lines figure $\boldsymbol{EMOK}$ is twice as big as the triangle $\boldsymbol{ACD}$. Therefore, the figure $\boldsymbol{AEMOIA}$, which is the composite of the mixed lines figures, equals twice the sum of the corresponding triangles that forms the segment $\boldsymbol{ADCB}$, which is what Duhre wanted to prove.

7 Duhre's method applied to the circle

In order to calculate the decimals of $\boldsymbol{\pi}$, or more specifically, in order to show that the proportion between the diameter and the circumference of a circle is approximately the same as 100 to 314, Duhre now wants to apply the proposition from Chapter XXX to a circle, i.e., instead of considering the circumference he considers the area of a circle. He begins Chapter XXXI with considering a half circle; the area under the corresponding curve to a half circle should be equal to the area of a full circle (see Figure 7.1). However, the corresponding curve$\boldsymbol{ASM}$ to the half circle $\boldsymbol{ACB}$in fact is an asymptote to the line $\boldsymbol{BV}$, and thus the "indescribable width"[8] of the area contained by the "indescribable" line $\boldsymbol{ASM}$ is equal to the area of the circle. However, the "undescribable width" is too difficult for Duhre to consider further. Therefore, he instead considers a quarter of a circle $\boldsymbol{ACD}$and its corresponding curve$\boldsymbol{ASR}$. Doing this, the area $\boldsymbol{ADRH}$ equals twice of the area of the segment $\boldsymbol{ACE}$ according to the proposition in Chapter XXX. By adding half of this area to the area of the triangle $\boldsymbol{ADC}$ and multiplying the expression by four, an expression of the area of the circle will be given.

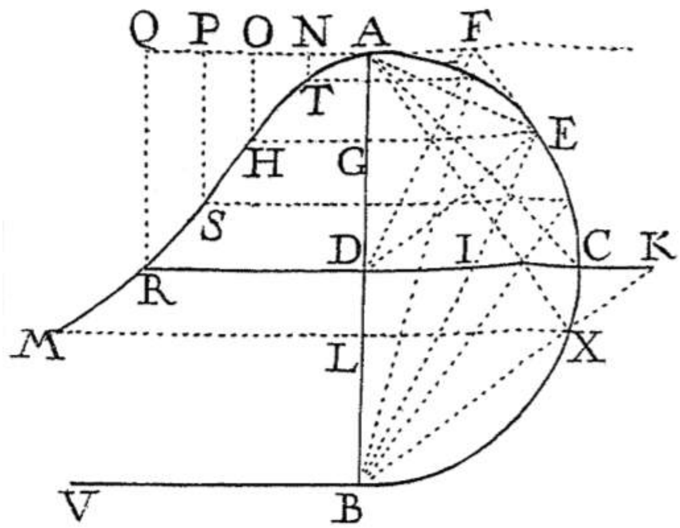


Figure 7.1: The area $ADRH$ equals twice of the area of the segment ACE (Duhre, 1721, p. 574).

Instead of calculating the area of the figure$\boldsymbol{ADRH}$, Duhre's idea is to calculate the area of the figure $\boldsymbol{ARQ}$. He states that the line $\boldsymbol{AQ}$, which is equal to the line $\boldsymbol{AD}$, can be divided into infinitely many equal parts, and the lines $\boldsymbol{NT}$, $\boldsymbol{OH}$, $\boldsymbol{PS}$ et cetera proceeding from these points of intersection will fill up the figure $\boldsymbol{ARQ}$.

Now Duhre introduces the variables $\boldsymbol{a}$, $\boldsymbol{x}$ and $\boldsymbol{y}$. He lets $\boldsymbol{AB = 2a}$, i.e., the radius of the circle equals $\boldsymbol{a}$, the ordinate $\boldsymbol{GH = AF = DI = x}$ and $\boldsymbol{AG = y}$. He wants to find an

[8] "[…] obeskrifweliga widden" (Duhre, 1721, p. 110).

expression for $\boldsymbol{y}$, which can be considered as a length that varies.He does this using proportional reasoning: He first concludes that $\boldsymbol{BG = 2a - y}$ and, because of properties of the circle the square of $\boldsymbol{GE}$ equals $\boldsymbol{AG \cdot BG}$ which is the same as $\boldsymbol{2ay - y^2}$. Considering the two uniform triangles $\boldsymbol{BDI}$ and $\boldsymbol{BGE}$, Duhre concludes that since$\boldsymbol{BD}$, $\boldsymbol{DI}$, $\boldsymbol{BG}$ and $\boldsymbol{GE}$ are geometrical proportional, i.e., $\boldsymbol{BD, DI :: BG, GE}$, the squares $\boldsymbol{BDq}$, $\boldsymbol{DIq}$, $\boldsymbol{BGq}$ and $\boldsymbol{GEq}$ will also be geometrical proportional, i.e., $\boldsymbol{BDq, DIq :: BGq, GEq}$.[9] From this it follows that $\boldsymbol{aa, xx :: 4aa - 4ay + yy, 2ay - yy}$, which can be simplified into $\boldsymbol{aa, xx :: 2a - y, y}$. He now uses the fact that the product of the two utmost in a geometrical progression equals the product of the two inners, i.e., $\boldsymbol{aay = 2axx - xxy}$. By adding $\boldsymbol{xxy}$ and dividing by $\boldsymbol{aa + xx}$ on both sides, Duhre now finally finds the expression $\boldsymbol{y = \frac{2axx}{aa+xx} = AG}$. This quotient can be expressed as an infinite series:

$$AG = y = \frac{2axx}{aa + xx} = \frac{2xx}{a} - \frac{2x^4}{a^3} + \frac{2x^6}{a^5} - \frac{2x^8}{a^7} \&c.$$

Furthermore, he concludes that if $GH = 2x$ then

$$AG = \frac{8xx}{a} - \frac{32x^4}{a^3} + \frac{128x^6}{a^5} - \frac{512x^8}{a^7} \&c.,$$

if $GH = 3x$ then

$$AG = \frac{18xx}{a} - \frac{162x^4}{a^3} + \frac{1458x^6}{a^5} - \frac{13122x^8}{a^7} \&c.,$$

and so on. Since $AQ = a$ is divided into infinitely many equal parts, where the first one is $AN = x$, $AO = 2x$, $AP = 3x$, and so on, the expressions above give the corresponding lengths of$AG = y$. These lengths could also be denoted NT, OH, PS according to Figure 7.1. The last of these lengths is $QR = a$.The infinitely many lengths together fill up the figure AQR, and therefore Duhre now has to compute the infinite sum of these infinitely many series. In order to compute the sum, i.e., the area of the figure AQR, Duhre now collects all terms of the same power of x. Thus, the area AQR will be:

$$\frac{2}{a}(xx + 4xx + 9xx\ \&c.) - \frac{2}{a^3}(x^4 + 16x^4 + 81x^4\&c.) + \frac{2}{a^5}(x^6 + 64x^6 + 729x^6\&c.)\&c.$$

In modern notation this expression can be interpreted as

$$\frac{2}{a}\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^2 - \frac{2}{a^3}\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^4 + \frac{2}{a^5}\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^6 - \cdots$$

To compute these sums, Duhre uses the results on infinite sums from his text book on algebra (see Section 5). First, he has to determine the *summa totidem terminorum maximo æqualium*. The *summa totidem terminorum maximo æqualium* to the infinite sum$xx + 4xx + 9xx\&c.$ must be $a \cdot aa$, since he considers a to be the number of terms in the infinite sum and aa to

[9] In modern notation: $\frac{BD}{DI} = \frac{BG}{GE}, i.e., \frac{BD^2}{DI^2} = \frac{BG^2}{GE^2}$.

be the biggest term in the sum. It follows that, in modern notation, $\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^2=\frac{1}{3}a^3$. In the same way $\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^4=\frac{1}{5}a^5$, $\lim_{n\to\infty}\sum_{k=1}^{n}(kx)^6=\frac{1}{7}a^7$ and so on. Therefore, the infinite sum of the infinite series above, i.e., the area of the figure AQR, will be equal to

$$\frac{2}{a}\left(\frac{1}{3}a^3\right)-\frac{2}{a^3}\left(\frac{1}{5}a^5\right)+\frac{2}{a^5}\left(\frac{1}{7}a^7\right)\&c.=$$

$$=\frac{2}{3}aa-\frac{2}{5}aa+\frac{2}{7}aa-\frac{2}{9}aa\ \&c.$$

Duhre can now easily find an expression for the area of the figure ARD; he just has to take the area of the square of AQ, i.e., a^2, and subtract the area of the figure AQR. Thus, the area of the figure ARD will be

$$aa-\frac{2}{3}aa+\frac{2}{5}aa-\frac{2}{7}aa+\frac{2}{9}\&c.$$

According to the method presented in Chapter XXX (see Section6), the area of the figure ARD is twice the area of the segment ACE, and therefore it follows that the area of the segmet ACE will be

$$\frac{1}{2}aa-\frac{1}{3}aa+\frac{1}{5}aa-\frac{1}{7}aa+\frac{1}{9}aa\ \&c.$$

Now, adding the area of the triangle ADC to this expression and multiply with four will finally give an expression for the area of the circle with radius a:

$$4aa-\frac{4}{3}aa+\frac{4}{5}aa-\frac{4}{7}aa+\frac{4}{9}aa\ \&c.$$

Duhre modifies this expression even further, in order to find an expression for the circumference of the circle. Since the area of a circle equals the area of a triangle where the base equals the circumference of the circle and the height equals the radius of the circle, he concludes that he will find an expression of the circumference of the circle if he divides the area of the circle with half of its radius, i.e., $\frac{1}{2}a$. Thus, he gets the following series expressing the circumference of the circle:

$$8a-\frac{8}{3}a+\frac{8}{5}a-\frac{8}{7}a+\frac{8}{9}a\&c.$$

Duhre now lets the diameter of the circle, i.e., $2a$, equal 1 and finds that the proportion between the diameter of a circle and its circumference is as one to the following series:

$$4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}\&c.$$

He finally modifies this series by merging the terms pairwise:

$$\frac{8}{3}+\frac{8}{35}+\frac{8}{99}+\frac{8}{195}+\frac{8}{323}+\frac{8}{483}+\frac{8}{675}\,\&c.$$

In modern notation we can interpret this result as

$$\pi = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{8}{(4k-2)^2-1}.$$

8 Duhre’s calculation of π

After finding the proportion of the diameter of the circle to its circumference, Duhre proceeds with computing this proportion. He starts with constructing a table (see Figure 8.1) with the first 315 denominators of the series $\sum_{k=1}^{n}\frac{8}{(4k-2)^2-1}$ .This table is actually not completely correct, possibly due to typesetting errors. For example, for k=100 it says 258.403 instead of 158.403 and for k= 50 it says 39.204 instead of 39.203.

116

En Tafla som innehåller nämnare åth 315 bråk harwande 8 til sin almänna täljare af hwilkas summa cirkelns omkretz består då samma cirkels diameter är 1.

3	14883	58563	131043	232323	362403
35	15875	60515	133955	236195	367235
99	16899	62499	136899	240099	372099
195	17955	64515	139875	244035	376995
323	19043	66563	142883	248003	381923
483	20163	68643	145923	252003	386883
675	21315	70755	148995	256035	391875
899	22499	72899	152099	260099	396899
1155	23715	75075	155235	264195	401955
1443	24963	77283	258403	268323	407043
1763	26243	79523	161603	272483	412163
2115	27555	81795	164835	276675	417315
2499	28899	84099	168099	280899	422499
2915	30275	86435	171395	285155	427715
3363	31683	88803	174723	289443	432963
3843	33123	91203	178083	293763	438243
4355	34595	93635	181475	298115	443555
4899	36099	96099	184899	302499	448899
5475	37635	98595	188355	306915	454275
6083	39204	101123	191843	311363	459683
6723	40803	103683	195363	315843	465123
7395	42435	106275	198915	320355	470595
8099	44099	108899	202499	324899	476099
8835	45795	111555	206115	329475	481635
9603	47525	114243	209763	334083	487203
10403	49283	116963	213443	338723	492803
11235	51075	119715	217155	343395	498435
12099	52899	122496	220899	348099	504099
12995	54755	125315	224675	352835	509795
13923	56643	128163	228483	357603	515523

117

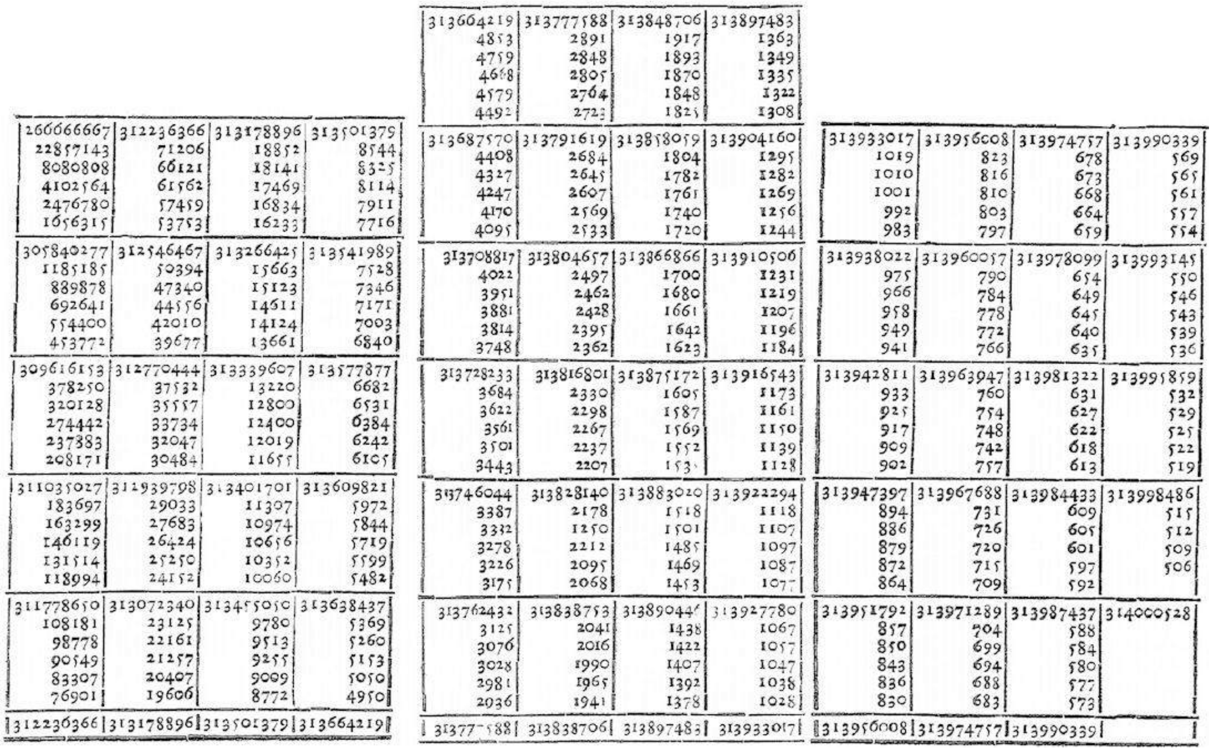
521283	708963	925443	1170723	1444803
527055	715715	933155	1179395	1454435
522899	622499	940899	1188099	1464099
538755	729315	948675	1196835	1473795
544643	736163	956483	1205603	1483523
550563	743043	964323	1214403	1493283
556515	749955	972195	1223235	1503075
562499	756899	980099	1232099	1512899
568515	763875	988035	1240995	1522755
574563	770883	996003	1249923	1532643
580643	777923	1004003	1258883	1542563
686755	784995	1012035	1267875	1552515
592879	992099	1020099	1276899	1562499
599075	799235	1028195	1285955	1572515
605283	306403	1036323	1295043	1582563
611523	813603	1044483	1304163	
617795	820835	1052675	1313315	
624099	828099	1060899	1322499	
630 35	835395	1069155	1331715	
636803	842723	1077443	1340963	
643203	850083	1085763	1350243	
649635	857475	1094115	1359555	
656099	864899	1102499	1368899	
662595	872355	1110915	1378275	
669123	879843	1119363	1387683	
675683	887363	1127843	1397123	
682275	894915	1136355	1406595	
688899	902499	1144899	1416099	
695555	910115	1153475	1425635	
702243	917763	1162083	1435203	

Figure 8.1: The table containing the first 315 denominators in Duhre’s infinite series of π (Duhre, 1721, pp. 116–117).

Duhre proceeds with constructing a second table, containing the first 315 terms and partial sums of the series (see Figure 8.2). However, he does not want to consider decimals and therefore he considers a circle with diameter 100.000.000 instead of 1, i.e., the general numerator in the series will be 800.000.000 instead of 8. In modern notation this new series can be written as

$$\mathbf{100.000.000}\boldsymbol{\pi} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\mathbf{800.000.000}}{(\mathbf{4}\boldsymbol{k}-\mathbf{2})^{\mathbf{2}}-\mathbf{1}}.$$

In this way the partial sums, after approximations, will be natural numbers. In the table in Figure 8.2 we can see that the proportion of the diameter of a circle to its circumference will be approximately as 100.000.000 to 314.000.528, or as 100 to 314.



266666667	312236366	313178896	313501379
22857143	71206	18852	8544
8080808	66121	18141	8325
4102564	61562	17469	8114
2476780	57459	16834	7911
1656315	53753	16233	7716
305840277	312546467	313266425	313541989
1185185	50394	15663	7528
889878	47340	15123	7346
692641	44556	14611	7171
554400	42010	14124	7003
453772	39677	13661	6840
309616153	312770444	313339607	313577877
378250	37532	13220	6682
320128	35557	12800	6531
274442	33734	12400	6384
237383	32047	12019	6242
208171	30484	11655	6105
311035027	312939798	313401701	313609821
183697	29033	11307	5972
163299	27683	10974	5844
146119	26424	10656	5719
131514	25250	10352	5599
118994	24152	10060	5482
311778650	313072340	313455050	313638437
108181	23125	9780	5369
98778	22161	9513	5260
90549	21257	9255	5153
83307	20407	9009	5050
76901	19606	8772	4950
312236366	313178896	313501379	313664219

313664219	313777588	313848706	313897483
4853	2891	1917	1363
4759	2848	1893	1349
4668	2805	1870	1335
4579	2764	1848	1322
4492	2723	1825	1308
313687570	313791619	313858059	313904160
4408	2684	1804	1295
4327	2645	1782	1282
4247	2607	1761	1269
4170	2569	1740	1256
4095	2533	1720	1244
313708817	313804657	313866866	313910506
4022	2497	1700	1231
3951	2462	1680	1219
3881	2428	1661	1207
3814	2395	1642	1196
3748	2362	1623	1184
313728233	313816801	313875172	313916543
3684	2330	1605	1173
3622	2298	1587	1161
3561	2267	1569	1150
3501	2237	1552	1139
3443	2207	1535	1128
313746044	313828140	313883020	313922294
3387	2178	1518	1118
3332	1250	1501	1107
3278	2212	1485	1097
3226	2095	1469	1087
3175	2068	1453	1077
313762432	313838753	313890446	313927780
3125	2041	1438	1067
3076	2016	1422	1057
3028	1990	1407	1047
2981	1965	1392	1036
2036	1941	1378	1028
313777588	313838706	313897483	313933017

313933017	313956008	313974757	313990339
1019	823	678	569
1010	816	673	565
1001	810	668	561
992	803	664	557
983	797	659	554
313938022	313960057	313978099	313993145
975	790	654	550
966	784	649	546
958	778	645	543
949	772	640	539
941	766	635	536
313942811	313963947	313981322	313995859
933	760	631	532
925	754	627	529
917	748	622	525
909	742	618	522
902	757	613	519
313947397	313967688	313984433	313998486
894	731	609	515
886	726	605	512
879	720	601	509
872	715	597	506
864	709	592	
313951792	313971289	313987437	314000528
857	704	588	
850	699	584	
843	694	580	
836	688	577	
830	683	573	
313956008	313974757	313990339	

Figure 8.2: Duhre's table showing the first 315 approximated termsand partial sums in the series $\sum_{k=1}^{n} \frac{800.000.000}{(4k+2)^2-1}$ (Duhre, 1721, pp. 119–121).

Duhre concludes Chapter XXX by noting that in practice, when minor computations have to be made, the proportion 100 to 314 or the Archimedean proportion 7 to 22 can be used, the requested proportion being smaller than the former and bigger than the latter. If larger computations have to be performed, however, he suggests that the proportion 100.000 to 314.159 should be used. Nevertheless, he does not perform the computations needed to find this proportion.

9 Concluding remarks

The series $\mathbf{4}-\frac{\mathbf{4}}{\mathbf{3}}+\frac{\mathbf{4}}{\mathbf{5}}-\frac{\mathbf{4}}{\mathbf{7}}+\frac{\mathbf{4}}{\mathbf{9}}\&c.$ which Duhre received before he merged the terms pairwise, we recognize as a Maclaurin series for $\mathbf{4}\tan^{-1}\boldsymbol{x}$ for$\boldsymbol{x}=\mathbf{1}$. Since $\mathbf{4}\tan^{-1}\mathbf{1}=\boldsymbol{\pi}$, we can conclude that Duhre's series is correct. However, it converges very slowly. This series is known as the Gregory–Leibniz' series after James Gregory (1638–1675) and Gottfried Wilhelm Leibniz (1646–1716). Leibniz was concerned with the quadrature and when he applied his method to the circle he received the series $\frac{\boldsymbol{\pi}}{\mathbf{4}}=\mathbf{1}-\frac{\mathbf{1}}{\mathbf{3}}+\frac{\mathbf{1}}{\mathbf{5}}-\frac{\mathbf{1}}{\mathbf{7}}+\cdots$. Leibniz found this result in 1673, but already in 1671 Gregory, who was concerned with infinite series representations of transcendental functions, had found the corresponding Taylor series. Also, an Indian mathematician, whose identity is not definitely known, found the series for $\mathbf{tan^{-1}}\,\boldsymbol{x}$ during the 15th century (Roy, 1990). This series, written in Sanskrit verse, is usually ascribed to Kerala Gargya Nilakantha (c.1450–c.1550) and can be found in the book *Tantrasangraha* composed around 1500.

Since Duhre follows Wallis' method of induction when he considers the infinite series, it may be surprising that he in his book on geometry does not proceed with studying Wallis' interpolation method to find the area of a circle in order to find an expression for $\boldsymbol{\pi}$. However, Duhre's method, where he from the circle constructs a corresponding curve where he can use the previously found infinite sums to find the enclosed area, is indeed ingenious. In his search for $\boldsymbol{\pi}$ Duhre also uses modern algebra that cannot be found in Wallis' *Arithmetica infinitorum*. Duhre considers algebra to be helpful, since it enables complicated expressions to be transformed into simpler ones, and thus convenience in calculations is obtained.

While Duhre primarily was an educator, his main pioneering achievement was that he brought knowledge of modern mathematics into the Swedish mathematical community. Of particular value is his choice to write in Swedish in order to find a greater audience. Twice he applied for a position as professor at Uppsala University, without success, but he still succeeded in inspiring several among the next generation of Swedish mathematicians. Certainly, also his students at Bergskollegium and the Royal Fortification Office had the opportunity to be introduced into modern mathematics thanks to Duhre.

Acknowledgement: This work was supported by the Swedish Research Council [Grant no. 2015-02043].

REFERENCES

Brandt, G. (1718). *En Grundelig Inledning til Mathesin Universalem och Algebram – Efter Herr And. Gabr. Duhres håldne prealectioner*. Stockholm: Kongl. Antiquit Archiv.

Duhre, A. G. (1721). *Första Delen af en Grundad Geometria – Bewijst uti de Föreläsningar som äro håldne på Swänska Språket Uppå Kongl. Fortifications Contoiret*. Stockholm: Kongl. Antiquit. Archiv.

Hebbe, Per (1933). *Anders Gabriel Duhres "Laboratorium mathematico-oeconomicum". Ett bidrag till Ultunas äldre historia*. Stockholm: Kungliga Lantbruksakademien.

Mancosu, P. (1996). *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*. New York, NY: Oxford University Press.

Nieuwentijt, B. (1695). *Analysis infinitorum, seu curvilineorum proprietates ex polygonorum natura deductæ.* Amstelædami: Joannem Wolters.

Pejlare, J. (2017). On the relationships between the geometric and the algebraic ideas in Duhre's textbooks of mathematics, as reflected via Book II of Euclid's Elements. In: K. Bjarnadóttir, F. Furinghetti, M. Menghini, J. Prytz, & G. Schubring (Eds.). *"D where you stand" 4. Proceedings of the fourth International Conference on the History of Mathematics Education* (pp. 263–273). Rome: Edizioni Nuova Cultura.

Rodhe, S. (2002). *Matematikens utveckling i Sverige fram till 1731*. Uppsala: Diss. Uppsala Universitet.

Roy, R. (1990). The discovery of the series formula for π by Leibniz, Gregory and Nilakantha. *Mathematics Magazine 63*(5), 291–306.

Stedall, J. (2004). Introduction: The arithmetic of infinitesimals by Jacqueline A. Stedall, In: J. A. Stedall (Ed.), *The arithmetic of infinitesimals: John Wallis 1695. Translated from Latin to English with an introduction by Jacqueline A. Stedall*. New York: Springer.

Stedall, J. (2005). John Wallis, Arithmetica infinitorum (1656), In: I. Grattan-Guinness, & R. Cooke

(Eds.), *Landmark Writings in Western Mathematics 1640–1940* (pp. 23–32). Elsevier Science Limited.
Vermij, R. H. (1989). Bernard Nieuwentijt and the Leibnizian Calculus. *Studia Leibnitiana 21*(1), 69–86.
Wallis, J. (1656). *Arithmetica infinitorvm, sive nova methodus inquirendi in curvilineorum quadraturam, aliaq*; *difficiliora matheseos problemata*. Oxoniense: Leon: Lichfield academiz typographi impensis Tho. Robinson.

The international *Scientific Program Committee* (SPC)

Evelyne Barbin, Université de Nantes (France), co-chair

Uffe Thomas Jankvist, Aarhus University (Denmark), co-chair

Tinne Hoff Kjeldsen, University of Copenhagen (Denmark), co-chair

Bjørn Smestad, OsloMet – Oslo Metropolitan University (Norway), co-chair

Constantinos Tzanakis, University of Crete (Greece), chair

Abraham Arcavi, Weizmann Institute of Science (Israel)

Janet Heine Barnett, Colorado State University (USA)

Aline Bernardes, Federal University of the State of Rio de Janeiro (Brazil)

Alain Bernard, Université Paris Est (France)

Kristín Bjarnadóttir, University of Iceland (Iceland)

Mónica Blanco, Universitat Politècnica de Catalunya, Barcelona (Spain)

Anne Boyé, IREM, Université de Nantes (France)

Kathleen Clark, Florida State University (USA)

Renaud Chorlay, IREM, Université Paris 7 (France)

Andreas Christansen, Western Norway University of Applied Sciences (Norway)

Ubiratan d'Ambrosio, Pontificia Universidade, Catolica de Sao Paulo, Sao Paolo (Brazil)

Elizabeth de Freitas, Manchester Metropolitan University (UK)

Cécile de Hosson, Université Paris Diderot (France)

Adriano Demattè, Liceo Rosmini, Trento (Italy)

Abdellah El Idrissi, Ecole Normale Supérieure, Marrakech (Morocco)

Florence Fasanelli, American Association for the Advancement of Science (USA)

Gail FitzSimons, Warrandyte, Victoria (Australia)

Michael Fried, Ben Gurion University of the Negev (Israel)

Fulvia Furinghetti, University of Genoa (Italy)

Katalin Gosztonyi, Eötvös Loránd University, Budapest (Hungary)

David Guillemette, University of Ottawa (Canada)

Thomas Hausberger, Université de Montpellier 2 (France)

Gavin Hitchcock, University of Stellenbosch (South Africa)

Wann-Sheng Horng, National Taiwan Normal University (Taiwan)

Sunwook Hwang, Soongsil University, Seoul (Korea)

Masami Isoda, University of Tsukuba (Japan)

Mikkel Johansen, University of Copenhagen (Denmark)

Rainer Kaenders, Universität Bonn (Germany)

Victor Katz, University of the District of Columbia, Washington, DC (USA)

Michael Kourkoulos, University of Crete (Greece)

Ladislav Kvasz, Charles University in Prague (Czech Republic)

Ewa Lakoma, Military University of Technology, Warsaw (Poland)

Snezana Lawrence, Anglia Ruskin University, Cambridge, (UK)

Po-Hung Liu, National Chin-Yi University of Technology (Taiwan)

Maria Rosa Massa-Esteve, University Politecnica of Catalunya (Spain)

Frédéric Métin, Université de Bourgogne (France)

Anne Michel-Pajus, IREM, Université Paris-Diderot (France)

Marc Moyon, Université de Limoges (France)

Kostas Nikolantonakis, University of Western Macedonia (Greece)

Antonio Oller-Marcen, Centro Universitario de la Defensa (Spain)

Maurice O'Reilly, Dublin City University (Ireland)

Daniel Otero, Xavier University (USA)

Johanna Pejlare, Chalmers University of Technology and University of Gothenburg (Sweden)

David Pengelley, Oregon State University (USA)

Helder Pinto, University of Aveiro (Portugal)

Luis Puig, University of Valencia (Spain)

Luis Radford, Université Laurentienne Sudbury, Ontario (Canada)

Peter Ransom, Bath Spa University (UK)

Leo Rogers, University of Oxford (UK)

Tatiana Roque, Universidade Federal do Rio de Janeiro (Brasil)

Sebastian Schorcht, Justus-Liebig-Universität Gießen (Germany)

Gert Schubring, University of Bielefeld (Germany)

Nicholas Scoville, Ursinus College (USA)

Man-Keung Siu, University of Hong Kong (China)

Henrik Kragh Sørensen, University of Copenhagen (Denmark)

Susanne Spies, University of Siegen (Germany)

Yi-Wen Su, University of Taipei (Taiwan)

Yannis Thomaidis, Regional Administration of Education, Thessaloniki (Greece)

Dominique Tournès, Université de La Réunion (France)

Ysette Weiss, Universität Mainz (Germany)

Greicy Winicki-Landman, California State Polytechnic University (USA)

Ingo Witzke, University of Siegen (Germany)

The *Local Organizing Committee* (LOC)

Bjørn Smestad, OsloMet – Oslo Metropolitan University, chair

Trude Sundtjønn, OsloMet – Oslo Metropolitan University

Lars Reinholdtsen, OsloMet – Oslo Metropolitan University

Eyvind Martol Briseid, OsloMet – Oslo Metropolitan University

Nikos Kapelonis, Department of Education, University of Crete, Greece

Author Index

Contributors	Country	email	Page
Adesso, Maria Giuseppina	Italy	mapinadesso@gmail.com	561
Affan, Abd elrahman Faheem	Israel	affanabd@gmail.com	575
Aktas, Mehmet Emin	USA	maktas@uco.edu	221
Almeida, Mária Cristina	Portugal	ajs.mcr.almeida@gmail.com	695
Barbin, Evelyne	France	evelyne.barbin@wanadoo.fr	69
Barnett, Janet H.	USA	janet.barnett@csupueblo.edu	221
Basyal, Deepak	USA	deepak.basyal@uwc.edu	697
Bella, Sandra	France	bellusky@hotmail.com	397
Bernard, Alain	France	alain.bernard@u-pec.fr	585
Bjarnadóttir, Kristín	Iceland	krisbj@hi.is	841
Blanco, Mònica	Spain	monica.blanco@upc.edu	397
Blaszczyk, Piotr	Poland	pb@up.krakow.pl	399
Bonissoni, Petronilla	Italy	nillarock@alice.it	143
Boyé, Anne	France	anne.boye@neuf.fr	401
Bråting, Kajsa	Sweden	kajsa.brating@edu.uu.se	711
Buchholtz, Nils	Norway	n.f.buchholtz@ils.uio.no	155
Can, Cihan	USA	cc15m@my.fsu.edu	221
Calandrino, Denise	Italy		601
Capone, Roberto	Italy	robertocapone69@gmail.com	81.561
Cazzola, Marina	Italy	marina.cazzola@unimib.it	143
Cecchi, Michele	Italy	michelececchi@diluvio.it	601
Chen, Tung-Shayan	Taiwan	chents@ncut.edu.tw	367
Chevalarias, Nathalie	France	n.chevalarias@orange.fr	205, 291
Chiorescu, Marcela	USA	marcela.chiorescu@gcsu.edu	437
Chorlay, Renaud	France	renaud.chorlay@espe-paris.fr	403
Christiansen, Andreas	Norway	andreas.christiansen@hsh.no	815
Clain, Teresa Costa	Portugal	tcostacaracol@gmail.com	483
Clark, Kathleen Michelle	USA	kclark@fsu.edu	205, 221, 223
Daval, Natalie	France	nathalie.daval@univ-reunion.fr	289
De Bock, Dirk	Belgium	dirk.debock@kuleuven.be	801
Del Sorbo, Maria Rosaria	Italy	marodel@gmail.com	81
Deligiannidis, Triantafyllos	Greece	soflak@yahoo.gr	303
Demattè, Adriano	Italy	adriano.dematte@scuole.provincia.tn.it, adriano.dematte@fastwebnet.it	23
Durmaz, Burcu	Turkey	drburcudurmaz@gmail.com	315
Ferrini, Attilio	Italy	attiliofe@virgilio.it	601
Fiore, Oriana	Italy	orianafio@gmail.com	561
Fried, Michael	Israel	mfried@bgu.ac.il, mfried4489@gmail.com	575
Geraniou Eirini	UK	e.geraniou@ucl.ac.uk	107
González Astudillo, María Teresa	Spain	maite@usal.es	499
Gu, Hui	China	guhuisky@163.com	317
Guevara-Casanova, Iolanda	Spain	iguevara@xtec.cat	449
Guillemette, David	Canada	guillemette.david@uqam.ca	23, 91
Guillet, Carene	France	carene.guillet@gmail.com	227
Guitart, René	France	rene.guitart@orange.fr	859
Halldórsson, Bjarni Vilhjálmur	Iceland	bjarnivh@ru.is	841
Hitchcock, Anthony Gavin	South Africa	aghitchcock@gmail.com	43, 609
Hou, Bin	China	1021000396@qq.com	317
Isolani, Laura	Italy	lauraiso@hotmail.com	601
Jahnke, Hans Niels	Germany	Niels.Jahnke@t-online.de	3
Jankvist, Uffe Thomas	Denmark	utj@edu.au.dk	107, 319
Karpińska, Karolina	Poland	karolinakarpinska001@gmail.com	727

Kvasz, Ladislav	Czech Republic	ladislavkvasz@gmail.com	619
Lakoma, Ewa	Poland	ewa.lakoma@wat.edu.pl	759
Languereau, Hombeline	France	hombeline.languereau@univ-fcomte.fr	245
Lappa, Eleni	Greece	elelappa@gmail.com	331
Lawrence, Snezana	UK	snezana@mathsisgoodforyou.com	65, 515
Lefort, Xavier	France	xavier.lefort@iutsn.univ-nantes.fr	401
Lemes, Ana Jimena	France	jimenalemes@gmail.com	371
Li, Xiaoni	China	370599934@qq.com	343
Liao, Chieh-Cheng	Taiwan	jenchengqq@gmail.com	621
Liu, Po-Hung	Taiwan	liuph@ncut.edu.tw	345
Longoni, Paolo	Italy	plong172@gmail.com	143
Man, Yiu-Kwong	China	ykman@eduhk.hk	355
Massa-Esteve, M. Rosa	Spain	m.rosa.massa@upc.edu	627
Mauntel, Matthew	USA	mmauntel@gmail.com	463
Menghini, Marta	Italy	marta.menghini@uniroma1.it	679
Métin, Frédéric	France	frederic.metin01@u-bourgogne.fr	375, 531
Michel-Pajus, Anne	France	annie.pajus@gmail.com	245
Misfeldt, Morten	Denmark	misfeldt@learning.aau.dk	319
Moureau, Marie-Line	France	marie-line.moureau@ac-nantes.fr	227
Mouyssinat, Michel	France	mmouyssinat@hotmail.com	289
Moyon, Marc	France	marc.moyon@unilim.fr	417
Mutanen, Arto	Finland	arto.mutanen@gmail.com	131
Narváez , Deissy Milena	Colombia	dmnarvaez@gmail.com	773
Natali, Vilma	Italy	vilma.natali@istruzione.it	601
Nikolantonakis, Kostas	Greece	knikolantonakis@uowm.gr	251, 303, 331
Ninni, Virginia	Italy	vninni@gmail.com	81
O'Reilly, Maurice	Ireland	maurice.oreilly@dcu.ie	481
Pejlare, Johanna	Sweden	pejlare@chalmers.se	861
Pinto,Hélder	Portugal	hbmpinto1981@gmail.com	483
Popotis, Apostolos	Greece	apopotis@gmail.com	251
Puig, Luis	Spain	luis.puig@uv.es	775
Puig-Pla, Carles	Spain	carles.puig@upc.edu	449
Ransom, Peter	UK	pransom@btinternet.com, p.ransom@bathspa.ac.uk	545
Riva, Gianstefano	Italy	gianstefano.riva@gmail.com	143
Rodin, Jennifer Marie	USA	jennirodin@gmail.com	557
Rodríguez Vásquez, Flor Monserrat	Mexico	flor.rodriguez@uagro.mx	499
Roelens, Michel	Belgium	michel.roelens@ucll.be	205, 273
Rogers, Leo	UK	leorogers@appleinter.net, lr@awaywithmaths.co.uk	623
Romero Vallhonesta, Fàtima	Spain	fatima.romerovallhonesta@gmail.com	627
Rottoli, Ernesto	Italy	ernerott@tin.it	143
Sánchez Aguilar, Mario	Mexico	mario.sanchez@me.com	319
Schorcht, Sebastian	Germany	sebastian.schorcht@math.uni-giessen.de	155
Shen, Zhongyu	China	344080566@qq.com	67, 177, 363
Siu, Man Keung	China	mathsiu@hku.hk	777
Sorgato, Sonia	Italy	sonia.sorgato@unimib.it	143
Stoffels, Gero	Germany	stoffels@mathematik.uni-siegen.de	223, 287
Struve, Horst	Germany	h.struve@uni-koeln.de	223
Sun, Dandan	China	18766173312@163.com,1427300047@qq.com	191
Tattersall, James Joseph	USA	tat@providence.edu	793
Tognaccini, Chiara	Italy	chiara.tognaccini@libero.it	601
Tortoriello, Francesco Saverio	Italy	fstortoriello@unisa.it	561

Tournès, Dominique	France	dominique.tournes@univ-reunion.fr	289
Tzanakis, Constantinos	Greece	tzanakis@edc.uoc.gr	641
Vanpaemel, Geert	Belgium	geert.vanpaemel@wet.kuleuven.be	801
Vicentini, Caterina	Italy	cater.vicentini@gmail.com	205
Voillequin, Isabelle	France	isabelle.voillequin@hotmail.fr	227
Wang, Xiaoqin	China	xqwang@math.ecnu.edu.cn	67
Wang, Xin	China	wangxinecnu@163.com, 1217312201@qq.com	357, 363
Witzke, Ingo	Germany	witzke@mathematik.uni-siegen.de	195, 223
Yan, Qi Chun	China	476890742@qq.com	361
Yue, Zengcheng	China	yuezengcheng@126.com	67, 363
Zou, Jiachen	China	jczou@math.ecnu.edu.cn	67, 363
Zubillaga Guerrero, Erika	Mexico	eguerrero@uagro.mx	499

Participants – Countries

Surname	Name	Country	email address
Adesso	Maria Giuseppina	Italy	mapinadesso@gmail.com
Affan	Abdelrahman	Israel	affanabd@gmail.com
Almeida	Mária Cristina	Portugal	ajs.mcr.almeida@gmail.com
Babutiu	Nicoleta	Canada	babutiun@brandonu.ca
Barbin	Evelyne	France	evelyne.barbin@wanadoo.fr
Basyal	Deepak	USA	deepak.basyal@uwc.edu
Bella	Sandra	France	bellusky@hotmail.com
Bernard	Alain	France	alain.bernard@u-pec.fr
Bjarnadóttir	Kristín	Iceland	krisbj@hi.is
Blanco	Monica	Spain	monica.blanco@upc.edu
Błaszczyk	Piotr	Poland	pb@up.krakow.pl
Bolognese	Chris	USA	bolognesechris@gmail.com
Boyé	Anne	France	anne.boye@neuf.fr
Boyé	Jean	France	cenub@club-internet.fr
Bråting	Kajsa	Sweden	kajsa.brating@edu.uu.se
Can	Cihan	USA	cc15m@my.fsu.edu
Capone	Robero	Italy	robertocapone69@gmail.com
Cecchi	Michele	Italy	michelececchi@diluvio.it
Chen	Tung-Shayan	China	chents@ncut.edu.tw
Chevalarias	Nathalie	France	n.chevalarias@orange.fr
Chiorescu	Marcela	USA	marcela.chiorescu@gcsu.edu
Chorlay	Renaud	France	renaud.chorlay@espe-paris.fr
Christiansen	Andreas	Norway	andreas.christiansen@hsh.no
Clark	Kathleen	USA	kclark@fsu.edu
De Bock	Dirk	Belgium	dirk.debock@kuleuven.be
Del Sorbo	Maria Rosaria	Italy	marodel@gmail.com
Demattè	Adriano	Italy	adriano.dematte@scuole.provincia.tn.it
Dorgiaki	Ioanna	Greece	joannadorgiaki@hotmail.com
Durmaz	Burcu	Turkey	drburcudurmaz@gmail.com
Ferrini	Attilio	Italy	attiliofe@virgilio.it
Fiore	Oriana	Italy	orianafio@gmail.com
Fried	Michael	Israel	mfried4489@gmail.com
Gander	Yvette	New Zealand	y.gander@middleton.school.nz
González Astudillo	María Teresa	Spain	maite@usal.es
Gu	Hui	China	guhuisky@163.com
Guevara-Casanova	Iolanda	Spain	iguevara@xtec.cat
Guillemette	David	Canada	guillemette.david@uqam.ca
Guillet	Carène	France	carene.guillet@gmail.com
Guitart	René	France	rene.guitart@orange.fr
Gustavsen	Trond	Norway	trond.gustavsen@uib.no
Haney	Zach	USA	zhaney@winsor.edu
Hitchcock	Gavin	South Africa	aghitchcock@gmail.com
Hou	Bin	China	1021000396@qq.com
Isolani	Laura	Italy	lauraiso@hotmail.com
Jahnke	Hans Niels	Germany	njahnke@uni-due.de

Jankvist	Uffe Thomas	Denmark	utj@edu.au.dk
Johansson	Bengt	Sweden	bengt.education@gmail.com
Kaenders	Rainer	Germany	R.Kaenders@uni-bonn.de
Kaphle	Krishna	USA	kaflek@gmail.com
Karpińska	Karolina	Poland	karolinakarpinska001@gmail.com
Katz	Victor	USA	vkatz@udc.edu
Kjeldsen	Tinne Hoff	Denmark	thk@math.ku.dk
Kvasz	Ladislav	Czech Republic	ladislavkvasz@gmail.com
Lakoma	Ewa	Poland	ewa.lakoma@wat.edu.pl
Lancisi	Silvio	Italy	michelececchi@diluvio.it
Languereau	Hombeline	France	hombeline.languereau@univ-fcomte.fr
Laptyeva	Nataliya	Canada	nataliya.laptyeva@gmail.com
Lawrence	Snezana	UK	snezana@mathsisgoodforyou.com
Lefort	Xavier	France	xavier.lefort@univ-nantes.fr
Lemes	Ana	France	jimenalemes@gmail.com
Levin	Svetlana	Israel	levinsveta@gmail.com
Li	Xiaoni	China	370599934@qq.com
Liao	Chieh-Cheng	Taiwan	jenchengqq@gmail.com
Liu	Po-Hung	Taiwan	liuph@ncut.edu.tw
Lunde	Linnea Marie Holst	Norway	lihlu0105@hotmail.com
Man	Yiu-Kwong	China	ykman@eduhk.hk
Martol Briseid	Eyvind	Norway	Eyvind-Martol.Briseid@hioa.no
Massa-Esteve	M. Rosa	Spain	m.rosa.massa@upc.edu
Mauntel	Matthew	USA	mmauntel@fsu.edu
Mazzeo	Elio	Canada	ronaldelio@gmail.com
McPolin	Peter	UK	p.mcpolin@smucb.ac.uk
Menghini	Marta	Italy	marta.menghini@uniroma1.it
Métin	Frederic	France	frederic.metin01@u-bourgogne.fr
Michel-Pajus	Annie	France	annie.pajus@gmail.com
Misfeldt	Morten	Denmark	misfeldt@learning.aau.dk
Moureau	Marie-line	France	mlmoureau@aol.com
Mouyssinat	Michel	France	michel@mouyssinat.com
Moyon	Marc	France	marc.moyon@unilim.fr
Mutanen	Arto	Finland	arto.mutanen@gmail.com
Narváez	Deissy Milena	Colombia	dmnarvaez@gmail.com
Natali	Vilma	Italy	vilma.natali@istruzione.it
Nikolantonakis	Konstantinos	Greece	knikolantonakis@uowm.gr
Opsal	Hilde	Norway	hilde.opsal@hivolda.no
O'Reilly	Maurice	Ireland	maurice.oreilly@dcu.ie
Pejlare	Johanna	Sweden	pejlare@chalmers.se
Pinto	Helder	Portugal	hbmpinto1981@gmail.com
Pogonowski	Jerzy	Poland	pogon@amu.edu.pl
Popotis	Apostolos	Greece	apopotis@gmail.com
Puig	Luis	Spain	luis.puig@uv.es
Puig-Pla	Carles	Spain	carles.puig@upc.edu
Qi	Chunyan	China	476890742@qq.com
Ransom	Peter	UK	pransom@btinternet.com
Reinholdtsen	Lars	Norway	Lars.Reinholdtsen@hioa.no

Roelens	Michel	Belgium	michel.roelens@ucll.be
Romero-Vallhonesta	Fàtima	Spain	fatima.romerovallhonesta@gmail.com
Rossi	Marzio	Italy	marzio.rossi50@alice.it
Rottoli	Ernesto	Italy	ernerott@tin.it
Schorcht	Sebastian	Germany	sebastian.schorcht@math.uni-giessen.de
Shen	Zhongyu	China	344080566@qq.com
Smestad	Bjørn	Norway	Bjorn.Smestad@hioa.no
Stoffels	Gero	Germany	stoffels@mathematik.uni-siegen.de
Sun	Dandan	China	18766173312@163.com
Tattersall	James	USA	tat@providence.edu
Tournès	Dominique	France	dominique.tournes@univ-reunion.fr
Tzanakis	Constantinos	Greece	tzanakis@edc.uoc.gr
van den Bogaart	Desiree	Netherlands	d.a.van.den.bogaart@hva.nl
Vicentini	Caterina	Italy	cater.vicentini@gmail.com
Voillequin	Isabelle	France	isabelle.voillequin@hotmail.fr
Wang	Xin	China	wangxinecnu@163.com
Weiss	Ysette	Germany	yweiss@uni-mainz.de
Witzke	Ingo	Germany	witzke@mathematik.uni-siegen.de
Yue	Zengcheng	China	yuezengcheng@126.com
Zubillaga Guerrero	Erika	Mexico	eguerrero@uagro.mx