

# **Björn Gunnlaugsson – Life and work Enlightenment and religious philosophy in nineteenth century Icelandic mathematics education**

Kristín Bjarnadóttir  
School of Education. University of Iceland

## **Abstract**

*Björn Gunnlaugsson (1788–1876) was a remarkable product of the Enlightenment movement in Iceland. He was never admitted to a school, but learnt mathematics by himself and from his tenant-farmer father before he entered the University of Copenhagen, where he twice won its gold medal. He taught mathematics for forty years and made valuable geodetic measurements as a basis of a map of Iceland. He wrote about various astronomical topics, and published his first book on mathematics at the age of 77. The book, which is the main subject of this paper, reveals his devotion and genuine knowledge of mathematics, and his philosophical and religious attitude towards mathematical concepts, structure and conventions. In order to clarify his way of thought, several topics will be explored: zero and infinity, divisibility, exponential laws and imaginary numbers.*

## **Introduction**

Iceland had lagged behind other European countries in many respects in the eighteenth century, due to various calamities. The Danish authorities made some efforts to gather information about life and conditions on the remote island with the aim of improving the situation, such as to arrange national censuses, have the land surveyed and commission reliable maps. Proponents of the Enlightenment movement initiated efforts in publishing educational material containing guidance to better farming and crafts, based on the scientific knowledge of that time.

Several arithmetic textbooks were written and distributed, both in manuscript form and in print, but the two Latin schools were not prepared to teach mathematics. Sources say that everyone who reached the upper class of one of them was given an arithmetic textbook, but that it was up to the pupils whether they ever opened the book or not (Helgason, 1907–1915, pp. 85–86). The University of Copenhagen was the natural continuation of the Icelandic Latin schools, and it only introduced more stringent requirements in mathematics under regulations of 1818.

Björn Gunnlaugsson (1788–1876), the son of a gifted tenant farmer, was a prominent product of the Enlightenment movement without ever being admitted into a school in Iceland. He enrolled in the University of Copenhagen in 1818, won two gold medals in mathematics, and returned home to teach for forty years at the Latin School, which by that time had become the only school in the country. In 1865, after his retirement, he wrote a book on advanced arithmetic, *Tölvísi*, that

Bjarnadóttir, K., Furinghetti, F., & Schubring, G. (Eds.) (2009). “Dig where you stand”. *Proceedings of the conference “On-going research in the History of Mathematics Education”*.

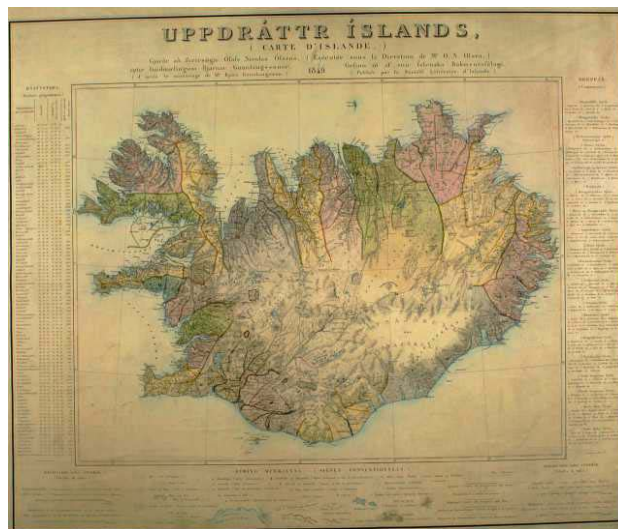
reflects his philosophical and religious attitudes to mathematics. *Tölvísi* is the main subject of this paper.

## Gunnlaugsson's background

Gunnlaugsson's father was a gifted man, who won prizes from the king for his inventions. It is not clear how he learned mathematics, but he is credited with teaching his son to such a level that the bishop said, in his testimonial about Gunnlaugsson's matriculation examination in 1808, that he had, by independent study, not only been through both the computing arts (probably arithmetic and algebra) but also geometry, trigonometry, stereometry, finite and infinite calculus, statics, mechanics, and parts of the measuring arts, and for this he deserved the strongest praise, *Egregia Laude* (Melsted & Jónsson, 1947, pp. 77–78).

During 1808–17 Gunnlaugsson became acquainted with Danish and Norwegian land surveyors, who gave him books and guided him in geodesy and mathematics, enough for him to earn the university's gold medal before being admitted in early 1818 (Melsted and Jónsson, p. 66). He studied mathematics and geodesy at the University of Copenhagen in 1818–22, for the first year with Professor C. F. Degen. In 1820 and 1821 he assisted Professor H. C. Schumacher, Gauss's collaborator, in land surveying in Altona, Germany. Gunnlaugsson is even conjectured to have met Gauss and told him that he had read Gauss's theory of celestial bodies (Björnsson, 1997).

Gunnlaugsson taught mathematics at Bessastaðir Latin School and later at Reykjavík Latin School in 1822–62. He surveyed Iceland during twelve summers in 1831–1843, forming the basis of a map of the country that was published by the Icelandic Literary Society. It was one of the society's greatest treasures, and for the following half-century it was the basis for most maps of Iceland.



Gunnlaugsson's map of Iceland, 1849 (National and University Library website)

During the same period, Gunnlaugsson composed his great astronomical, philosophical and religious poem *Njóla* in 518 verses about ‘easy examination of the world and the highness of God and his global intensions or his purpose with the world’. The poem may be considered as a contribution to natural theology, in which the author concentrated on finding the purpose of the creation of the world (Guðmundsson, 2003).

The Latin School was situated at Bessastaðir in the vicinity of Reykjavík from 1805 to 1846. The building had previously been the residence of the governor of Iceland and could only accommodate a limited number of pupils: initially 24, but rose to 40 by 1840. The rooms were so small that the pupils were likened to lambs in a pen (*Report 1840–1841*). The teachers did not live at the school, but were farmers in the surrounding area, including Gunnlaugsson. In 1846, when Gunnlaugsson was 58 years old, the school moved to the growing capital of Reykjavík. The change in environment, in addition to trends from European revolutionary movements of 1848, proved difficult for the school, but Gunnlaugsson’s gentleness ensured him his pupils’ continued respect.

### ***Tölvísi***

In 1865, when Gunnlaugsson was 77 years old, the Icelandic Literary Society published his book on advanced arithmetic, *Tölvísi*. The Literary Society had commissioned Gunnlaugsson as early as 1855 to write a theoretical arithmetic book (Minutes, Feb. 19). He was in correspondence with the president of the Copenhagen chapter of the Society and the leader of Iceland’s independence movement, Jón Sigurðsson. In August 12, 1861 he wrote to Sigurðsson:

I have been this summer working hard on the arithmetic textbook, as the Society calls it, and it is planned that I show 40 quires [each 16 pages], written by my hand, at the next society meeting. It is the theoretical arithmetic, even if with the practical mixed in, I valued it less, as that is contained in the Icelandic ones. I call this arithmetic textbook *Tölvísi* [Number wisdom] [...] as in [the ancient manuscript] Snorra Edda. In its course I have inserted many things which are not common, and intend that there will be everything that is taught in schools and even more (Lbs. 2590, 4to, transl. by me, K. B.<sup>1</sup>).

In 1862 (Minutes, Sept. 17) Gunnlaugsson presented 800 handwritten pages and the members of the society concluded that printing should commence the following winter. Gunnlaugsson presented 96 printed pages in April 1864 (April 20). In 1865, 400 printed pages were published, stopping abruptly in the midst of continued fractions. Gunnlaugsson handed in the second part of his work in manuscript (Oct. 5, 1867) together with introduction and a list of contents, but took it back for further examination, to return it finally in 1868 (May 30). The minutes of the Literary Society reveal nothing more about the fate of the manuscript, but the fact remains that it was never printed. In the autumn of 1868

---

<sup>1</sup> All quotes in this article are translated by the author.

(Sept. 18) Gunnlaugsson offered the Society the opportunity to purchase a manuscript of *Plane Geometry* which he had begun to write but not finished. *Plane Geometry* was discussed both in the Reykjavík and the Copenhagen chapters of the Society (Sept. 10, 1869) but was finally turned down. Gunnlaugsson was highly respected, not the least for his maps, but apparently the Society refrained from publishing the second part of the book, its means being limited.

Who would read *Tölvísi*? The Icelandic Literary Society was a forum for the Icelanders' campaign for independence from Denmark in the nineteenth century and it enjoyed great support. In a population of less than 70,000 the Literary Society had over 700 subscribers in 1865 (Líndal, email), who were obliged to buy *Tölvísi*. Gunnlaugsson's students during his 40 years at the Latin School, around 300 in total, were the most likely to understand the text, but there was also a tradition of attaining knowledge among farmers who could afford books. There were no schools in the country besides the Latin School, which only few could afford to attend.

All topics in *Tölvísi* are thoroughly explained from first principles with several examples, but they develop, however, quite steeply into higher mathematics. The book is said to have turned out to be the 'book that everyone praises but no one reads' (Melsted & Jónsson, 1947).

### **The content of *Tölvísi***

*Tölvísi* comprises two parts:

288 paragraphs on 400 printed pages.

251 paragraphs on 810 pages of an unpublished manuscript (Lbs. 2397, 4to).

The text flows onwards in logical order, but with several detours to topics which the author wants to share with the reader in order to be able to explain in full depth the subject of the section in question. We shall now look into the content of *Tölvísi*, the printed part and the manuscript:

Introduction – 4 printed pages.

On numbers.

Numeration – 8 p.

Number notation in base ten and various other bases.

The four species in whole numbers and algebra.

Addition and subtraction – 30 p.

Multiplication and division – 60 p.

Fractions – 34 p.

On the nature of whole numbers – Introduction to number theory – 75 p.

Modular arithmetic, congruences, prime numbers.

Euclid's Algorithm, Fermat's Little Theorem.

Decimal fractions – 102 p.

Periodic fractions, emphasis on numerical methods and error bounds.

Exponents and roots – 73 p.

- Exponential laws, extracting square root and cubic root.
- Irrational and imaginary roots, imaginary numbers, binomial theorem.
- Continued fractions – 12 p. + 63 manuscript pages.
  - Approximation to fractions and irrational numbers.
- Greatest common divisor of algebraic expressions – 11 ms.p.
- Ratio and proportions – 141 ms.p.
  - Arithmetic and geometric ratios, the Rule of Three.
  - Harmonic proportions, interests, density.
- Equations – 482 ms.p.
  - 1st and 2nd degree equations, one or more unknowns.
  - Square root equations, polynomials, imaginary roots, approximations.
  - The general but indirect solution of all higher equations in numbers.
  - Differential calculus (Lagrange function calculus), Taylor series.
  - Fourier's rule of a double sign.
  - Cubic equations (Cardan's rule) and biquadratic equations.
  - Indeterminate Diophantine equations solved by continued fractions.
  - Exponential equations, logarithms.
- Progressions – 47 ms.p.
  - Arithmetic, geometric and harmonic sequences and series.
- Functions with indeterminate coefficients – 46 ms.p.
  - Logarithms.
  - Convergent and divergent series.
- Interests, annuity – 13 ms.p.
- Permutations, combinations – 8 ms.p.

The list of contents reflects Gunnlaugsson's view of life. He grew up in the spirit of the Enlightenment and land surveying, which is expressed in his emphasis on numerical methods, accuracy in calculations and on error bounds. His life as a land-surveyor and a farmer is also reflected in his concern for the agricultural environment and his choice of metaphors in problems chosen to demonstrate theories or techniques. The list reflects also Gunnlaugsson's attitude towards mathematics as a divine science and his philosophical and religious views on the nature of mathematics, especially in his treatment of zero and infinity, the imaginary numbers and the laws of exponents.

During his studies Gunnlaugsson is known to have read works by Euler, Lagrange and Kästner, professor in Göttingen (Björnsson, 1997). *Tölvísi* reflects these works, but works by Fermat, Leibniz, Maclaurin, d'Alembert, Fourier and Gauss are also mentioned, all of which were developed before 1822, when Gunnlaugsson left Copenhagen for Iceland. No later nineteenth century mathematicians are mentioned but several contemporary Danish textbooks, such as those by L. S. Fallesen and C. Ramus, and eighteenth and nineteenth-century Icelandic arithmetic textbooks are quoted.

## Gunnlaugsson as land-surveyor and calculator

Throughout the whole work, Gunnlaugsson's background as land-surveyor and calculator is apparent. He gave a very thorough treatment of decimal fractions, but his most elegant topic was the continued fractions. They seem to have been much used at the time. Degen, Gunnlaugsson's teacher at the University of Copenhagen, wrote about Pell's equation, which Euler had tried to solve by continued fractions. Continued fractions are, e.g., found in Fallesen's textbook for the Latin School: *Begyndelsesgrunde i den rene Mathematik* (1834, pp. 135–145, 213–216). Gunnlaugsson used continued fractions to approximate irreducible fractions and irrational quantities like Fallesen, but also to solve Diophantine equations. He thus went further than Fallesen, as he promised in his letter to Sigurðsson (Lbs. 2590, 4to).

The word *Tölvísi* means 'number wisdom', presentation of which is the book's main aim. However, Gunnlaugsson borrowed (in his own words) theorems from other sections of mathematics, such as from trigonometry in order to factor the equations  $x^n \pm a^n = 0$  (before he proved the general solution of a quadratic equation) and thus be able to solve cubic equations. He also introduced differential calculus in order to be able to locate roots of functions. He explained the difference between the presentations of differential calculus by Maclaurin and d'Alembert on one hand and by Lagrange on the other hand, in the same manner as explained by Katz (1993, p. 530). However, Gunnlaugsson's main aim is to use the calculus to help him locate roots with the aid of Fourier's rule of a double sign.

## Gunnlaugsson as a traditional Icelandic farmer

The reader may notice that Gunnlaugsson was a farmer, firmly positioned in the old Icelandic farming society, as witnessed by his use of metaphors.

When introducing unknown quantities (1865, p. 2) he told a story about three groups of horses used for transport. The first group was large, the second one still larger and the third one was equal in number to the sum of the two other groups.



Gunnlaugsson then discussed a number of conclusions: The total number of horses was even. If the first group contained 12 horses, then the total number was at least 50. The total exceeded four times the number in the first group plus two, etc.

Gunnlaugsson knew the ancient puzzle of thirty birds, ducks, swans and sparrows, each with its price, worth 30 pence, which are the wages of a worker hunting them, given by his master (Lbs. 2397, p. 1375). The puzzle is hidden in verses, but he used it to demonstrate solutions to Diophantine equations by continued fractions. It gives two equations in three unknowns:

$$x + y + z = 30$$

$$1/2x + 2y + 1/10 z = 30$$

where  $x, y$  and  $z$  denote the numbers of birds.

The equations reduced to  $4x + 19y = 270$

Gunnlaugsson found several solutions

Ducks	$x$	1	20	39	58
Swans	$y$	14	10	6	2
Sparrows	$z$	15	0	-15	-30

The first solution is the only valid one, but the author could justify the others, such as by counting the negative numbers of sparrows as debts that the worker owes his master for allowing him to hunt the great number of other birds.

Gunnlaugsson knew a number of old problems, but he did not present them for their own sake but in order to demonstrate theories or methods. Problem solving in general does not seem to have appealed to him.

A textbook by one of Gunnlaugsson's students indicates that he may have taken the grazing of cattle as an example where proportions in the form of the Rule of Three did not apply. In *Tölvísi* this problem was attacked in detail and used to illustrate a solution of two equations with two unknowns (Lbs. 2397: §421, p. 1165–1173). The grazing per square fathom meadow in one day of a certain number of cattle  $m$  in a certain space  $p$  with the grazing  $g$  of one cow is

$$(m/p)g$$

from which one must subtract the grass-growth  $b$  in one day in a square fathom. This gives the decrease in height of the grass in one day as

$$(m/p)g - b$$

Using two given cases he finds that  $b/g = 1/250$  and the original grass-height  $G$ , when the grass has been bitten down to the roots in  $n$  days, was

$$G = ((m/p)g - b)n$$

The number  $G = 196$ , which has no unit, is found from the two given cases so the number  $m''$  of cows grazing on the grass of a  $p'' = 600$  square fathom meadow in  $n'' = 84$  days is

$$m'' = (G + bn'')p''/(gn'') = ((196 + 1 \cdot 84) \cdot 600)/(250 \cdot 84) = 8$$

The solution is developed and discussed in eight handwritten pages. The discussion might have been of interest to reflective farmers, but regrettably this part of the work was never published.

### **Gunnlaugsson as a philosopher and eighteenth century mathematician**

Gunnlaugsson is a remarkable example of a deeply thinking person who developed his ideas in isolation. All his life he was the only mathematician in Iceland and could not share his thoughts with anyone but his pupils. He wrote papers of various kinds, mainly astrological in nature, but not about mathematics, and he is not known to have been in correspondence with European mathematicians (Guðmundsson, 2003). His religious disposition created a unique

system of ideas and attitudes which are reflected in his poem *Njóla* and in a different way in *Tölvísi*. We shall look at several topics - zero and infinity, divisibility, exponential laws and imaginary numbers - and explore how his religious and philosophical attitudes affected his presentation.

### *Zero and Infinity*

Gunnlaugsson developed his own ideas on zero, as well as on mathematical infinity, which he tied to religious ideas of eternity.

In the series

$$1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \dots + x^n/(1-x)$$

he inserted  $x = 1/2$  and wrote:

[...] it is [...] obvious that 2 is exactly the number the series aims at and never exceeds; in order to reach 2 eternity is needed, but to reach further, more than eternity is needed. The infinite series reaches eternity as it lies in it always and has been lying in it eternally, even if the one who calculates never reaches it by going on. What is eternal is eternally eternal and does not need to reach eternity, but what is not eternal can never become eternal (p. 94).

He continued investigating the series, replacing  $x^n$  by  $x^\infty$  to reach in the case  $x = 1$

$$1/0 = 1 + 1 + 1 + 1 + 1 + \dots + 1^\infty/0$$

which led him to the conclusion that the infinite cannot receive any addition, as then one has exceeded the limits of quantity, which are 0 and  $\infty$ . Eternity would not be any longer, even if months or years were added to it; the eternity awaiting Alexander the Great is no longer than the eternity awaiting Napoleon the Great, even if he lived later (p. 96), so that

$$\infty + \infty = \infty \quad \text{just as} \quad 0 + 0 = 0$$

Gunnlaugsson stated that a number is a discrete quantity's comparison to the unit (p. 2). But the zero is a digit, denoting an empty seat, no unit, nothing. He introduced negative numbers in the sequence ... -3, -2, -1, 0, +1, +2, +3, ... and said about the zero:

[§35] [...] Between these two infinite sequences stands 0 in the middle, as neither positive nor negative, looking to neither direction, or rather as well, both positive and negative and looking to both directions [...]  $\infty$  (infinitely large) is more than all finite, positive quantities, 0 less than all positive quantities, the negative quantities less than 0, and finally the  $-\infty$  (negative infinitely large) less than all finite negative quantities (p. 26).

Farther on in the book, after having introduced division, he said:

The quotient  $\infty$  cannot become greater, as the divisor 0 cannot become smaller. Both 0 and  $\infty$  are therefore the limits of quantity. The zero is actually no quantity but it is less than all quantities; similarly  $\infty$  is no quantity but larger than all quantities. That 0 is less than any quantity should here be taken in another sense than before (§35), where the negative quantities were said to be less than 0. From §35 it is in fact obvious that 0 must be less than the negative numbers in division, even if it is not so in subtraction, as it stands in the centre between two infinite sequences, and the negative



numbers are in fact the opposite quantities calculated from this same centre in the opposite direction to the positive ones. From that one can see that 0 or the centre itself is no quantity, but the origin of the quantities, having no quantity itself and no sign, + or – (p. 97).

About quantities one can say that they are equal, but not about 0 or  $\infty$ , which are not quantities but the limits of quantity, and it cannot enter them or reach out to them as 0 lies further in than any quantity and the  $\infty$  outside it. It is also questionable to say that zero (nothing) is infinitely small and the  $\infty$  infinitely large, (even if people do so), as small and large belong to quantities and not to what is not quantity. It is better to call them inner infinite and outer infinite or the quantity's inner and outer limit. The zero could also be called the quantity's *terminus a quo* and the  $\infty$  its *terminus ad quem* (p. 99).

In continuation Gunnlaugsson treated  $\infty$  of first or higher order.

Gunnlaugsson is known to have read the series of *Anfangsgründe* by the German Professor A.G. Kästner (Björnsson, 1997) a work of great importance. In *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspectiv* it says:

This term 'less than nothing' presupposes [...] a meaning of the word Nothing which relates to a certain manner to regard the 'something' (nihilum relativum) and which discerns it from the Nothing, regarded without any relation (nihilum absolutum) [...] If one does not regard the term 'less than nothing' in this meaning it becomes wrong [...] and, in fact, mathematical experts have been seduced to erroneous conceptions of negative quantities [...] (Kästner, 1792, p. 73).

This quotation relates to differentiation between the philosophical/metaphysical Nothing and the mathematical relative zero which d'Alembert had rejected but which the Germans have propagated (G. Schubring, email).

Gunnlaugsson did not mention this distinction of the 'Nothing', but he gave the zero and the negative numbers much thought. He had doubts about the expression 'less than nothing', which he used originally about the negative numbers (p. 26), but opposed himself by writing that '0 must be less than the negative numbers' (p. 97) when he introduced division. His trouble is related to the discrepancy between introducing the zero within a set of numbers in the sequence ... -3, -2, -1, 0, +1, +2, +3, ... (p. 26), on one hand, and on the other hand to regard the zero as no quantity (p. 99). This dilemma was first solved in the 1880s when the construction of natural numbers was realized in the works of Dedekind, Peano and Frege. The quotes reveal that Gunnlaugsson was considering the same questions as many other mathematicians in the eighteenth and nineteenth centuries.

### *Divisibility*

When leaving this difficult subject of 0 and  $\infty$ , Gunnlaugsson moved on to fractions and further on to divisibility and modular arithmetic, which was one of his favourite subjects. After thoroughly explaining divisibility and finding the prime

modulus of every whole number, Gunnlaugsson warned that e.g. the methods of casting out nine and casting out eleven are not infallible. If both methods were applied, one error out of 99 would not be discovered. However, he wrote, we want to become acquainted with the nature of numbers. In science one should not measure the value in shillings as *Man Does Not Live by Bread Alone* (p. 193).

### *Law on exponents*

The following topic was decimal fractions and error bounds, treated very thoroughly, and thereafter exponents. Gunnlaugsson was concerned about remarks on exponents in Danish textbooks by Fallesen and others. In his understanding it was an inviolable law that e.g.  $a^0 = 1$ , and he wrote: ‘This law on the exponents is higher than any human agreement; it lies in the nature of the quantities, established by the eternal (p. 320).’

Gunnlaugsson objected again to Fallesen, who wrote that the exponents could be extended to include fractions and that one agreed upon that

$$a^{n/m} = \sqrt[m]{a^n}$$

Gunnlaugsson complained that Fallesen and others treated this topic as a convention (*placita*) when it should be a theorem (p. 331). That gave the impression that man could have anything to do with the quantities’ eternal law. In his understanding these learnings were higher than laws set by human beings.

### **The non-existence of imaginary numbers**

Gunnlaugsson’s extensive knowledge of mathematics appears well in his treatment of imaginary numbers. He explained clearly why the square root of  $-1$  has to be as it is, even if he had difficulties in finding its right placement:

This coefficient [of  $\sqrt{-1}$ ] is [...] the number that counts how often the imagined unit  $\sqrt{-1}$  is taken. In this sense we can call the imagined quantities quantities [...] When they appear in calculations, they denote non-existence of quantities, for which an uncountable number of examples may be found in the analytic geometry [...]

The non-existence of quantities that the imaginary quantities denote in the analytic geometry is different from the nonexistence that the zero denotes. The zero is the origin of quantity and from there it grows either in a *positive* or *negative* direction. But the imaginary quantity is a complete denial of all quantity so that it can neither be *positive*, zero nor *negative*, and not  $\infty$  either, as  $\infty$  is the other path between the *positive* and the *negative* (p. 347–348).

At this point, Gunnlaugsson introduced the Cartesian coordinate system and analytic geometry. He took an example of a formula to be presented in the system:  $y = \pm\sqrt{a^2 - x^2}$ , the equation of a circle, where  $a$  is half the diameter. When inserting  $x = \pm a$ , then  $y = 0$ , which means that both points of the circle lie on the axis, while when  $x > a$ , the  $y$  becomes an imaginary number. This illustrates, he wrote, the difference between the quantitative denial of the zero and the quantitative denial of the imaginary quantity. The zero points to a certain point on the axis,

while the imaginary quantity says that it is nowhere (p. 349). Gunnlaugsson continued with examples on the ellipse and the parabola, referring to two new textbooks on astronomy and physics, which the reader might know, translated into Icelandic, and then he concluded with an example on the hyperbola.

The virtue of the imaginary numbers was, he said, however, not only included in the denial of the quantity, but

[...] they are in mathematics similar to the airship of physics, as the mind can on the imaginary airship like elevate itself from the earth and sail back and forth in the *Ginnungagap*<sup>2</sup> of non-existence and then return to the earth when it chooses (p. 350–351).

Here one must recall that the year was 1865 and airships were still only imaginary phenomena.

The existence of imaginary numbers was not completely accepted until the geometrical interpretation of complex numbers had been described. This was first done by Caspar Wessel (1745–1818), a Norwegian who went to Denmark for further studies. He was employed as a surveyor and it was the mathematical aspect of surveying that led him to explore the geometrical significance of complex numbers. He had been developing more and more sophisticated mathematical methods of surveying, which he explained in a report in 1787. This report already contains Wessel's innovation. His fundamental paper, *Om directionens analytiske betegnning*, was published in 1799 by *Det Kongelige Danske Videnskabernes Selskab* / The Royal Danish Academy of Sciences and Letters. Since the paper was in Danish, it passed almost unnoticed in mainland Europe, and the same results were later independently found by Argand in 1806 and Gauss in 1831 (University of St. Andrews, website).

Wessel's ideas may not have been as unnoticed in Denmark as elsewhere and Gunnlaugsson may have become acquainted with them, either while assisting Danish surveyors in Iceland before his stay in Copenhagen or during his studies there. He may even have met Wessel during the winter of 1817–18. However, Gunnlaugsson wrote about imaginary numbers as an isolated phenomenon and he did not mention complex numbers. His text above reveals that he was thinking about imaginary numbers in connection with geometric concepts, but he could not position them in the geometric plane, so he did not present imaginary numbers geometrically.

Another author, who wrote about imaginary numbers and whose work Gunnlaugsson might have known, is Wenceslaus J. G. Karsten (1732–87), mathematics professor at Bützow, and, later, at Halle in Germany. In a paper, first published in 1768 and later in a revised version in 1786, he discussed the possibility of a geometrical construction of imaginary quantities. He gave as an example a particular relation between a hyperbola with the equation  $x^2 - y^2 = 1$  and the related circle  $x^2 + z^2 = 1$ , where all values of  $y$  of the hyperbola for  $x$  between  $-1$

---

<sup>2</sup> *Ginnungagap* belongs to the Norse mythology as the vast, primordial void that existed prior to the creation of the manifest universe.

and 1 are imaginary. All ordinates of the circle are imaginary ordinates of the hyperbola, and vice versa, all ordinates of the hyperbola are imaginary ordinates of the circle (Schubring, 2001, pp. 142–143). This paper by Karsten bears some resemblance to Gunnlaugsson's ideas. Karsten's publication of 1786, *Mathematische Abhandlungen, theils durch eine Preisfrage der Königl. Pr. Acad. vom Jahr 1784 über das Mathematisch-Unendliche, theils durch andre neuere Untersuchungen veranlassen*, exists in the Royal Library in Copenhagen, and may well have been there at Gunnlaugsson's time in Copenhagen.

It is likely that Gunnlaugsson knew Wessel's and Karsten's writings even if his ideas do not seem to be fully identical to theirs, at least not Wessel's geometrical interpretations. At any rate, Gunnlaugsson was, in his solitude, challenging his mind with the same problems as mathematicians elsewhere in Europe had done.

### **The impact of Björn Gunnlaugsson's work**

What was the impact of the 400-page book *Tölvísi*? About 700 copies of it were distributed, but sources say that few read it. It was never used as a textbook in the Latin School to replace the current Danish textbooks or in any other school. For that purpose it must have been considered too advanced. Possibly it was also considered wiser to use the Danish textbooks authorized or recommended by the Royal Directorate of the University and the Learned Schools for the Latin School.

What impact had Gunnlaugsson's 40 years of teaching? None of his students studied mathematics at a university, but three of them studied at the Polytechnic Institute in Copenhagen. Two of his students published mathematics textbooks for the general public. Two other students translated the textbooks on astronomy and physics mentioned earlier and thanked him cordially for his good advice. He was respected and loved by his students and the public and was honoured in many ways in his late life.

However, few could follow his thoughts. He was a loner in this respect and had to work alone on mathematics all his life after he left Copenhagen. He was far ahead of his Icelandic contemporaries in mathematics, but inevitably fell increasingly behind his European colleagues.

### **Summary**

Gunnlaugsson was nearly 30 years old when he could enter university studies in mathematics. His attitude to life, learning and mathematics was shaped by two very different worlds: his upbringing as a son of a tenant farmer in the old Icelandic culture, rooted in the sagas and Norse mythology, pious Christendom and the Enlightenment, and his experience in land-surveying and the eighteenth century mathematics books that he read.

He was all his life considered a unique but highly respected person, called the wise man with the heart of a child. His book *Tölvísi* illustrates sincere and honest reflections of a person who has lived in intellectual solitude.

Before Gunnlaugsson's time there was no one to teach mathematics at the Latin School and after his time, in 1877, mathematics teaching at the Latin School was cut back by a political decision, and it did not rise again until 1919.

Therefore the history of Gunnlaugsson and his book, *Tölvísi*, and the works of his former pupils, is nearly the complete history of creative secondary level mathematics education in nineteenth century Iceland.

**Acknowledgment.** The author wants to express her gratitude to Gert Schubring for valuable hints and references.

## References

### Electronic sources

- Líndal, S., president of the Icelandic Literary Society, email, August 15, 2009.  
National and University Library, website: <http://kort.bok.hi.is/kort.php?a=gm&id=599>, accessed August 14, 2009.  
Schubring, G. email, June 6, 2009.  
University of St. Andrews, website: [www-history.mcs.st-andrews.ac.uk](http://www-history.mcs.st-andrews.ac.uk).

### Manuscripts from the National and University Library Archives

- Lbs. 2397, 4to. *Tölvísi* I (printed) and II (manuscript).  
Lbs. 2590, 4to. Jón Sigurðsson's correspondence.  
Minutes book of the Icelandic Literary Society 1816 to 1879.

### Printed sources

- Björnsson, O. J. (1997). *Varð Gauss á vegi Björns Gunnlaugssonar?* Reykjavík: Raunvísindastofnun Háskólans.  
Fallesen, L. S. (1834). *Begyndelsesgrunde i den rene Mathematik*. Copenhagen: C.A. Reitzel.  
Guðmundsson, E. H. (2003). Björn Gunnlaugsson og náttúruspekin í Njólu. *Ritmennt. Ársrit Landsbókasafns Íslands – Háskólabókasafns*, 8, 9–78.  
Gunnlaugsson, B. (1865). *Tölvísi*. Reykjavík: HÍB (The Icelandic Literary Society).  
Helgason, Á. (1907–1915). Frásagnir um skólalíf á Íslandi um aldamót 18. og 19. aldar. 1. Skólahættir í Skálholti og í Reykjavíkurskóla hinum forna. In *Safn til sögu Íslands og íslenskra bókmennta að fornu og nýju* (Vol. IV, pp. 74–98). Copenhagen, Reykjavík: HÍB.  
Katz, V. (1993). *A history of mathematics. An introduction*. New York: Harper Collins.  
Kästner, A. G. (1758, 1792). *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspectiv*. Göttingen: Vandenhof und Ruprecht.  
Melsted, P., & Jónsson, B. (1947). Björn Gunnlaugsson. In Þorkell Jóhannesson (Ed.), *Merkir Íslendingar* (Vol. I, pp. 65–78). Reprint from *Andvari, tímarit hins íslenska þjóðvinafélags*, 9 (1883), 3–16. Reykjavík: Bókfellsútgáfan.  
*Report on Bessastaðir-School for 1840–1841*. Printed as an appendix to *Bòðsrit*. Viðeyjarklaustur.  
Schubring, G. (2001). Argand and the early work on graphical representation: New sources and interpretation. In J. Lützen (Ed.), *Around Caspar Wessel and the geometric representation of complex numbers. Proceedings of the Caspar Wessel Symposium at The Royal Danish Academy of Sciences and Letters, Copenhagen, August 11–15, 1998* (pp. 125–146). Copenhagen: C. A. Reitzels forlag.

