# REGULA DE TRI

# ITS ORIGIN AND PRESENTATIONS IN ICELANDIC TEXTBOOKS

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*Regula de Tri, the Rule of Three, is in its simplest form a way of computing the fourth proportional in an equation of two ratios. It was an indispensable topic in Iceland from the first textbooks in arithmetic in the 18th century until the 1970s. Instruction in the Rule of Three in compulsory education developed into a ritual procedure for exercises of certain types and content, often without relevance to contemporary life or reference to ratios and proportions. It was subjected to severe criticism at times of changes in compulsory education, in the early 20th century and at the introduction of the "New Math" in the 1960s. At that time the Rule of Three was quashed once and for all, and attention was turned to the plain concepts of ratio and proportions.*

## INTRODUCTION

At times of crossroads in the history of mathematics education in Iceland *Regula de Tri, the Rule of Three,* was called an unnecessary system, applied to problems which might be solved by a simple reasoning. It was even called an insult to common sense.

Where does this *Rule of Three* come from, and why did it have such a long life in arithmetic textbooks, longer than in other countries? What are its merits? We will investigate what has been written about this rule and trace it in Icelandic textbooks.

## ORIGIN OF THE REGULA DE TRI

The most ancient example of the Rule of Three is found in the Rhind Papyrus (1650 BC) (Tropfke, 1980, pp. 359–363). Problem 69 states that from 3 ½ hekats of flour 80 loaves can be made. How much flour is needed for 1 loaf? How many loaves can be made from 1 hekat of flour? In the Chinese *Jiuzhang suanshu*/*Nine Chapters on the Mathematical Art* (Period of Han Dynasty, 200 BC–200 AD) this rule is described generally:

> Mit der Menge des Vorhandenen multipliziert man die Messzahl des Gesuchten: es ist der Dividend. Nimm die Messzahl des Vorhandenen; es ist der Divisor. Teile den Dividenden durch den Divisor (Tropfke, p. 360).

The terms "Vorhandenes/available", "Gesuchtes/sought after" and "Messzahl/-measured value" are general and may be applied to a whole group of problems.

The Indian Brahmagupta (7th century) spells the rule out as well. Arabic study of the Rule of Three is derived from the Indian, while al-Kwarizmi (c. 780–850) improved its vocabulary (Tropfke, p. 361). Heath (1956, II, p. 215) calls the construction of the fourth proportional in *The Elements*, prop. 12, Book VI, the geometrical equivalent of the Rule of Three. Leonardo of Pisa states in *Liber Abaci* (Siegler, 2002, p. 127) that four proportional numbers are always found in all negotiations, of which three are known and one unknown. Leonardo used alphabetical symbols to express the extended rule into the Rule of Five (Tropfke, p. 362).

In subsequent European arithmetic textbooks, the Rule of Three becomes more and more schematised into a mathematical formula. The Rule was highly esteemed in the 15th and 16th century as a powerful mathematical technique applicable to many problem-solving situations. The intellectual impact of practical arithmetic textbooks of this period influenced the composition and context of subsequent arithmetic texts. The arithmetic taught in Europe until the 20th century was primarily business arithmetic, reflecting commercial interests of an earlier age (Swetz, 1992).

Hatami (2007, pp. 185–199) divided Swedish textbooks into three groups with respect to the Rule of Three, as to whether they emphasized algorithms, demanding more remembering than understanding; proportionality, relating to earlier definitions; or went back to find the unit, where the calculations are built on logical reasoning and verbal argumentation. In his final comments Hatami discusses as an interesting question whether the rhetoric of the Rule of Three, with its links to languages and argumentation, could not be a doorway to algebra for young pupils in the present day.

## THE RULE OF THREE IN ICELANDIC ARITHMETIC TEXTBOOKS

The Rule of Three is found in most arithmetic textbooks known to have been used in Iceland since the 17th century, up to the 1970s. Their content concerns weight, measure and monetary exchanges, the typical problems in trade. We shall look at a few examples of the Rule of Three from influential Icelandic arithmetic textbooks, as well as criticism on the mechanical presentation of its solution process.

### ***Arithmetic*** **from 1780 by Olavius**

The oldest Icelandic printed arithmetic textbook with the Rule of Three is *Greinileg vegleiðsla til Talnalistarinnar*/*A Clear Guidance to the Number Art with its four Species and the Rule of Three* by philosopher Ó. Olavius (1780). The author says of the Rule:

> ... it is generally called Regula trium (terminorum) or regula de-tri ... [or] (regula proportionum), as it is built on proportions (§. 197). also ... (regula aurea), due to the great use of it in society and all fields of knowledge (Olavius, 1780: p. 290).

The Rule of Three is presented within a section on arithmetic and geometric ratios of numbers (Olavius, 1780, p. 170). Examples of both kinds of ratio are presented in six pages. As a general solution to the problems of finding the fourth proportional number, when three are known, in the equation a : b = c : d (presented only in examples such as 12 : 4 = 17 : __), a three-step procedure is presented, where the first number is called ‘front term’, the second one ‘middle term’ and the third ‘rear term’:

> 1° The ratio of the front term to the middle term is as the rear term is to the sought term.
>
> 2° Multiply the middle term and the rear term together.
>
> 3° Divide the above product by the front term and the answer will be the fourth term, as the product of the middle term and the rear term is equal to the product of the first and the fourth terms (as explained previously) (Olavius, 1780, p. 176–177).

A couple of examples follow, together with an explanation that one may divide the front term into the middle term, or divide the front term into the rear term, to obtain the same solution. However, by using these procedures one may reach fractional numbers, which is not so convenient; hence their use is not customary, as it cannot always be foreseen if the divisor will divide the other term evenly.

Ratio and proportions are covered on nine pages in small format. Further back in the book, when fractions have been treated properly, the Rule of Three is taken up again with explanations and examples. Three more rules on how to write down such problems are presented. After eight pages it says that clearly a number cannot be found by the Rule of Three unless one knows that the numbers in question obey geometric ratio. Examples follow, such as price proportional to the quantity of merchandise and money exchange. After 17 pages, procedures on solving Rule-of-Three problems are summarised in six rules, such as to ensure that the numbers are of similar kind, consider fractions, try to reduce the numbers by cancelling etc., followed by 40 pages of examples. The *Clear Guidance* treats the Rule of Three thoroughly and sensibly, by explaining its background and each step in the procedures. The book contains no exercises.

### ***Arithmetic* by E. Briem**

*Reikningsbók*/*Arithmetic* by the Rev. Briem was an influential textbook for nearly half a century, from its publication in 1869 until after its last edition in 1911. It says:

> The Rule of Three is a method to find how much a number makes, when another number of the same kind makes so and so much; for example: when 2 pounds cost 3 marks, how much do 4 pounds cost? Answer: 1 rix dollar ... there are thus three terms, and the fourth is sought after; in the example taken, 2 pounds, 3 marks and 4 pounds are the three available terms, while the price of 4 pounds is the fourth term sought after. The first term … is called *front term*, the second … *middle term*, and the third one *rear term*; … in a direct Rule of Three one is seeking the number which is in the same ratio to the rear term as the middle term is to the front term, or *is as many times greater or less than middle term as the rear term is greater or less than the front term*. ... there are always two sentences (ratios), ... in the definite sentence there are the front term and the middle term, but in the question sentence the rear term, corresponding to the front term in the definite sentence; … there must always be the same kind in the front term and the rear term; as that what is sought for must be of the same kind as the middle term (Briem, 1869: p. 69).

This is further explained in an example and summarized in three rules:

> 1st rule. The front term and rear term have to be made into the same name if they are not.
>
> 2nd rule. Multiply the middle term by the rear term and divide the product by the front term.
>
> 3rd rule. Front term and rear term may be multiplied by the same number; the same may be done to the front term and the middle term (Briem, 1869: p. 70),

followed by four pages of examples. Three more rules were introduced about how to treat the numbers according to their relative size, followed by 160 exercises, before the inverse rule and the composite rule are explained. Lastly, about 45 pages and 160 exercises after the introduction, the reader is warned to check if the quantity which is sought has indeed the same ratio to the rear term as the middle term to the front term, followed by examples, where the Rule of Three may not apply, or the inverse rule is the correct application.

The upper-secondary Reykjavík School, a necessary preparation for university studies and official positions, was the only school of its kind in Iceland when E. Briem's *Arithmetic* was published. The book was used there for some time, for independent study, e.g. by youngsters who were preparing themselves for its entrance examination, and in the first lower-secondary schools, established in the 1880s. G. Finnbogason, a young psychologist, was entrusted to write a proposal for education legislation, passed in 1907. He prepared the ground by writing a book, where he expressed his views about schools, education and instruction. He said:

> … one could throw out many of the computation rules, still found in some arithmetic textbooks, with a pompous look, as if they had stepped down from heaven. He who can think and use commonsensically the four main … operations, can … solve any Rule-of-Three problem, even if he/she has heard neither of the Rule of Three, a front term, middle term or rear term, nor the rules about their use (Finnbogason, 1994/1903: p. 93).

Finnbogason is referring to Briem's textbook. The quotation indicates that the syllabus of the Reykjavík School was being transferred to primary level. Briem stated in his foreword to the book that he emphasized memorizing. The rules on the three terms and their names seem to have been the focus of the study of the Rule of Three.

### *Arithmetic* by Ó. Daníelsson

Ó. Daníelsson completed a doctorate in mathematics at the University of Copenhagen in 1909. The third edition of his *Reikningsbók*/*Arithmetic* (1920) became very influential and had hardly any competition at the lower-secondary level for thirty years. The book was written against the background of regulations of 1904 for the Reykjavík School; a translation of Danish regulations for the middle school, based on school legislation for the Danish state of 1903 (Hansen, 2002: p. 132; National Archives of Iceland).

> A fraction is named 'Ratio' when both its numerator and denominator have the same name, for example $\frac{6m}{4m}$; it is read 6m 'against' 4 m. As now 4m·1½ = 6 m, the 1½ is called the 'numerical value' of the ratio … The numerical value is the number by which the denominator must be multiplied to have the numerator (Daníelsson, 1920, p. 43).

The author proceeded by explaining how to write the problem down in sentences. The initial terms are called a 'conditional sentence', while the second sentence is called a 'question'. The way of writing the problem is then:

> 4 m cost 3 *krónur*
>
> 6 m cost x *krónur* (Daníelsson, 1920: p. 44).

Now the ratio, $\frac{6m}{4m}$, is to be multiplied by 3, which means 3 *krónur*$\cdot \frac{6}{4}$ = 4.50 *krónur*.

The second example concerns 4 men, who complete a piece of work in 3 days. How many days would it take 6 men to complete the same work? In both cases the answer is to be found by multiplying the number in the 'conditional sentence' above the x by the numerical value of the ratio of the numbers in the other column.

> But how that ratio turns depends on whether the outcome is to be larger or smaller than the number in the conditional sentence above the x (Daníelsson, 1920: p. 44).

Daníelsson's rule of procedure remained a standard for about half a century. The front, middle or rear terms were not mentioned, while the stress was on a procedure on how to write the problem down and considering whether the quantity in question should be larger or smaller than the initial one. More examples and 14 exercises follow, first on prices of the new measuring units: metres, litres and kilograms of undefined merchandize, and the number of men performing undefined work in a number of days. Later they become more diverse. The text continues with the composite Rule of Three, society rule, inverse proportions, chain rule, percentages, interests and monetary rates. After 42 pages a warning appears that the Rule of Three may only be applied where the numbers increase or decrease proportionally. Problems on a free fall could e.g. not be solved by the Rule of Three, and those on interest could also involve fallacies. However, no exercises illustrate these important issues.

### *Arithmetic* by Gissurarson and Guðmundsson

Legislation in 1946 introduced compulsory schooling up to the age of 15, with a consequent expansion of 'mathematics for all'. A provisional curriculum only (*Drög að námsskrám*, 1948) followed the legislation, recommending certain pages in Daníelsson's *Arithmetic* with exercises of the Rule of Three. By the legislation the number of pupils increased, and the high difficulty level of Daníelsson's *Arithmetic* became an obstacle. The exercises were hard but ingenious.

In 1950 J. Á. Gissurarson and S. Guðmundsson published an alternative, *Reikningsbók*/*Arithmetic IIA*, which remained widespread for about a quarter of a century. It was taught side-by-side with Daníelsson's *Arithmetic*, but to a more diverse group of pupils. Gissurarson completed teacher training in Germany, while Guðmundsson became theologian in 1917 in order to be eligible as secondary teacher. He was in charge of the national entrance examination into high schools from 1946 to 1962. Gissurarson & Guðmundsson's treatment of the Rule of Three is short. The rule is introduced by an example where 5 kg cost 125 *krónur* as the 'conditional sentence', and then a 'unit sentence' by: How much does 1 kg cost? The question is how much 8 kg will cost. This is summarized in four rules:

> 1) Always write the conditional sentence such that the name to be found is behind.
>
> 2) In the unit sentence you must find out whether you should multiply or divide.
>
> 3) In the questioning sentence one should use an operation opposite to the one in the unit sentence.
>
> 4) The name in the final answer will always be the same as the name of the latter term in the conditional sentence (Gissurarson and Guðmundsson, 1949–1950: p. 29).

The total introduction and explanation of the Rule of Three takes one page. Exercises with examples in between and more variations of the rule follow on 50 pages. The topics of the exercises were similar to Daníelsson's; commerce, workdays, percentages, interests, etc. Nothing is said about ratio or proportions. The limitations were not mentioned, nor cases where the quantities are not proportional, or cases where larger quantities might be bargained down to lower prices than small ones.

A primary teacher, Gestur O. Gestsson, a former pupil of Dr. Daníelsson, criticized this textbook, stating that not a word was offered to lead to a sensible effort on the part of the pupils to find out solutions for themselves (Gestsson, 1962).

### *Arithmetic* by K. Gíslason

A national curriculum introduced in 1960 for lower-secondary education stated: 'Rule of Three with two terms' (Menntamálaráðuneytið, 1960, p. 22) without further explanation. Gíslason, a primary teacher by education and profession, wrote his *Arithmetic* in 1962. Gíslason introduced the Rule of Three by several exercises on prices of merchandise and men digging ditches, to be computed mentally, before proceeding to a formal introduction. He emphasized defining an 'initial sentence' leading to a 'unit sentence' and thereafter an 'answer sentence' after carefully coordinating the measuring units. This is done in five pages, followed by 120 exercises and further sections on interests and percentages with similar procedures. The treatment is clear, and takes care to explain and prevent common mistakes. However, no mention is made of ratio or proportions, or of cases when using the Rule of Three is inappropriate.

## CRITICISM FOLLOWING THE 'NEW MATH' MOVEMENT

The 'New Math' reached Iceland in the mid-1960s. Mathematics education was submitted to consideration and revision. H. Elíasson, a young mathematician, wrote:

> In the present textbooks there are a good many examples of bad computation tricks ... The calculation of proportions and ratios is an absolute insult to common sense in the form it is taught … the use of the Rule of Three ensures that the pupils have no idea of what they are doing, and are in no position to judge if it is really correct to use the Rule of Three ... Some knowledge is always needed to be able to solve a problem, knowledge of the concepts that occur. … one should teach the handling and use of concepts such as price, length, interest, etc. In that way one can get rid of the Rule of Three and come closer to the core of the concept of proportional calculations (Elíasson, 1966).

New textbooks were written in accord with the 'New Math' in the early 1970s. The Rule of Three was no longer mentioned. Problems involving ratios and proportions were solved by equations and graphs, while topics such as men digging ditches were replaced with more up-to-date ones.

## SUMMARY AND CONCLUSIONS

The Rule of Three is a legacy from medieval times, a guide to merchants and their customers in trade. It was useful for adult self-instruction in Iceland, not least while the metric system, introduced by law in 1907, had not come into common use. In 1946, when two years of secondary education became compulsory, the target group of arithmetic education changed, but the topics remained the traditional commercial arithmetic.

By Hatami's classification into emphasis on algorithms, proportionality or finding the unit, the textbooks by Olavius, Briem and Daníelsson fall within the middle group, emphasizing proportionality. Mathematician Daníelsson takes proportionality most seriously, while Briem, after introducing proportionality, places emphasis on memorizing algorithms. When the Rule of Three became a compulsory element of the syllabus in 1946, textbook authors Gissurarson & Guðmundsson and Gíslason turned to finding the unit by verbal argumentation, as the most didactic method for the new, young target group of pupils, and did not mention ratios and proportions until later.

However, instruction on proportions tended to develop into a ritual procedure for exercises of certain types, with content of little relevance to pupils. The ever-increasing number of exercises reflects the perceived difficulty of the topic, while its volume pushed out more relevant topics in the short academic year. Criticism did have an effect, but outside influence was needed.

Iceland was culturally isolated in the inter- and post-war periods, after gaining sovereignty in 1918. The problems of a newly-independent nation exerted an influence on educational matters, reflected in a neglect of mathematics education, meagre curriculum documents intended to describe ‘mathematics for all,’ and lack of renewal of teaching material. The ‘New Math’ finally brought long-needed currents from other countries, and new thinking about mathematics education.

However, Hatami’s reasoning on the merits of the Rule of Three is well justified (2007, p. 198). It did serve as a doorway to algebra, so inaccessible to many pupils, in the post-war period in Iceland. Maintaining and renewing the tradition might have been useful to Icelandic mathematics education, if the tasks had had more relevance to the pupils’ life and been more diverse, including cases when the Rule did not apply. But revolutions, as the ‘New Math’ turned out to become, tend to create an entirely new framework, which may not initially have room for old traditions.

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